How to achieve universality in a CA using the same local function but different neighborhoods

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Outline

Introduction

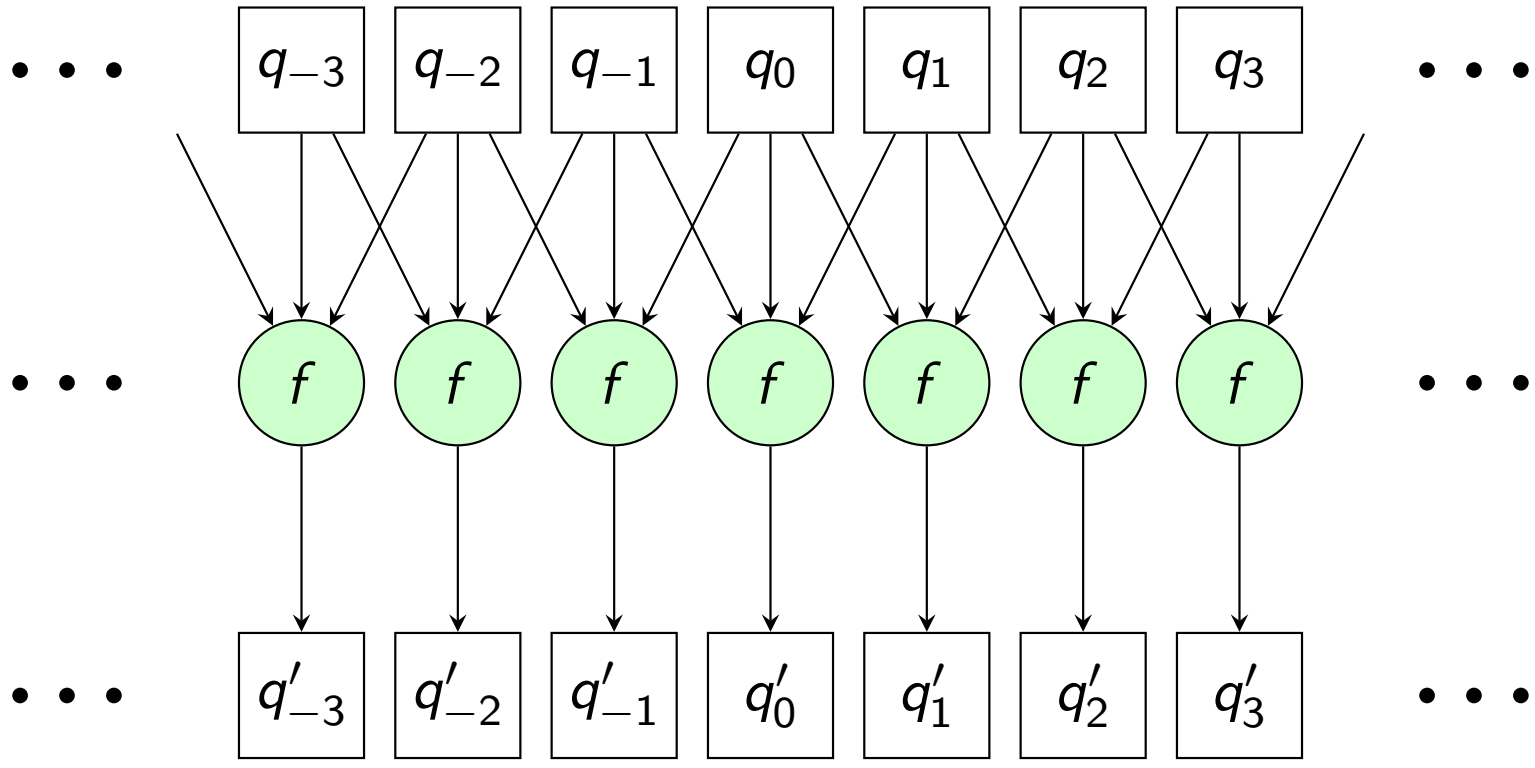
How to simulate a finite number of arbitrary CA with one local function

How to achieve universality

Discussion of the embeddings used

Outline

The basic picture



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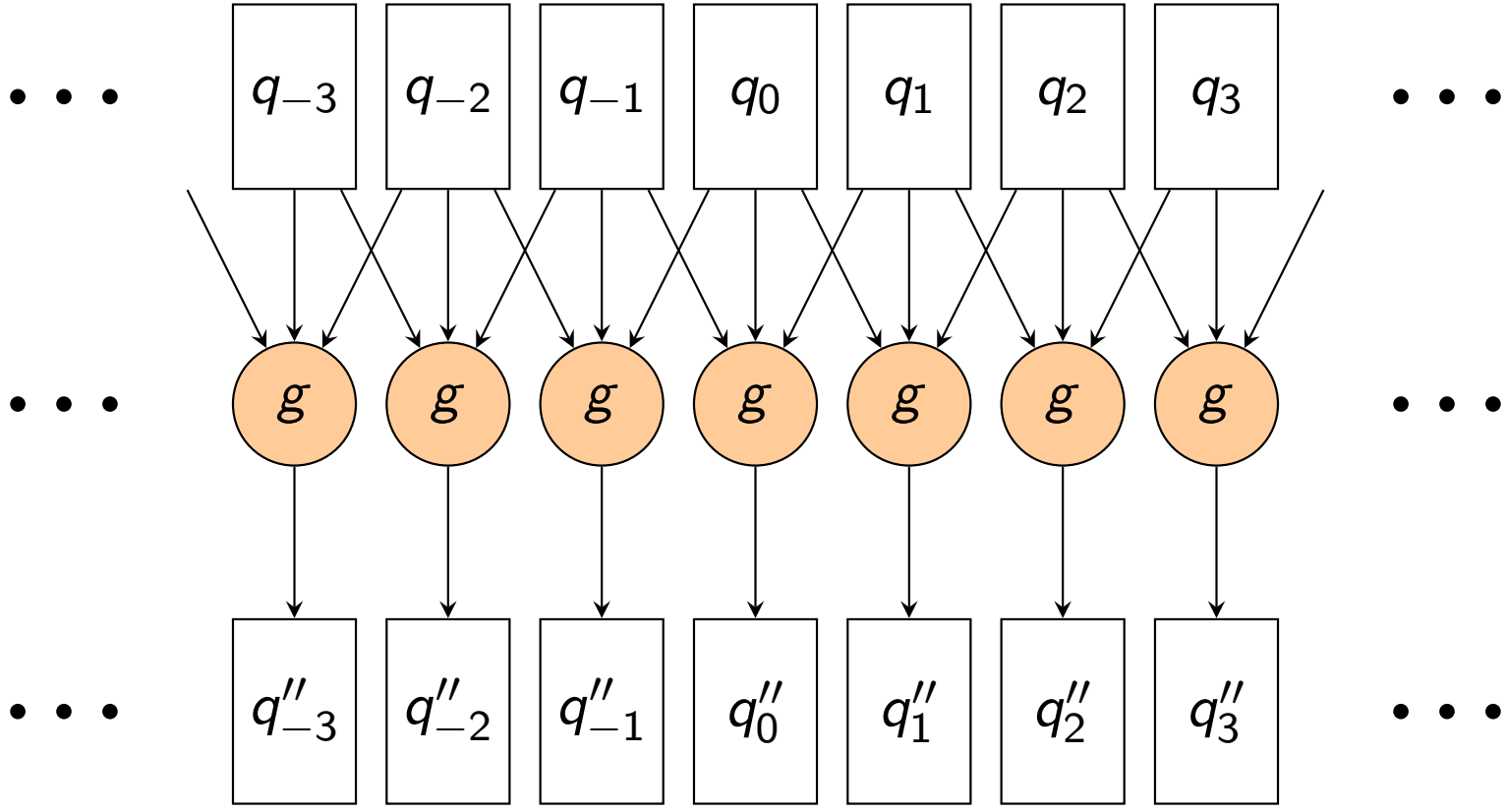


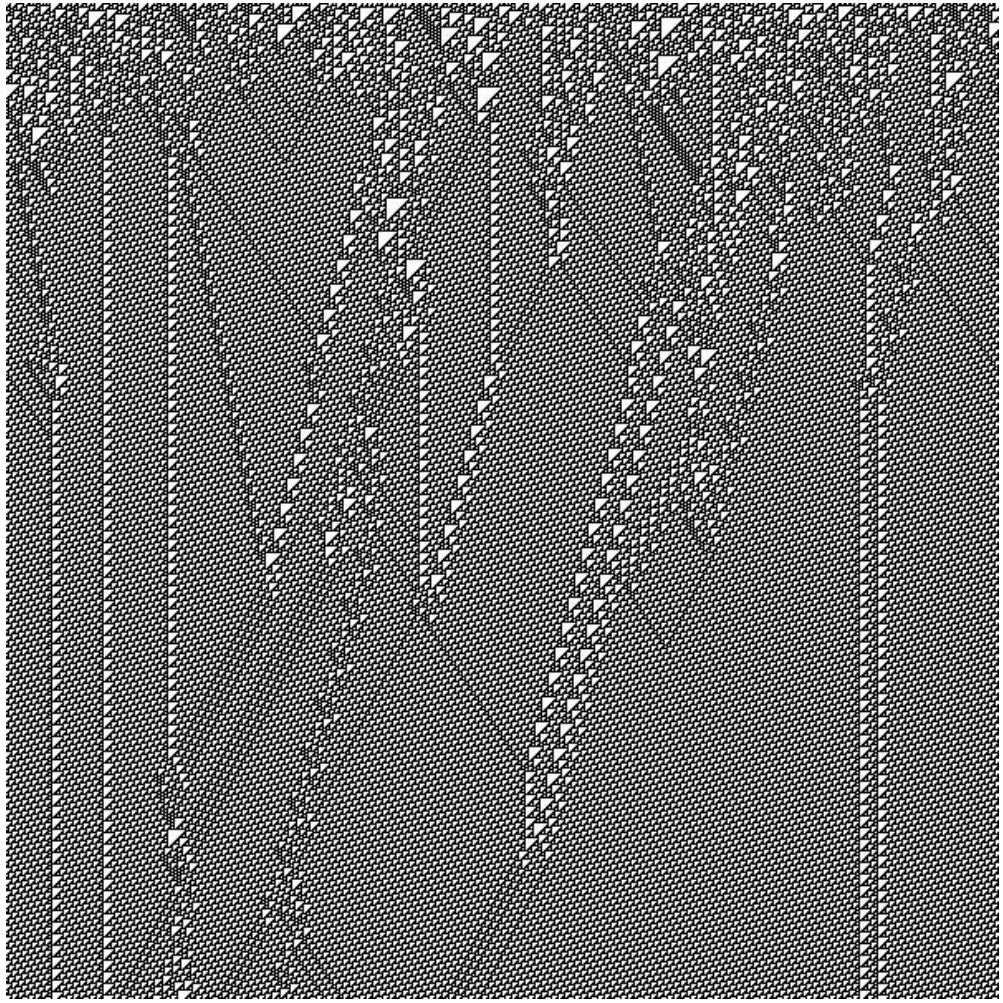
Basic notation

- R : set of all cells
- Q : finite set of states for each cell
- $c : R \to Q$ global configuration
  - $c(j)$ is the state of cell j in configuration c
  - Q^R set of all global configurations
- $f : Q^n \to Q$: local function
- $\nu : \{0, \ldots, n-1\} \to R$: neighborhood
  - neighbors of cell j are $j + \nu(0), j + \nu(1), \ldots, j + \nu(n-1)$
- f induces the global function $\mathcal{A} : Q^R \to Q^R$ via: $\forall c \ \ \forall j$

$$\mathcal{A}(c)(j) = f(c(j + \nu(0)), c(j + \nu(1)), \ldots, c(j + \nu(n-1)))$$

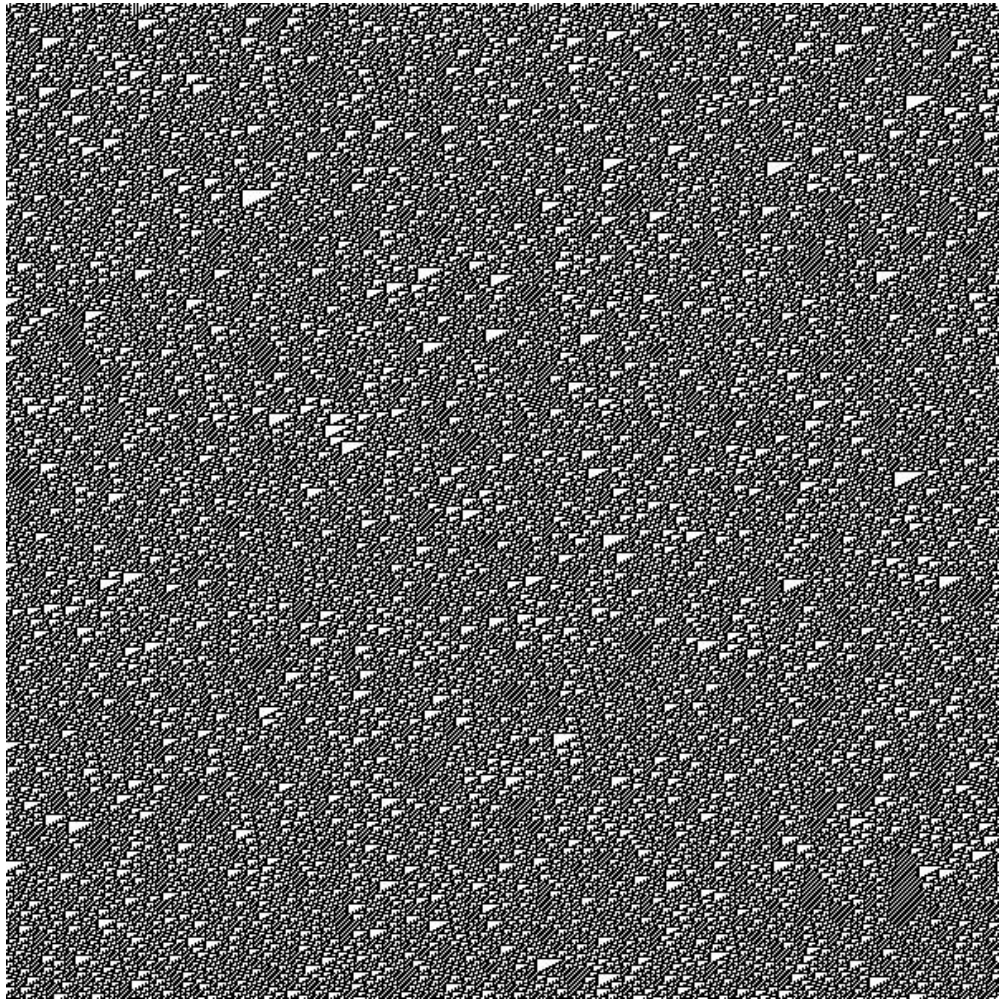
Example: Rule 110 with different neighborhoods

$\nu = (-1, 0, 1)$



Example: Rule 110 with different neighborhoods

$\nu = (-1, 0, 2)$



Observation

Fact

Each non-constant local function induces an infinite number of different global functions by changing the positions of neighbors.

Question

"How different" can those different global functions be/look like?

Answer

in general not known (to us) . . .
. . . but if one selects the right local functions . . .

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Goal

Given m CA $\mathcal{A}_i = (R, Q_A, \nu_A, f_i)$ for $0 \leq i < m$

- the *same* set of states, "$\mathcal{A}$-configurations"
- w.l.o.g. the same ν_A
- *different* local functions

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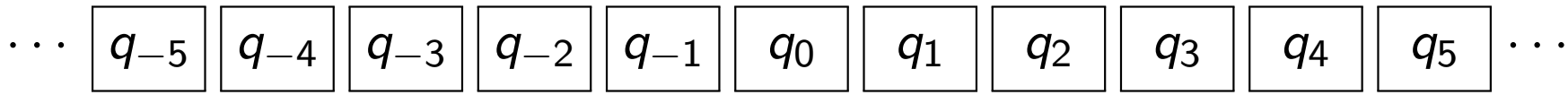
such that $\mathcal{B}_i$ simulates $\mathcal{A}_i$ in an obvious sense.

Simulation: embedding, e.g. for $m = 3$

embed $\mathcal{A}$-configuration



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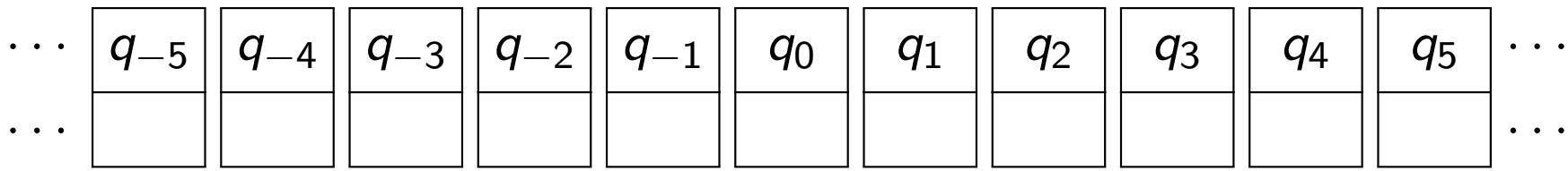

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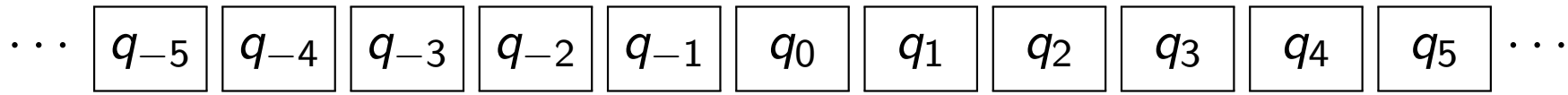

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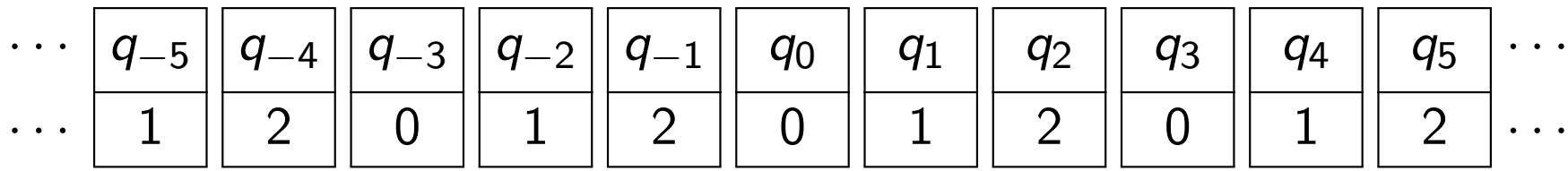

formally:

- $Q_B = Q_A \times \{0, 1, \ldots, m-1\}$
- $E : Q_A^R \to Q_B^R$ where $E(c)(j) = (c(j), j \bmod m)$

Simulation: an example for von Neumann neighborhood

- for $\mathcal{A}_i$ assume

j	0	1	2
$\nu_A(j)$	-1	0	1

for $\mathcal{B}_i$ use

j	0	1	2	3
$\nu_0(j)$	-1	0	1	6
$\nu_1(j)$	-1	0	1	5
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- example: use ν_1 for simulation of f_1

$\cdots$	q_{-5}	q_{-4}	q_{-3}	q_{-2}	q_{-1}	q_0	q_1	q_2	q_3	q_4	q_5	$\cdots$
$\cdots$	1	2	0	1	2	0	1	2	0	1	2	$\cdots$

q'_{-2}
1

$= f_{1-0}(q_{-3}, q_{-2}, q_{-1})$

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q'_{-1}
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$= f_{2-1}(q_{-2}, q_{-1}, q_0)$

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q'_0
0

$= f_{0-2}(q_{-1}, q_0, q_1)$

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Universality can be considered from different points of view:

- "... for each computable function ..."
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Here we only

- show the trick
  which allows to do "all" of the above.
- The rest is tedious routine work.

Starting point

Consider all CA $\mathcal{A}_i$ with

- some fixed Q_A,
- arbitrary local functions $f_i : Q_A^{n_i} \longrightarrow Q_A$ and
- arbitrary neighborhoods ν_i of size n_i (matching f_i).

Each computable function can be computed by such a CA for some reasonable definition of initial and final configurations.

Observe, that in contrast to the previous section one must allow for different neighborhoods.

Goal

Devise

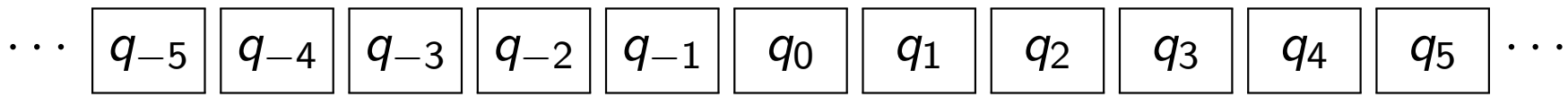
- one set Q_B of states,
- a simple embedding $E : Q_A^R \to Q_B^R$ of configurations and
- one local function f_B of fixed arity b,

such that

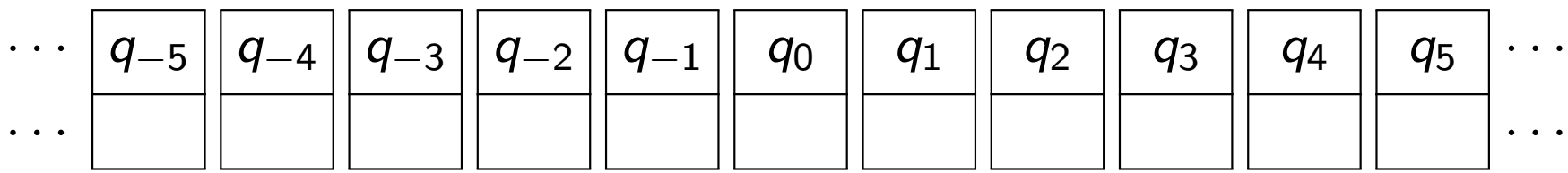
- for each $\mathcal{A}_i$
- there is a computable neighborhood ν_j of size b such that
- the CA $\mathcal{B}_j = (R, Q_B, \nu_j, f_B)$
- for each $\mathcal{A}$-configuration c_A
- when started with the embedded configuration $c_B = E(c_A)$
- simulates each step of each cell of $\mathcal{A}_i$ for c_A.

Embedding of configurations

$\mathcal{A}$-configuration



corresponding $\mathcal{B}$-configuration



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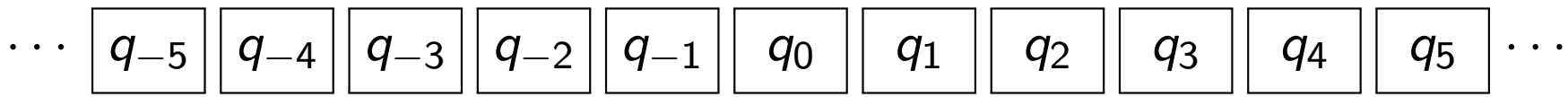

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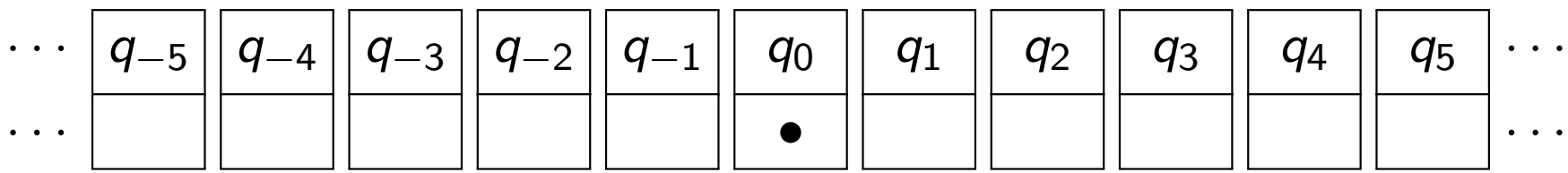

The trick

- Neighborhood $N_r = \{-r, -1, 0, 1, r\}$, where $r \geq 2$.
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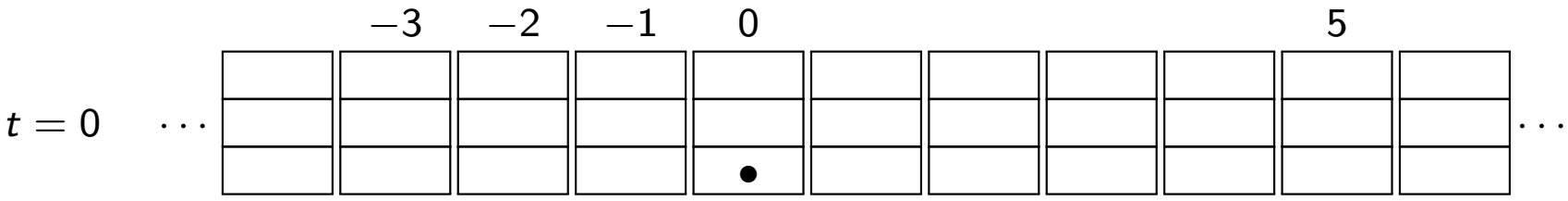
- Call the neighbor at position r the *"remote" neighbor*.
- Use the marker $\bullet$ to compute the distance of the remote neighbor
- and represent it as a binary number.
- Use this representation as that of the CA $\mathcal{A}_i$ (if any) to be simulated.

The trick for $N_5 = \{-5, -1, 0, 1, 5\}$



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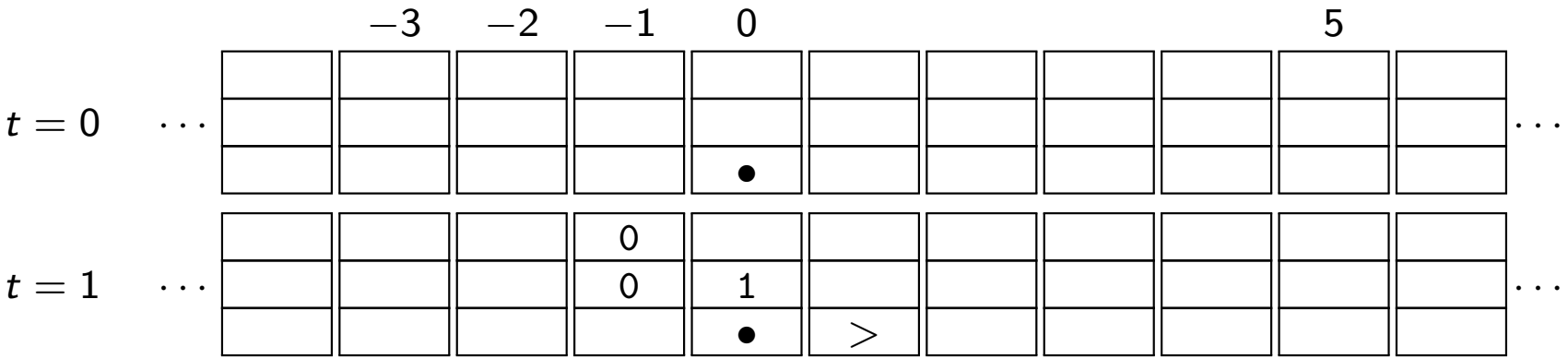

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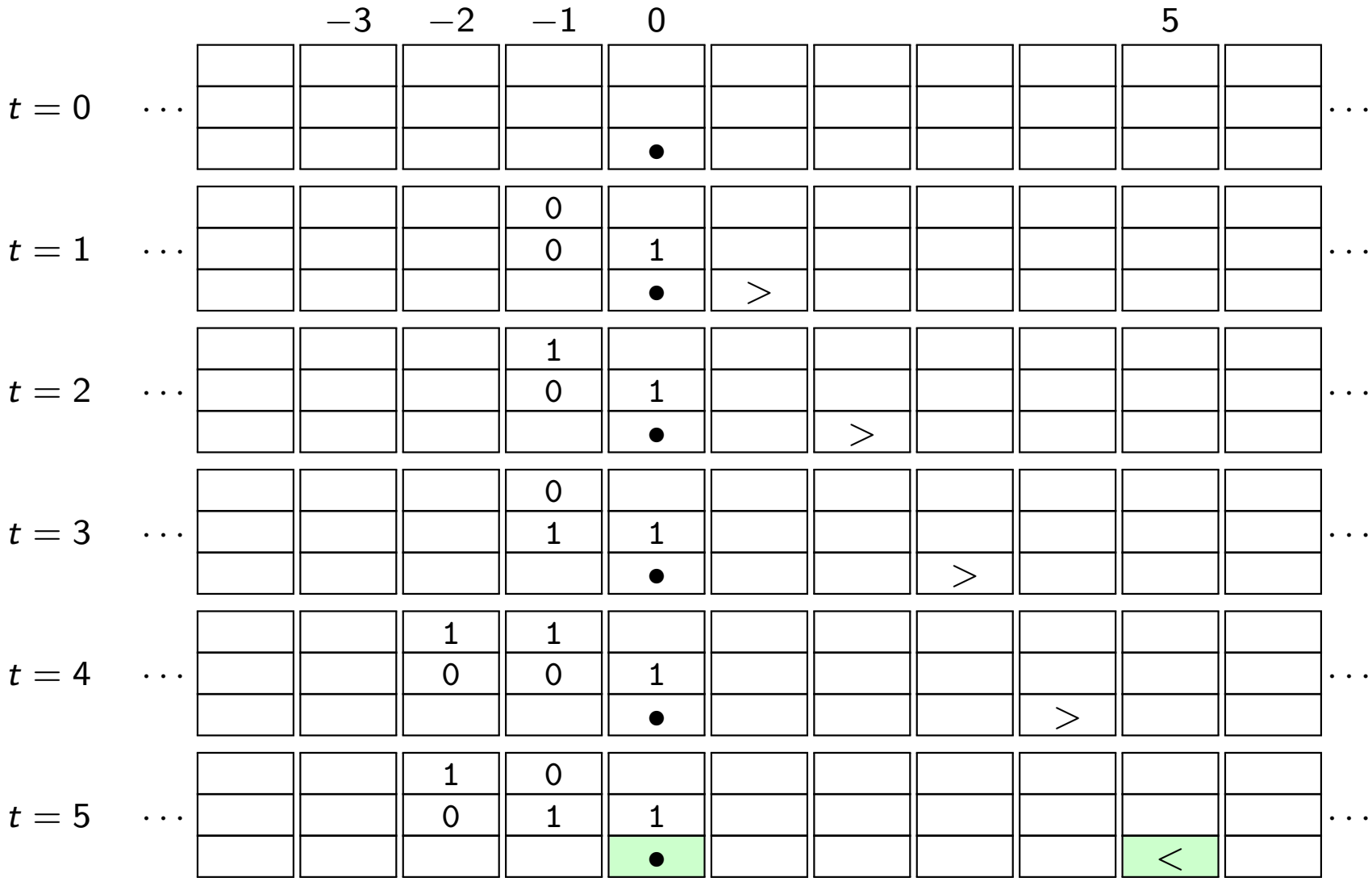

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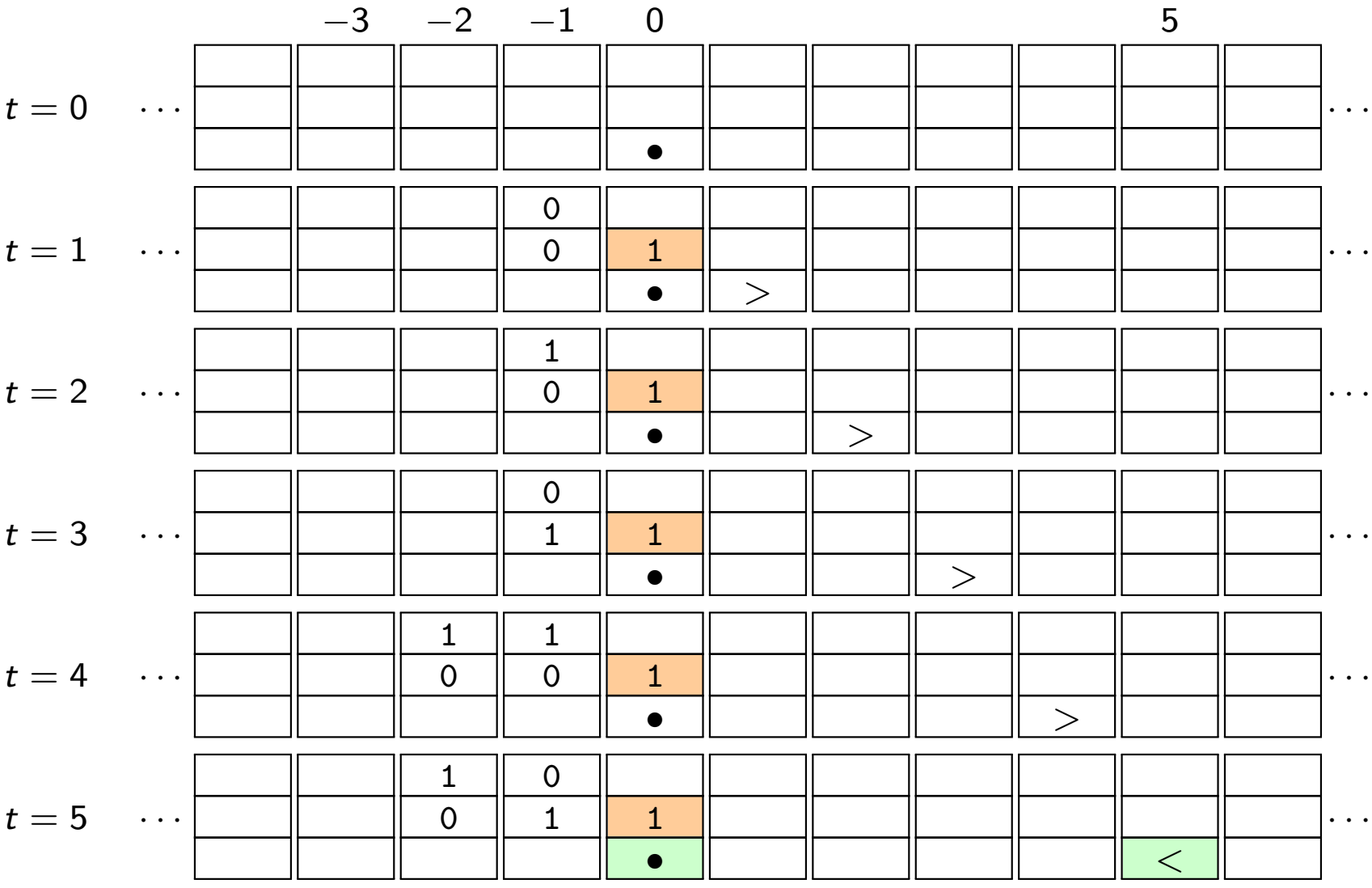

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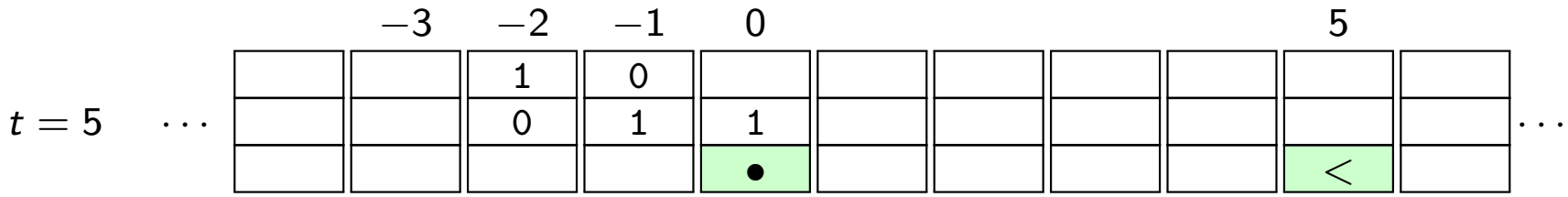

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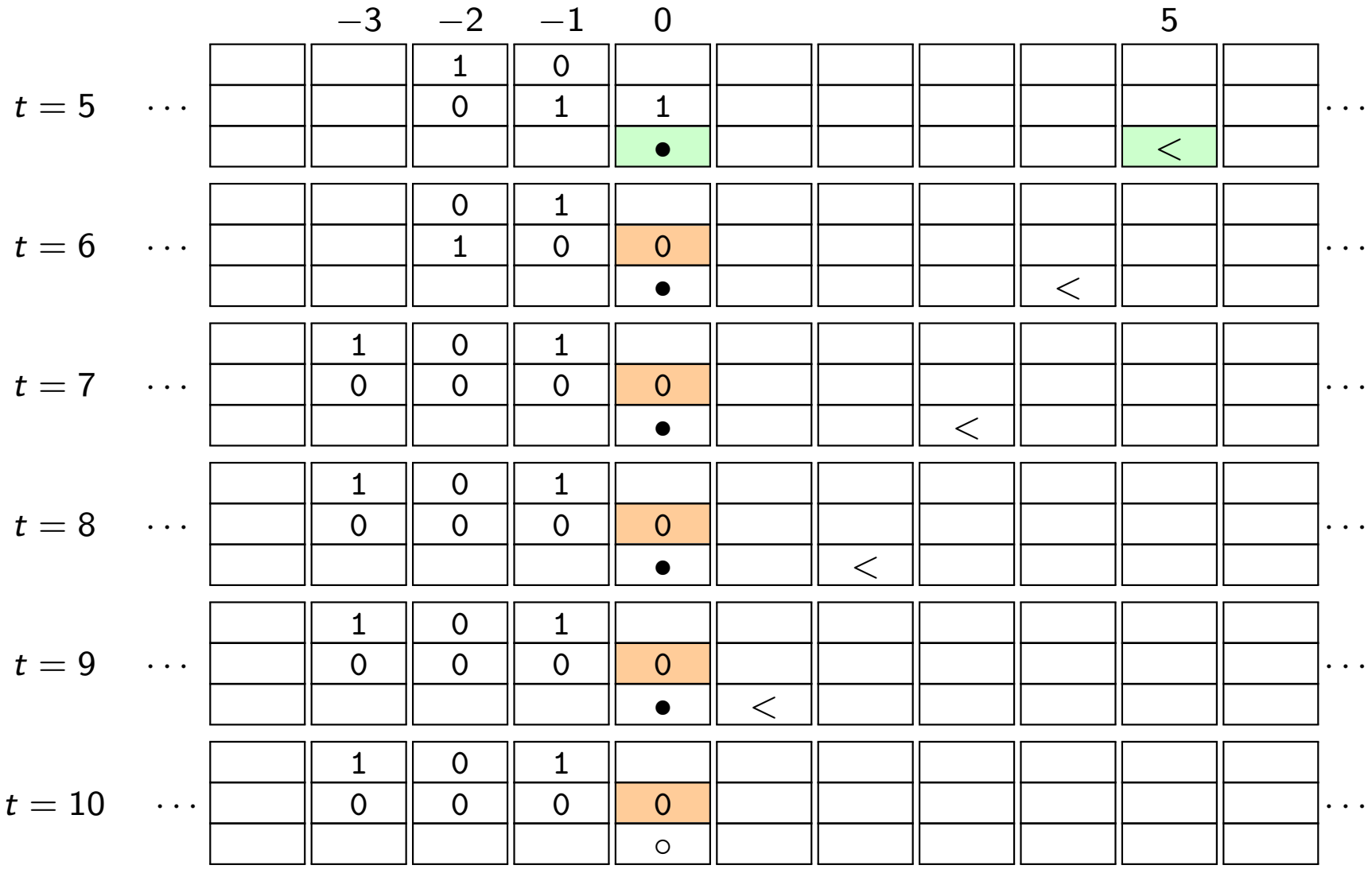

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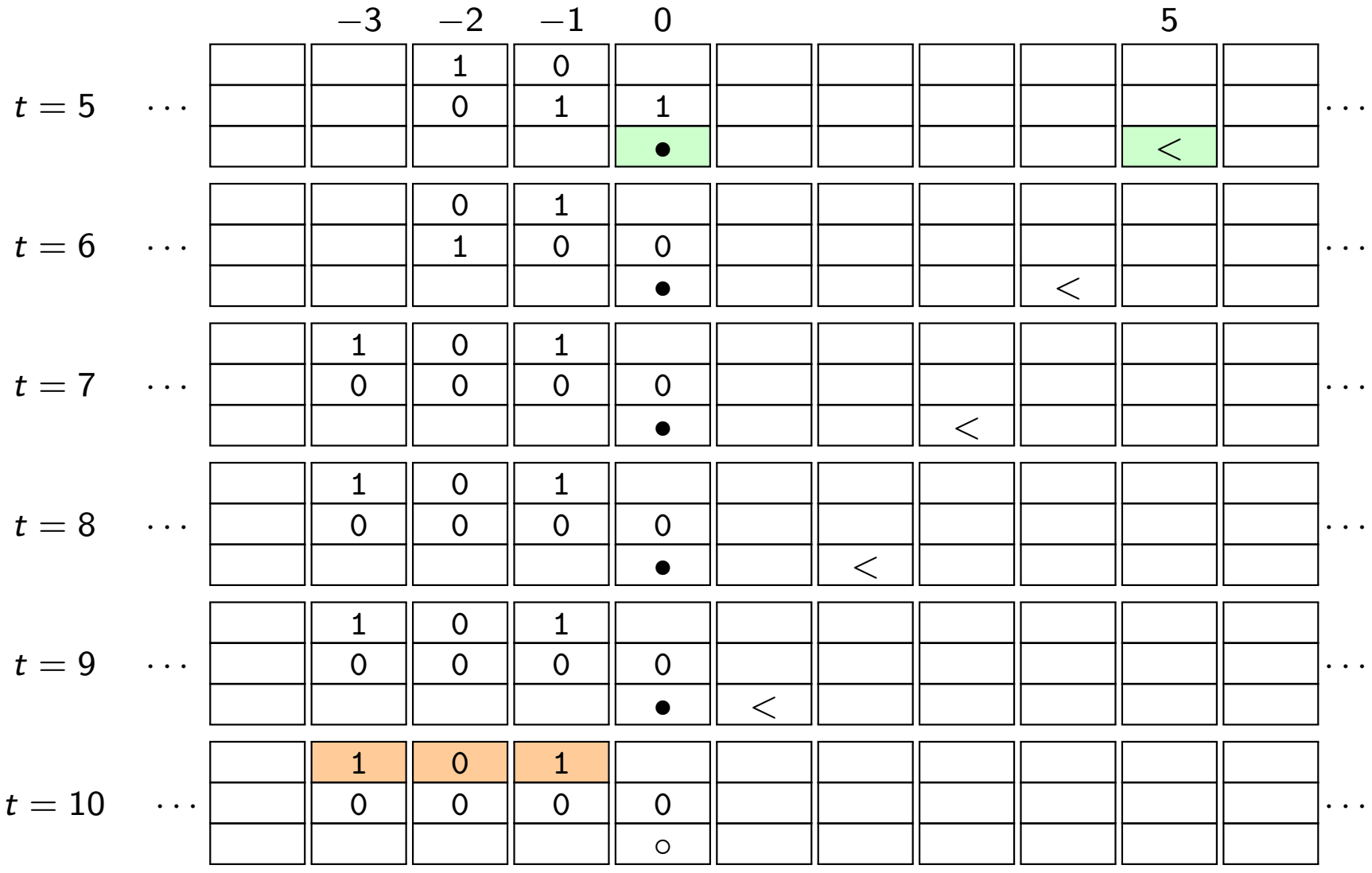

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A step aside:
simulation of irreversible CA by reversible CA for infinite configurations

- Theorem (Hertling): This is impossible.
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Both are right

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- Theorem (Hertling): This is impossible.
- Theorem (Durand-Lose): This is possible.

Both are right for their respective notions of simulation:

- Hertling requires the embedding of configurations to commute with shifts ...
- Durand-Lose uses one marker bit.

What about our embeddings?

First: Does not commute with the shift, but at least preserves spatial periodicity.

Second: Even destroys spatial periodicity.

Conjecture

If one wants to simulate an infinite number of $\mathcal{A}_i$ using only different neighborhoods, one *must* use an embedding which destroys spatial periodicity of configurations.

Proof

Just to be sure, let me sleep a few more nights about this.

Conclusion

- There is a set of states and a local function for CA such that by choosing the appropriate neighborhood any CA (with Q_A) can be simulated (no matter which local function and neighborhood the latter uses).
- The above technique can be generalized to higher dimensions.
- In both cases the embeddings do not commute with the shift, in the second case even spatial periodicity is broken.

- So, this was a first step ...

Thank you very much for your attention!