A Cellular Automata Approach for Discrete-Time Distributed Parameter Systems

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- **Introduction to systems theory**

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- **Discrete-time DPS statement by means of CA formalism**

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  - **Cellular Automata models**
  - **The new state equation**
  - **Control in cellular automata**

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- **Concluding remarks**

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In the case of distributed parameter systems (DPS), f and h as well as all components (state, inputs, outputs, coefficients, etc) are dependent on time and space variables.

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- Its applications ranging from life sciences to industrial processes.

Introduction to systems theory

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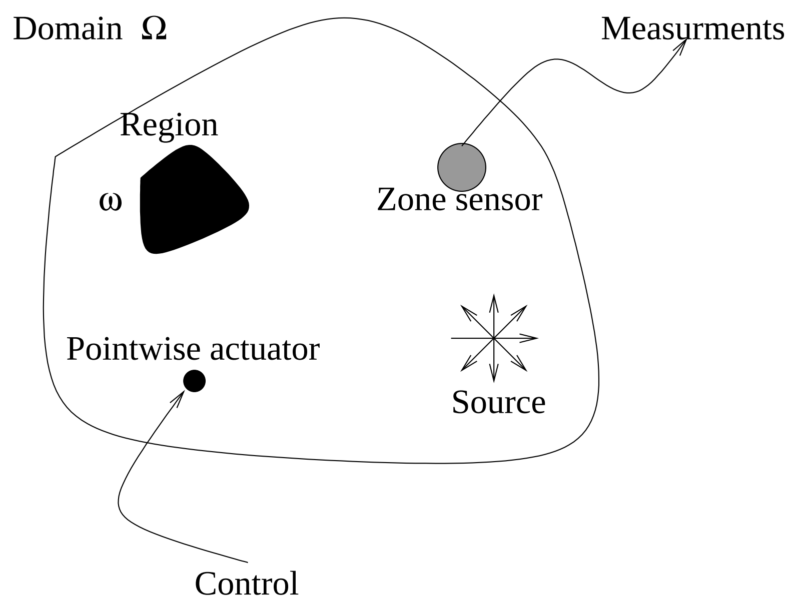

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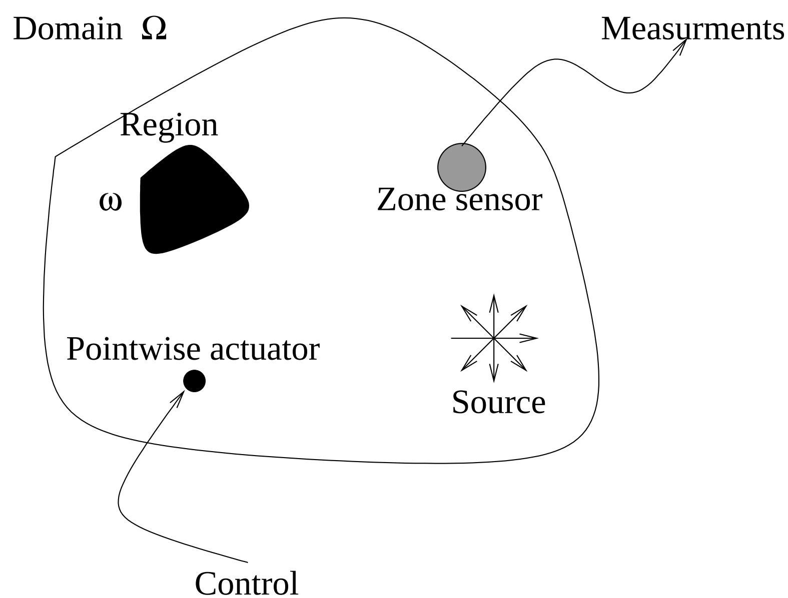

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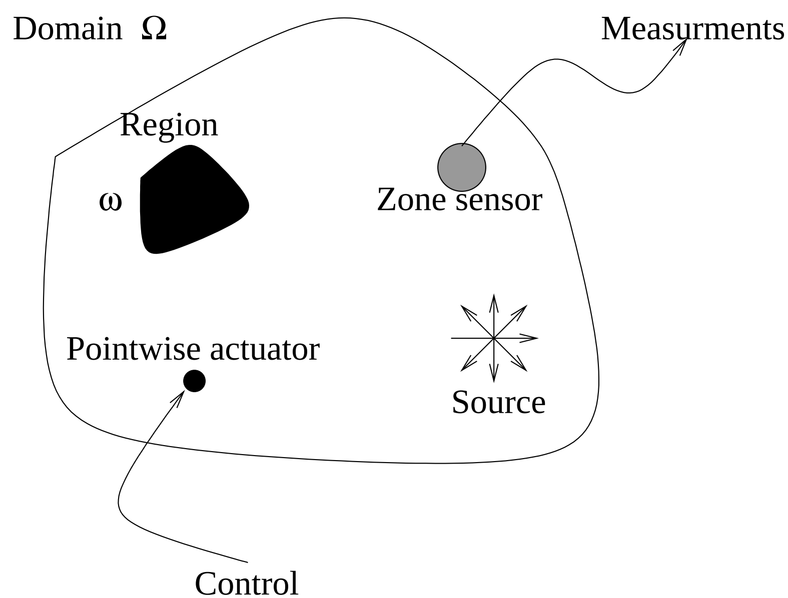

- Conventional models for DPS : Partial differential or integro-differential equations, continuous or discrete version
- Proposed models: Cellular Automata can be considered as a possible alternative for modelling and analysing DPS.

Discrete-time DPS statement by means of CA formalism

Cellular Automata models (CA)

A CA is classically defined by a quadruple $\mathcal{A} = (\mathcal{L}, \mathcal{S}, N, f)$ where

- $\mathcal{L}$ is a d-dimensional lattice of cells c which are arranged depending on space dimension and cell shape. In the infinite case, $\mathcal{L} = \mathbb{Z}^d$.
- $\mathcal{S}$ denotes a discrete state set. It's a finite commutative ring given by $\mathcal{S} = \{0, 1, \cdots, k-1\}$ in which the usual operations use modular arithmetics.
- N is a mapping which defines the cell's neighborhood. It's usually given by:

Discrete-time DPS statement by means of CA formalism

$$\begin{array}{rccl} N: & \mathcal{L} & \longrightarrow & \mathcal{L}^n \\ & c & \longrightarrow & N(c) = \{c' \in \mathcal{L} \mid \|c' - c\|_i \leq r\} \end{array}$$

where $\|c\|_i, i \in \{1, \infty\}$ indicate the sum and the maximum respectively, of the absolute value of the components of cell c.

- f is a transition function which can be defined by :

$$\begin{array}{rccl} f: & \mathcal{S}^n & \longrightarrow & \mathcal{S} \\ & s_t(N(c)) & \longrightarrow & s_{t+1}(c) = f(s_t(N(c))) \end{array}$$

where $s_t(c)$ designates the c cell state at time t and $s_t(N(c)) = \{s_t(c'), c' \in N(c)\}$ is the neighborhood state.

Discrete-time DPS statement by means of CA formalism

The new state equation

- A (DPS) is generally defined by a triple of operators (A, B, C) corresponding to the system dynamics, the control and observation operators respectively.

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- Consider the case $\mathcal{L} = \mathbb{Z}^d$, ($d \geq 1$) and introduce a metric over $X = \mathcal{S}^{\mathbb{Z}^d}$ as : $d_\delta(x, y) = \sum_{c \in \mathbb{Z}^d} \frac{\delta(x(c), y(c))}{2^{||c||_\infty}}$ where $\delta : \mathcal{S} \times \mathcal{S} \rightarrow \{0, 1\}$ is defined by : $\delta(i, j) = \begin{cases} 0 & \text{if} \quad i = j \\ 1 & \text{if} \quad i \neq j \end{cases}$

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- The set $X = \mathcal{S}^{\mathcal{L}}$ equipped with the distance d_δ is a compact metric space and the global dynamics $F : \begin{array}{ccc} X & \to & X \\ s & \to & F(s) \end{array}$ is continuous according to the topology induced by d_δ.

Discrete-time DPS statement by means of CA formalism

Result : In the context of DPS,

- the compact configurations set X defines the state space of the autonomous CA.
- the sequence of continuous global maps F^i defined as the i^{th} iteration under F, plays the same role than the semi-group, usually denoted by (Φ_t) generated by the operator A.

In a similar way to linear discrete-time DPS, the evolution of an autonomous CA starting from a given initial configuration s_0 can be defined in terms of the global dynamics by the state equation :

$$\begin{cases} s_{t+1} & = & F s_t \\ s_0 & \in & X \end{cases}$$

Discrete-time DPS statement by means of CA formalism

whose solution given by : $s_t = F^t(s_0), \ t \in I$
has the same form than the solution of discrete-time DPS in the autonomous case, given by

$$x(t) = \Phi_t(x_0), \ t \geq 0$$

where $(\Phi_t)_{t \geq 0}$ is the semi-group generated by an operator defining the system dynamics A :

$$Ax = \lim_{t \to 0^+} \frac{\Phi_t(x) - x}{t}$$

where this limit exists.

Discrete-time DPS statement by means of CA formalism

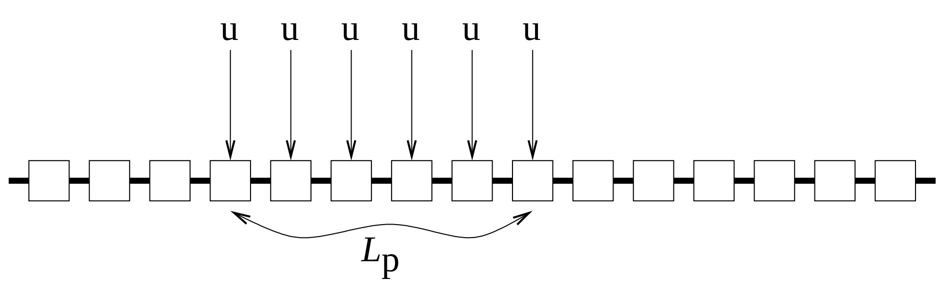
Control in cellular automata

The CA model will be completed by control and measurement functions. For the control aspects, it is done via inputs (actuators) which have a spatial structure (number, spatial location and distribution).

Let us consider for a 1-D CA :

• $I_T = \{0, 1, \cdots, T\}$ is a discrete time horizon.

• $\mathcal{L}_p$ is a sub-domain which defines the region of the lattice $\mathcal{L}$ where the CA is excited. It contains p cells which may be connected or not.



Discrete-time DPS statement by means of CA formalism

- The study of CA in terms of DPS needs the introduction of some specific spaces and operators related to the control and observation.
- The Control space is defined by

$$\mathcal{U} = \ell^2(\mathcal{L}_p, \mathbb{R})$$

where $\ell^2(\mathcal{L}_p, \mathbb{R}) = \{u \ : \mathcal{L}_p \longrightarrow \mathbb{R} \mid \sum_{c \in \mathcal{L}_p} u^2(c) < \infty\}$.

The inner product in ℓ^2 is defined by :

$$< u_1, u_2 >_{\ell^2} = \sum_{c \in \mathcal{L}_p} u_1(c) u_2(c)$$

with the associated norm $||u||_{\ell_2} = \sqrt{< u, u >_{\ell^2}}$

Discrete-time DPS statement by means of CA formalism

- The control operator G which defines the way how the control excites the CA through the cells of $\mathcal{L}_p$, is given by :

$$\begin{array}{rccc} G: & \mathcal{U} & \longrightarrow & \mathcal{S}^{\mathbb{Z}^d} \\ & u & \longrightarrow & Gu \end{array}$$

- The CA is then considered as a controlled system denoted by $\mathcal{A}_c$ defined by the local transition function :

$$s_{t+1}(c) = f_c(s_t(N(c)), u_t(c)\chi_{\mathcal{L}_p})$$

- The corresponding state equation is:

$$\begin{cases} s_t & = & \mathcal{F}(s_{t-1}, u_{t-1}) \ , \ t \in I_T \\ s_0 & & \in X \end{cases}$$

Discrete-time DPS statement by means of CA formalism

where $\mathcal{F}$ is defined according to the global dynamics F and the global control operator G.

- The observation problem can be considered by duality where an observation space and a global observation operator have to be defined.
- This leads to a complete description of CA in terms of inputs and outputs where the state equation is augmented with

$$\theta_t = Hs_t \ , \ t \in I_m$$

  and then defines the so-called distributed CA (DCA).
- The obtained CA statement is very close to the usual discrete-time distributed parameter systems formulation augmented by the output function.

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The rules are as follows: An ant sits on a bit of graph paper where all the squares are initially empty. It moves into a neighbouring square and does one of two things, based on the colour of the square;

- If the square is white, it turns to the left and colours the square black.
- If the square is black, it turns to the right and colours the square white.

The interesting thing is that after a fixed number of steps, the ant builds a highway and goes very quickly to infinity.

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After 9000 iterations

After 11000 iterations

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With a control applied on cells $c_{i,j}$ such that $|i - j| = 2$:

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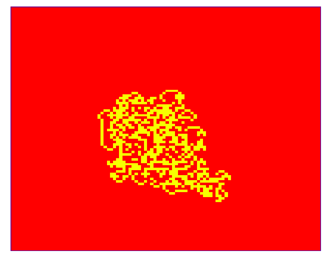

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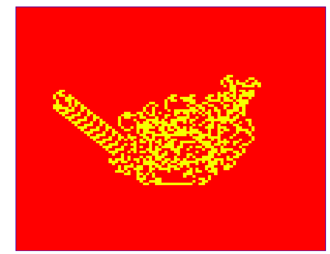

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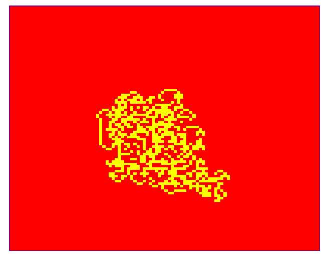

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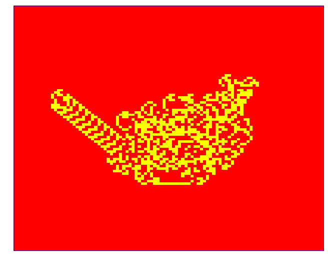

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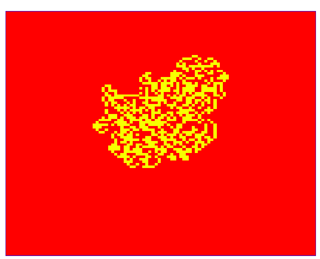
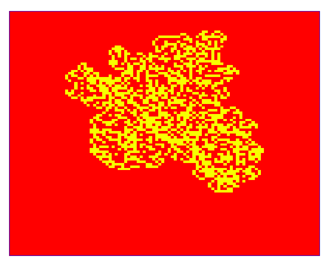
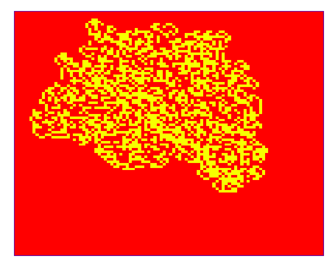


After 10000 iterations



After 30000 iterations



After 45000 iterations

Important result : The ant stays much longer in a bounded area with the control.

Application to Regional Controllability

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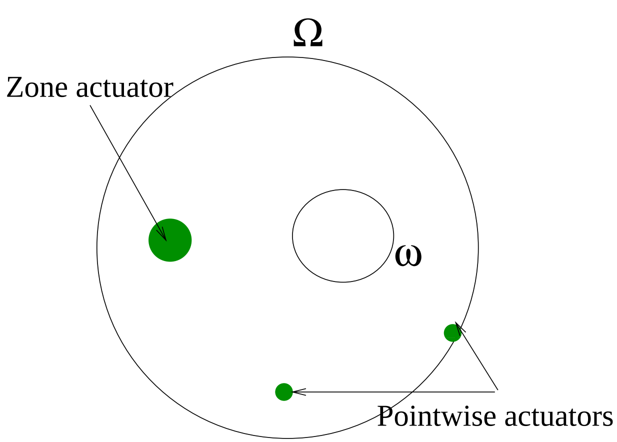
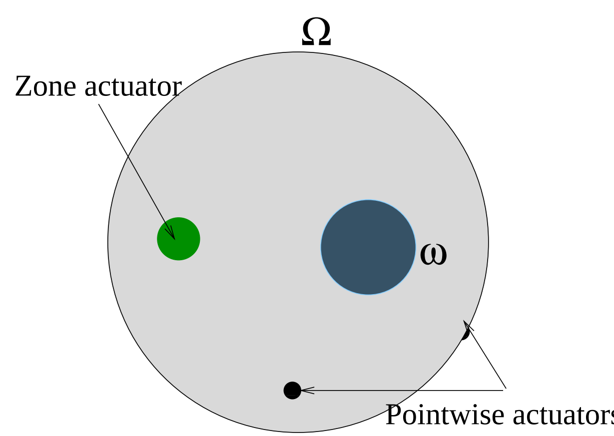


System (S) at t = 0



System (S) at t = T

Application to Regional Controllability

For the statement by means of CA formalism :

- a DCA defined on a discrete lattice $\mathcal{L}$ with state set $\mathcal{S}$ and described by the state equation above given.
- $\omega \subset \mathcal{L}$
- s_ω the restriction to ω of the CA configuration s.
- $\mathcal{S}^\omega = \{s \ : \omega \to \mathcal{S}\}$

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The DCA is said to be regionally controllable if for a given $s_d \in \mathcal{S}^\omega$ there exists a control $u = (u_0, \cdots, u_{T-1})$ with $u_i \in \mathcal{U}$ such that

$$s_T = s_d \quad \text{on} \quad \omega$$

where s_T is the final configuration at time T and $\mathcal{U}$ is the control space.

Application to Regional Controllability

A weak regional controllability is achieved if for $s_d \in \mathcal{S}^\omega$ and $\varepsilon > 0$, there exists a control $u = (u_0, u_1, \cdots, u_{T-1})$ such that

$$d_\delta(s_T, s_d) \leq \varepsilon \quad \text{on } \omega$$

where d_δ is the distance defined on X.

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The main characterization is based on :

- The target region ω and its geometry
- The space location of the region ω
- The actuators exciting the system, number, location, etc.

Application to Regional Controllability

A characterization result

Let K be the operator defined by

$$\begin{array}{rcccc} K & : & \mathcal{U}^{I_T} & \longrightarrow & \mathcal{S}^{\mathcal{L}} \\ & & u & \longrightarrow & s_T \end{array}$$

where s_T is the configuration of the CA at time T.

Application to Regional Controllability

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The problem of regional controllability consists in finding a control $u \in \mathcal{U}^{I_T}$ such that $\chi_\omega K(u) = s_d$ where χ_ω from $\mathcal{S}^{\mathcal{L}} \longrightarrow \mathcal{S}^{\omega}$ defines the restriction mapping.

$$\Leftrightarrow \mathcal{I}m(\chi_\omega K) = \mathcal{S}^{\omega}$$

Application to Regional Controllability

We obtain the following necessary and sufficient condition for exact regional controllability :

Proposition:
The DCA is exactly regionally controllable if and only if

$$Ker\chi_\omega + \mathcal{I}mK = \mathcal{S}^{\mathcal{L}}$$

where $Ker\chi_\omega = \{s \in \mathcal{S}^{\mathcal{L}} \mid s_\omega = \underline{0}\}$ denotes the kernel of χ_ω and $\underline{0}$ indicates the null configuration. The operation $+$ in the rank condition is interpreted as follows :
$\forall\ s\ \in\ \mathcal{S}^{\mathcal{L}}$, $\exists\ s_1 \in Ker\chi_\omega$ and $s_2 \in \mathcal{I}mK$ such that

$$s = s_1 + s_2$$

Application to Regional Controllability

Proof :

(1) $Ker\chi_\omega + \mathcal{I}mK = \mathcal{S}^{\mathcal{L}} \Rightarrow \mathcal{I}m(\chi_\omega K) = \mathcal{S}^\omega$.
The inclusion $\mathcal{I}m(\chi_\omega K) \subseteq \mathcal{S}^\omega$ is immediate.

Let $s \in \mathcal{S}^\omega$ and consider $s' \in \mathcal{S}^{\mathcal{L}}$ given by : $s' = \begin{cases} s & \text{on } \omega \\ 0 & \text{on } \omega^c \end{cases}$

Due to relation (1), we can write $s' = s_1 + s_2$ where :

- $s_1 \in ker\chi_\omega \Rightarrow s_1|_\omega = 0$
- $s_2 \in \mathcal{I}mK \Rightarrow \exists u \in \mathcal{U}^{I_T}$ such that $K(u) = s_2 \Rightarrow \chi_\omega K(u) = s_2|_\omega$

$= s_2|_\omega + s_1|_\omega = s'|_\omega = s \Rightarrow s \in \mathcal{I}m(\chi_\omega K)$ and then $\mathcal{S}^\omega \subseteq \mathcal{I}m(\chi_\omega K)$.

Application to Regional Controllability

($\Leftarrow$)

$Ker\chi_\omega + \mathcal{I}mK \subset \mathcal{S}^{\mathcal{L}}$ is obvious since $ker\chi_\omega \subset \mathcal{S}^{\mathcal{L}}$ and $\mathcal{I}mK \subset \mathcal{S}^{\mathcal{L}}$.

$Ker\chi_\omega + \mathcal{I}mK \supset \mathcal{S}^{\mathcal{L}}$. Let us introduce the quotient space $\dfrac{\mathcal{S}^{\mathcal{L}}}{Ker\chi_\omega}$ and the canonical projection π of $\mathcal{S}^{\mathcal{L}}$ onto $\dfrac{\mathcal{S}^{\mathcal{L}}}{Ker\chi_\omega}$ and consider $\overline{\chi}_\omega$ defined by the commutative diagram :

$$\begin{array}{ccc} \mathcal{S}^{\mathcal{L}} & \xrightarrow{\chi_\omega} & \mathcal{S}^{\omega} \\ \pi \downarrow & & \downarrow I \\ \dfrac{\mathcal{S}^{\mathcal{L}}}{Ker\chi_\omega} & \xrightarrow{\overline{\chi}_\omega} & \dfrac{\mathcal{S}^{\omega}}{Ker\chi_\omega} \end{array}$$

Let $s \in \mathcal{S}^{\mathcal{L}} \Rightarrow \chi_\omega(s) \in \mathcal{S}^{\omega} = \mathcal{I}m(\chi_\omega K) \Rightarrow \exists u \in \mathcal{U}^{I_T}$ such that $\chi_\omega(s) = \chi_\omega(K(u)) \Rightarrow s - K(u) \in Ker\chi_\omega \Rightarrow s = s_1 + s_2$ with $s_1 = s - K(u)$ and $s_2 = K(u) \in \mathcal{I}mK$.

Application to Regional Controllability

Case of additive cellular automata

For additive CA, the global dynamics F satisfies

$$F(s_1 + s_2) = F(s_1) + F(s_2) \ ; \ s_1, s_2 \in \mathcal{S}^{\mathcal{L}}$$

Application to Regional Controllability

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The state equation can be expressed in terms of F and the global control operator G by:

$$\begin{cases} s_t & = & F(s_{t-1}) + G(u_{t-1})\ ,\ t \in I_T \\ s_0 & & \in X \end{cases}$$

where F and G are linear bounded operators with state space $X = \mathcal{S}^{\mathcal{L}}$ and input-value space $\mathcal{U} = \ell^2(\mathcal{L}_p, \mathbb{R})$. Moreover, F is the generator of a discrete-time semi-group $\{F^t\}_{t\geq 0} \subset \mathbf{L}(X)$ (linear bounded operators).

Application to Regional Controllability

With the above hypothesis, we can express the solution s_t for $t \geq 0$ in the following compact form :

$$s_t = F^t(s_0) + \sum_{\tau=0}^{t-1} F^{t-\tau-1} G(u_\tau)$$

which is similar to the linear continuous case where the system is given by

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t), \quad t \in]0, T[\\ z(0) = z_0 \in \mathcal{D}(A) \end{cases}$$

and its solution

$$z(t, u) = \Phi(t) z_0 + \int_0^t \Phi(t - \tau) B u(\tau) d\tau$$

where $(\Phi(t))_{t \geq 0}$ is a strongly continuous semi-group generated by the operator A.

Application to Regional Controllability: Additive case

- In 1-D CA, the state transition may be expressed in terms of a characteristic matrix M of order $(N_{\mathcal{L}} \times N_{\mathcal{L}})$ given by :

$$s_{t+1} = M s_t$$

where $N_{\mathcal{L}} = |\mathcal{L}|$ and s_t is represented by a $N_{\mathcal{L}}$-vector.

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- The matrix M is then constructed as follows :

$$M_{ij} = \begin{cases} a_{j-i} & \text{if } j \in [i-r, i+r] \\ 0 & \text{elsewhere} \end{cases}$$

Application to Regional Controllability

- Let ω be a subregion of $\mathcal{L}$ which has n_ω cells and s_0, $s_T|_\omega$ are the two vectors representing the CA configurations at time $t = 0$ and T respectively.

Application to Regional Controllability

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where u_t is at the position i_p and can take values in $\mathcal{S}$ as the $\mathcal{S}$-valued control variables.
- Let M be the transition matrix corresponding to the additive CA. We can easily show that the CA state is given at time T by :

$$s_T = M^T s_0 + \sum_{i=0}^{T-1} M^{T-i} V_i$$

Application to Regional Controllability: Additive case

Let s_d be a desired state on ω. The problem of regional controllability in terms of characteristic matrix, consists in finding a vector V_t, $t = 0, \cdots, T-1$, such that $s_T|_\omega = s_d$ or

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This problem can be formulated as a system of n_ω equations with T unknowns $(u_0, u_1, \cdots, u_{T-1})$:

$$\left\{\begin{array}{lllllll}
M^T_{\alpha,i_p} k_0 & +M^{T-1}_{\alpha,i_p} k_1 & +\cdots & +M_{\alpha,i_p)} k_{T-1} & = & s_d(\alpha) - (M^T s_0)_\alpha \\
M^T_{\alpha+1,i_p} k_0 & +M^{T-1}_{\alpha+1,i_p} k_1 & +\cdots & +M_{\alpha+1,i_p} k_{T-1} & = & s_d(\alpha+1) - (M^T s_0)_{\alpha+1} \\
 & & \vdots & & & \\
M^T_{\beta,i_p} k_0 & +M^{T-1}_{\beta,i_p} k_1 & +\cdots & +M_{\beta,i_p} k_{T-1} & = & s_d(\beta) - (M^T s_0)_\beta
\end{array}\right.$$

where $\beta = \alpha + n_\omega - 1$, $s_d(i) = s_d(c-i)$.

Application to Regional Controllability: Additive case

The system's solution which is not unique, gives the control $(V_0, V_1, \cdots, V_{T-1})$ which steers the CA to the state s_d at time T on ω.
This will be illustrated through the following numerical example.
With $n_{\mathcal{L}} = 100$, $\mathcal{S} = \{0, 1, 2\}$, $N(c_i) = \{c_{i-1}, c_i, c_{i+1}\}$ with an additive rule defined by :

$$s_{t+1}(c_i) = s_t(c_{i-1}) + 2s_t(c_{i+1}) \mod 3$$

Let s_0 be an arbitrary initial configuration and suppose that the control is active only in the cells c_{23}. We will determine the appropriate action which is able to steer the system from the initial configuration s_0 to a desired configuration s_d, on the region $\omega = \{c_{21}, \cdots, c_{25}\}$ at time $T = 5$ where s_d is defined by :

$$s_d(c_i) = 0 \ , \ 21 \leq i \leq 25$$

Application to Regional Controllability: Additive case

The regional controllability problem is equivalent to find $u_0, u_1, \cdots, u_4$ solutions of the system :

$$\left\{ \begin{array}{l} M^5_{21,23}u_0 + M^4_{21,23}u_1 + M^3_{21,23}u_2 + M^2_{21,23}u_3 + M_{21,23}u_4 = s_d(21) - (M^5 s_0)_{21} \\ M^5_{22,23}u_0 + M^4_{22,23}u_1 + M^3_{22,23}u_2 + M^2_{22,23}u_3 + M_{22,23}u_4 = s_d(22) - (M^5 s_0)_{22} \\ M^5_{23,23}u_0 + M^4_{23,23}u_1 + M^3_{23,23}u_2 + M^2_{23,23}u_3 + M_{23,23}u_4 = s_d(23) - (M^5 s_0)_{23} \\ M^5_{24,23}u_0 + M^4_{24,23}u_1 + M^3_{24,23}u_2 + M^2_{24,23}u_3 + M_{24,23}u_4 = s_d(24) - (M^5 s_0)_{24} \\ M^5_{25,23}u_0 + M^4_{25,23}u_1 + M^3_{25,23}u_2 + M^2_{25,23}u_3 + M_{25,23}u_4 = s_d(25) - (M^5 s_0)_{25} \end{array} \right.$$

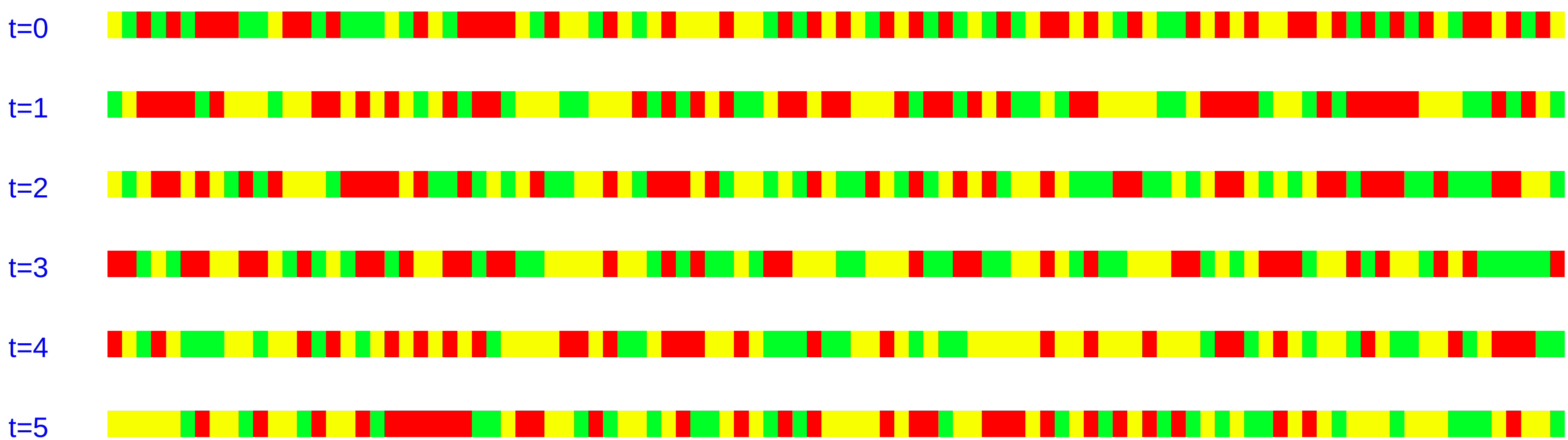
where M is a sparse transition matrix given by :

$$M = \begin{pmatrix} 3 & 5 & 0 & 0 & . & . & . & . & . & 4 \\ 4 & 3 & 5 & 0 & . & . & . & . & . & 0 \\ 0 & 4 & 3 & 5 & 0 & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ 5 & 0 & 0 & . & . & . & . & 0 & 4 & 3 \end{pmatrix}$$

Application to Regional Controllability: Additive case

Using the congruence modulo 3, a solution of this problem is given by :

$$(u_0, u_1, u_2, u_3, u_4) = (2, 0, 1, 2, 0)$$



Application to Regional Controllability

Particular 2-D Additive case

Let us consider a square lattice $\mathcal{L} = \{c_{i,j},\ i,j = 1, \cdots, N_{\mathcal{L}}\}$, a discrete set $\mathcal{S} = \{0, 1, \cdots, k-1\}$ and the following transition rule :

$$s_{t+1}(c_{i,j}) = \sum_{c' \in \dot{N}(c_{i,j})} s_t(c') \text{ mod } k$$

where $N(c)$ is the von Neumann neighbourhood of radius r and $\dot{N} = N - \{c\}$. The corresponding transition matrix M is constructed as follows :

$$M_{i,j} = \begin{cases} 1 & \text{if } \ c_{j,k} \in N(c_{i,j}) \quad \text{for every } k = 1, \cdots, N_{\mathcal{L}} \\ 0 & \qquad \text{elsewhere} \end{cases}$$

Application to Regional Controllability

We can easily give the following relation between s_{t+1} and s_t by means of the transition matrix M :

$$s_{t+1} = M s_t + s_t M$$

By stacking the columns of the matrix s_t in a vector and using the Kronecker product, i.e. using

$\sigma_t := Vec(s_t) \in \mathcal{S}^{(n_{\mathcal{L}})^2}$ and $\mathcal{M} := (I \otimes M) + (M^* \otimes I) \in \mathcal{S}^{(n_{\mathcal{L}})^2 \times (n_{\mathcal{L}})^2}$

The equation can be made to read :

$$\sigma_{t+1} = \mathcal{M} \sigma_t$$

which leads to the final state in terms of $\mathcal{M}$ and σ_0 : $\sigma_T = \mathcal{M}^T \sigma_0$

Application to Regional Controllability

If the control is active only on c_{i_p,i_p}, its effect may be expressed by :

$$W_t = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} \begin{array}{ccccc} & & i_p^{\text{th}} & & \end{array} & \\ \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ & \vdots & & \vdots & \\ 0 & \cdots & u_t & \cdots & 0 \\ & \vdots & & \vdots & \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} & \begin{array}{c} \\ \\ i_p^{\text{th}} \\ \\ \\ \end{array} \end{array}$$

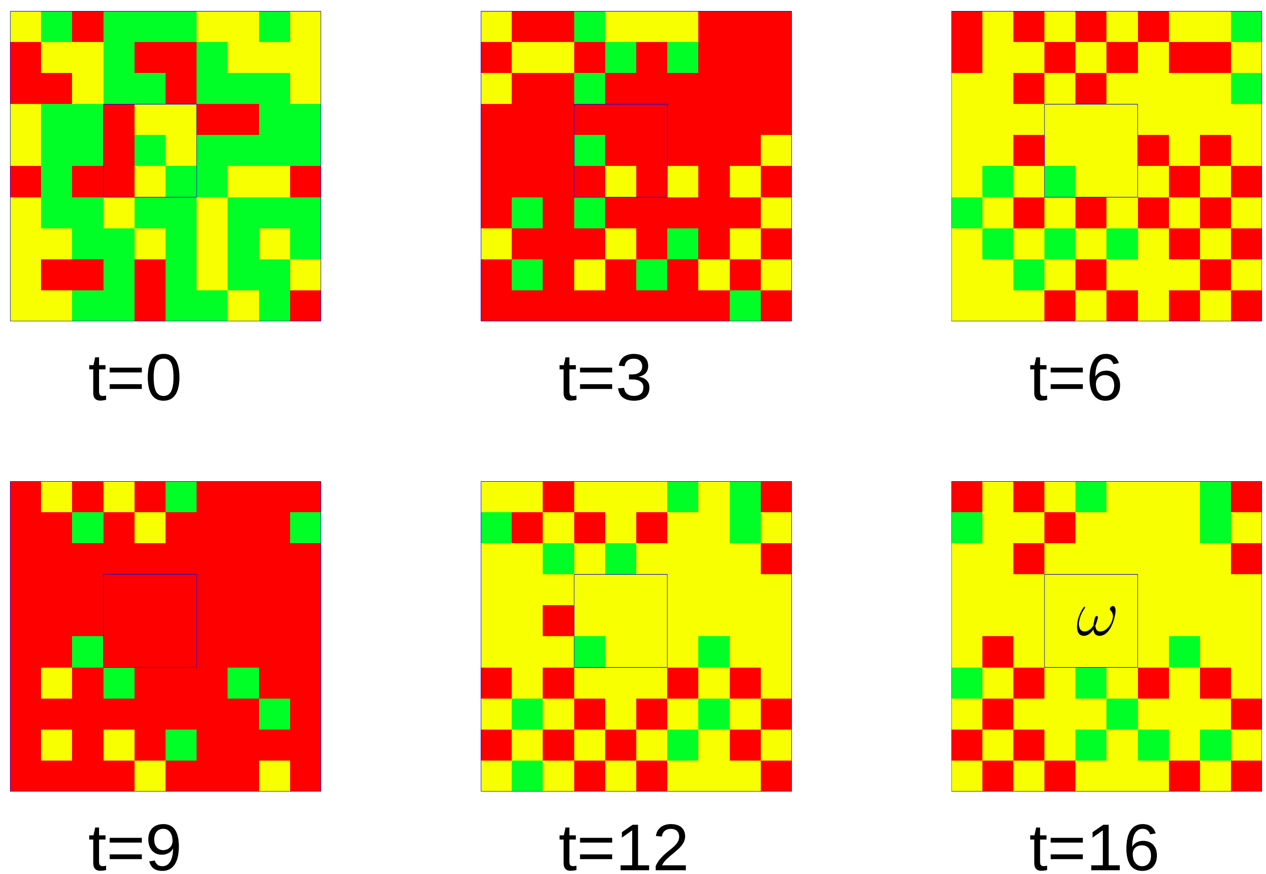
A similar result as for the 1-D case is then obtained.

$$\mathcal{W}_t := Vec(W_t) \in \mathcal{S}^{(n_{\mathcal{L}})^2}$$

and the CA state at time T is

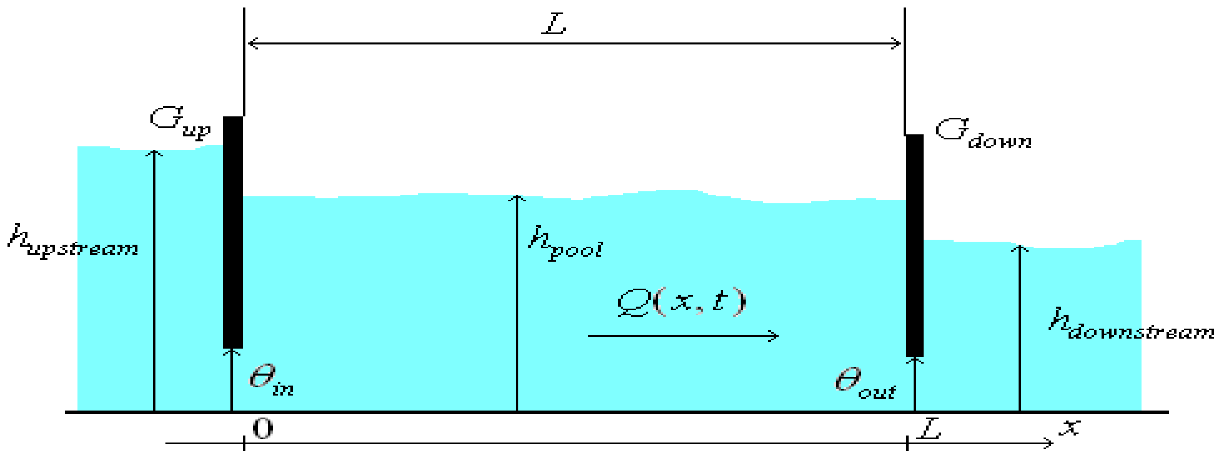
$$\sigma_T = \mathcal{M}^T \sigma_0 + \sum_{i=0}^{T-1} \mathcal{M}^{T-i} \mathcal{W}_i$$

Regional Controllability in a 2-D example



Some real applications

Modelling and control of irrigation channels



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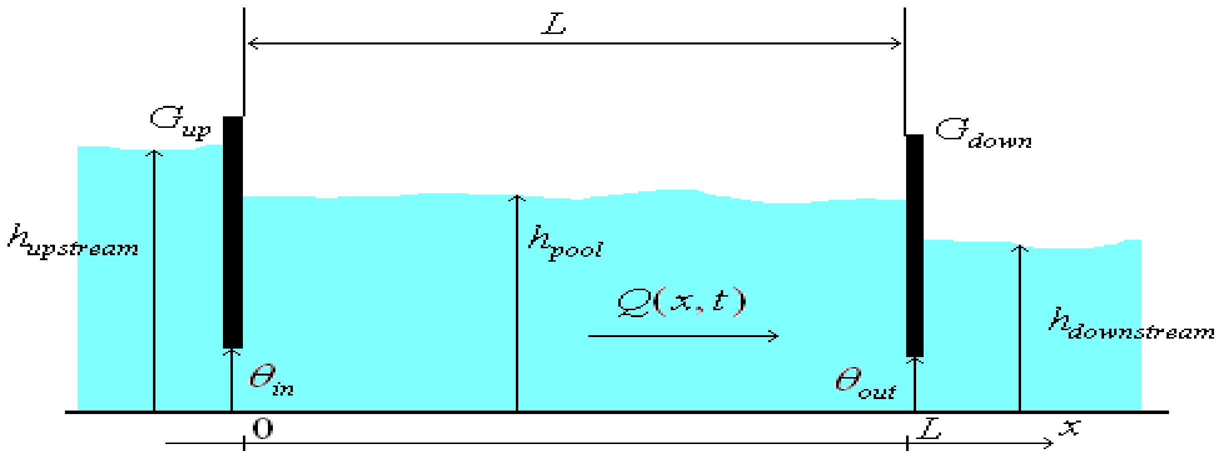

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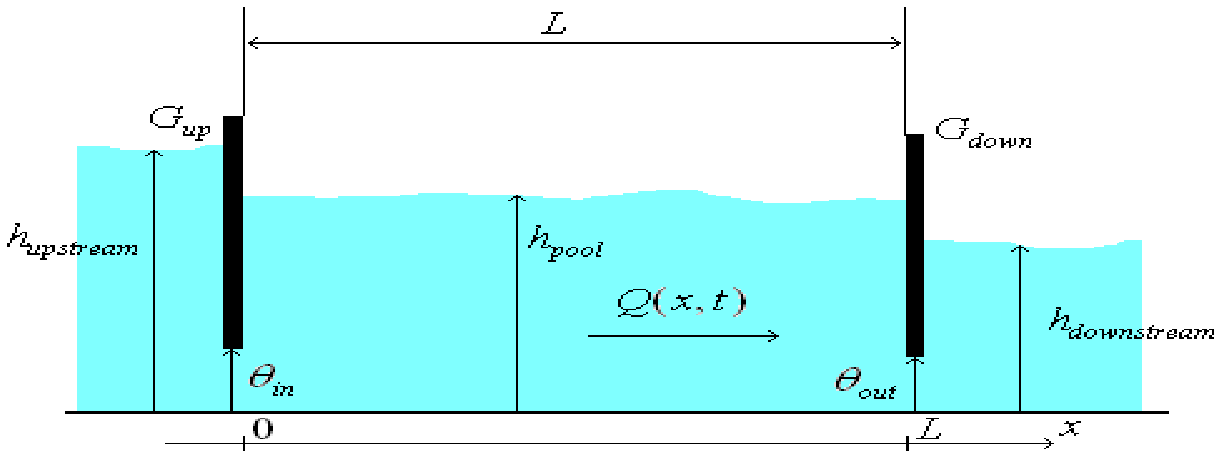

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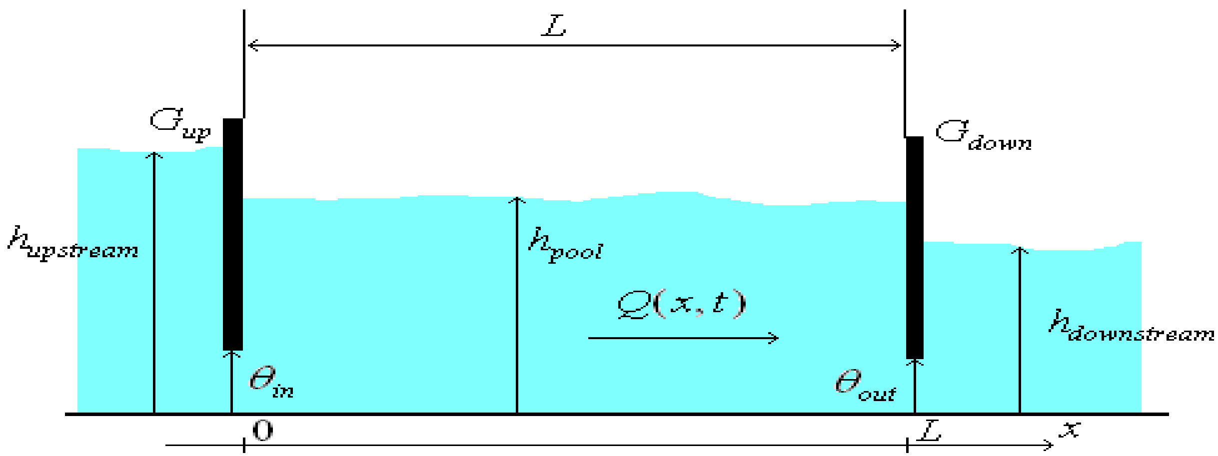

- A Lattice Boltzmann approach is used instead of the usual Saint-Venant equation
- The system is controlled through two gates whose opening allow an inlet and outlet discharge of water Q_{in} and Q_{out}.
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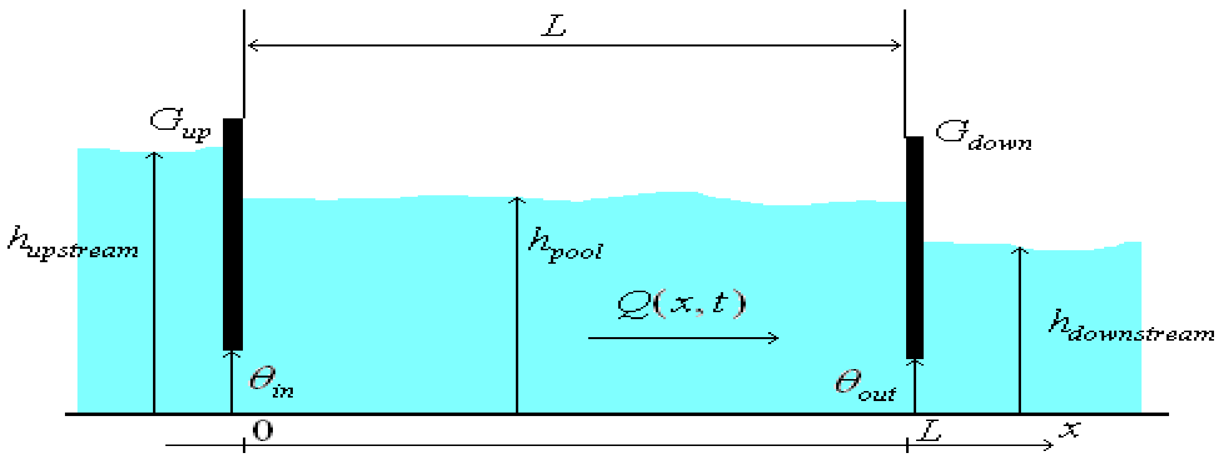

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- The water level in the pool reaches a given height profile $h(x,t)$ which has to be maintained between two desired levels h_{min} and h_{max}.
- Parameter identification has been done using experimental data provided by the micro-canal of Valence (ESISAR).

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Modelling and control of the population dynamics of the insect vectors, responsible of Chagas disease in a village of the Yucatan peninsula.

Some real applications

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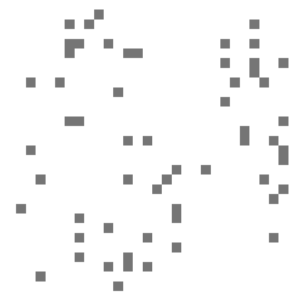
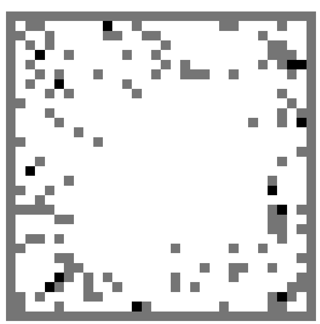
The proposed CA model gives the following evolution :

5^{th} day

90^{th} day

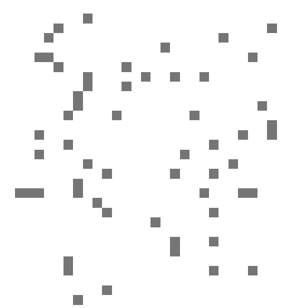


180^{th} day



370^{th} day

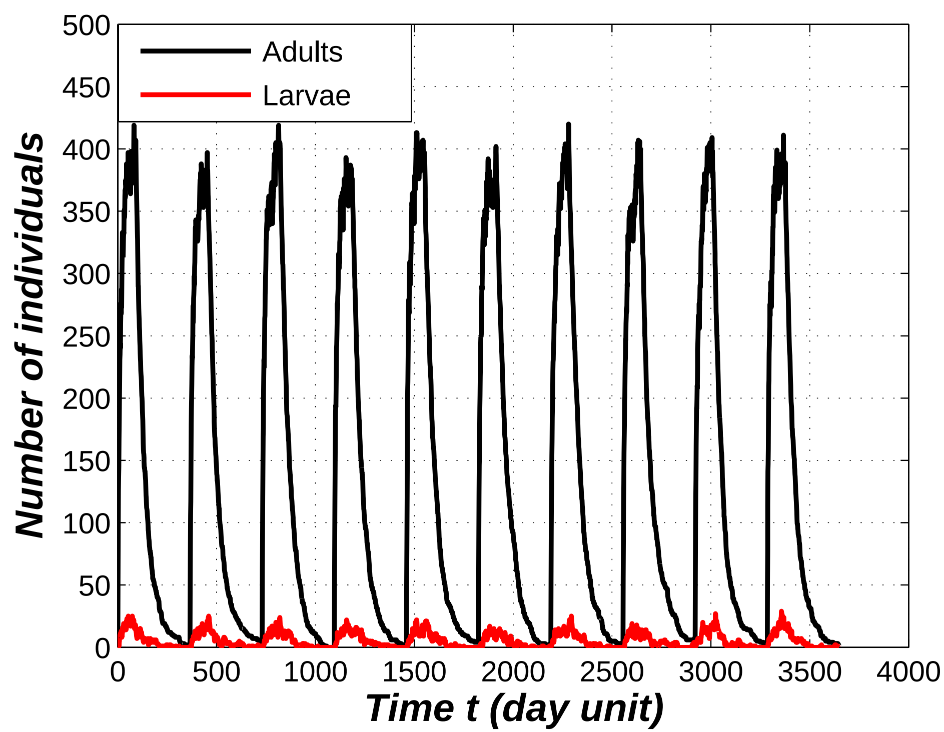
455^{th} day



545^{th} day

Some real applications

The variations in abundance and proportion of adults at the village scale did reproduce the patterns observed in several villages of Yucatan.



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- A rigorous analysis of the possibility to control the periodic spread of non-domiciliated vectors by an optimal spatial strategy is under consideration.
- This will be performed within the framework of control theory. It will be a kind of boundary control which will act on the border of the village in order to prevent the invasion and then the insects spread.

Concluding remarks

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- Simulation of DPS and control implementation can be overcome
- The control problems which are related to optimization techniques will use learning algorithms rather than classical ones which can't be applied in the context of CA.
- The so-called intelligent control which uses artificial intelligence computing approaches : neural networks, machine learning, evolutionary computation, etc, would be considered.