Phase Space Equivalences of Sequential Dynamical Systems

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Automata XIII, Toronto, August 2007

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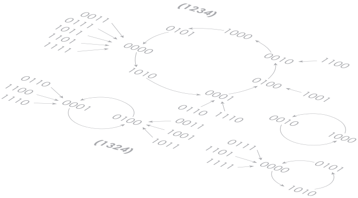
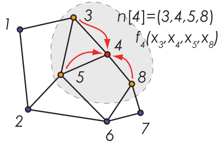


Sequential Dynamical Systems (SDS) – Definitions



- An SDS is a triple consisting of:
  - A graph Y with vertex set $\mathrm{v}[Y] = \{1, 2, \ldots, n\}$.
  - For each vertex i a state $x_i \in K$ (e.g. $\mathbb{F}_2 = \{0, 1\}$) and a *Y-local function* $F_i \colon K^n \longrightarrow K^n$

$$F_i(x = (x_1, x_2, \ldots, x_n)) = (x_1, \ldots, x_{i-1}, \underbrace{f_i(x[i])}_{\text{vertex function}}, x_{i+1}, \ldots, x_n) \, .$$

  - A word w of length k over $\mathrm{v}[Y]$.

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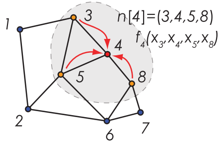

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- The *SDS map* of $(Y, \mathbf{F}_Y = (F_i)_1^n, w)$ is

$$\boxed{[\mathbf{F}_Y, w] = F_{w(k)} \circ F_{w(k-1)} \circ \cdots \circ F_{w(1)}\ .}$$

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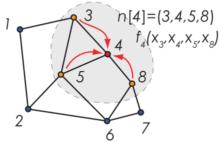

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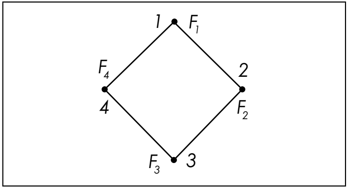
- Comments.

SDS – A Basic Example

- Circle graph on 4 vertices, $Y = \mathrm{Circ}_4$.
- Permutation update order $\pi = (1, 2, 3, 4)$.
- Vertex functions given by $\mathrm{nor}_3(x, y, z) = (1 + x)(1 + y)(1 + z)$.

Thus $F_1(x_1, x_2, x_3, x_4) =$

$$\mathrm{nor}_3(x_1, x_2, x_4), x_2, x_3, x_4).$$



▶ We have for example

$$(x_1, x_2, x_3, x_4) = (0, 0, 0, 0) \overset{F_1}{\mapsto} (1, 0, 0, 0) \text{ and}$$

$$(1, 0, 0, 0) \overset{F_2}{\mapsto} (1, 0, 0, 0) \overset{F_3}{\mapsto} (1, 0, 1, 0) \overset{F_4}{\mapsto} (1, 0, 1, 0),$$

and thus:

$$\boxed{[\mathbf{F}_Y, \pi](0, 0, 0, 0) = (1, 0, 1, 0)}$$

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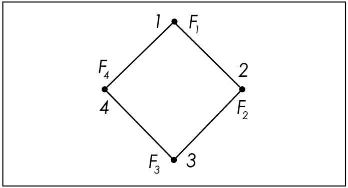

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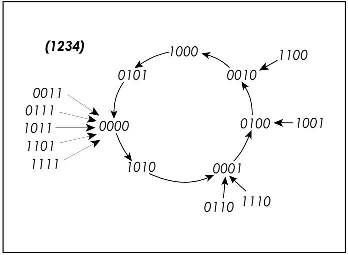

SDS – Dynamics and Phase Space

- The *phase space* of an SDS map $[\mathbf{F}_Y, w]$ is the finite, directed graph $\Gamma[\mathbf{F}_Y, w]$ given by:

$$\begin{aligned} \mathrm{v}(\Gamma[\mathbf{F}_Y, w]) &= \{x = (x_1, x_2, \ldots, x_n) \in K^n\}, \\ \mathrm{e}(\Gamma[\mathbf{F}_Y, w]) &= \{(x, [\mathbf{F}_Y, w](x)) \mid x \in K^n\} \, . \end{aligned}$$

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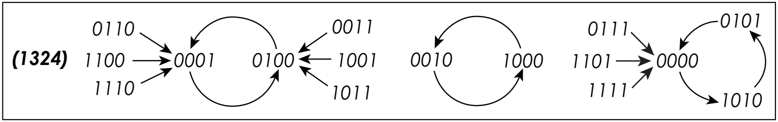
Example: $\Gamma[\mathbf{Nor}_{\mathrm{Circ}_4}, (1, 3, 2, 4)]$



Equivalence Types:

- Can compare two (SDS) phase spaces at many levels:
  - Functional Equivalence: phase spaces are identical
  - Dynamical Equivalence: phase spaces are isomorphic
  - Cycle Equivalence: phase spaces have isomorphic cycles

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- In the remainder:
  - Review of results on functional and dynamical equivalence for SDS.
  - Results from initial work on cycle equivalence for SDS and relations to Coxeter theory.

Functional Equivalence I

- Given two permutation update orders π and σ. When are the SDS maps identical, i.e.

$$[\mathbf{F}_Y, \pi] = [\mathbf{F}_Y, \sigma] \ ?$$

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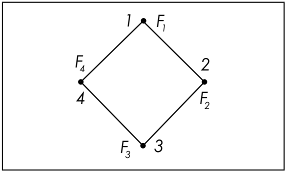

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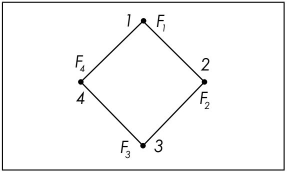

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▶ An answer is given by the *update graph* $U(Y)$.

Definition

The update graph of Y has vertex set S_Y (all permutations of v$[Y]$). Two permutations are connected if they differ by exactly one flip of two consecutive elements i and j such that $\{i, j\} \notin$ e$[Y]$.

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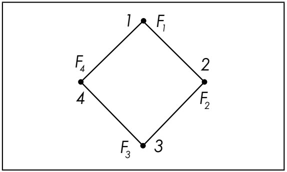

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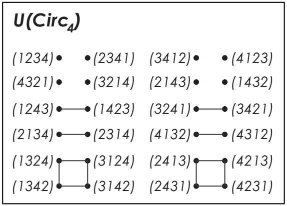

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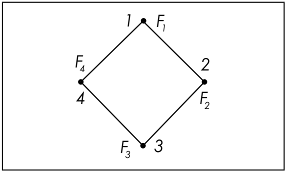

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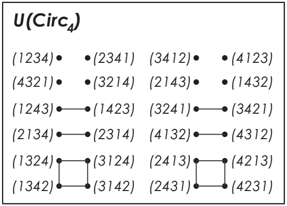

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Proposition

(*i*) *The permutations in a (connected) component of $U(Y)$ induce identical SDS maps.*
(*ii*) *There is a bijection $f_Y : S_Y / \sim_Y \longrightarrow \mathrm{Acyc}(Y)$.*
(*iii*) *The upper bound $a(Y) = |\mathrm{Acyc}(Y)|$ is always realized for* Nor*-SDS.*

Dynamical Equivalence I

Two maps $\phi, \psi \colon K^n \longrightarrow K^n$ are dynamically equivalent if there is a bijection $h \colon K^n \longrightarrow K^n$ such that

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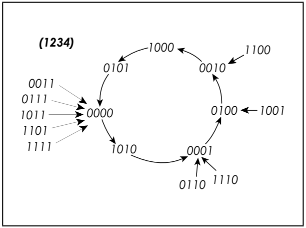

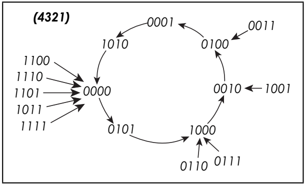

Dynamical Equivalence II

- Can also have equivalence as a result of vertex functions (or both function and update order):

$$[\mathbf{Nand}_Y, \pi] \circ \mathsf{inv}_n = \mathsf{inv}_n \circ [\mathbf{Nor}_Y, \pi]$$

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Proposition

An upper bound for the number of dynamically nonequivalent SDS that can be created through rescheduling is:

$$\Delta(Y) = \frac{1}{|\,\mathrm{Aut}(Y)|} \sum_{\gamma \in \mathrm{Aut}(Y)} a(\langle \gamma \rangle \setminus Y),$$

where $\langle \gamma \rangle \setminus Y$ is the orbit graph of Y under $\langle \gamma \rangle$.

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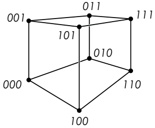
Example: Let Y be the three-dimensional cube, let f be a fixed function, and consider the induced SDS. Then there are $8! = 40320$ permutation update orders, there are at most 1862 functional equivalence classes, and at most $\Delta(Y) = 54$ dynamical equivalence classes (when f is outer-symmetric). Both bounds are sharp (realized by $f = \mathrm{nor}_4$).



Equivalence: Examples

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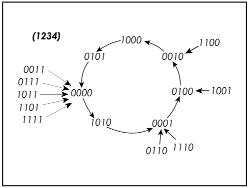

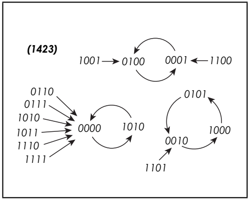

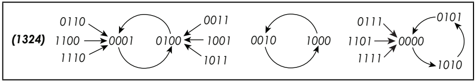

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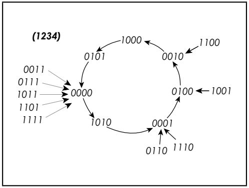

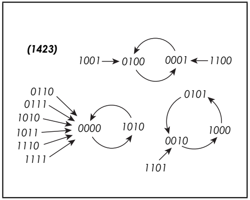

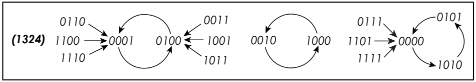

There are $\Delta(\mathrm{Circ}_4) = 3$ dynamically inequivalent phase spaces and $\delta(\mathrm{Circ}_4) = 2$ cycle inequivalent phase spaces.

Cycle Equivalence under Shifts

Set $\sigma = (n, n-1, \ldots, 2, 1)$,

$\tau = (1, n)(2, n-1)\cdots(\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor + 1)$

$\boldsymbol{\sigma}_s(w) = \sigma^s \cdot w = (w_{s+1}, w_{s+2}, \ldots, w_n, w_1, \ldots, w_s)$, and

$\boldsymbol{\rho}(w) = \tau \cdot w = (w_n, w_{n-1}, \ldots, w_2, w_1)$.

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Theorem

For any $w \in S_Y$, the SDS *maps $[\mathbf{F}_Y, w]$ and $[\mathbf{F}_Y, \boldsymbol{\sigma}_s(w)]$ are cycle equivalent.*

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Remark: Holds for any graph and any choice of functions over a finite state space K.

Proof.

Set $P_k = \mathrm{Per}[\mathbf{F}_Y, \boldsymbol{\sigma}_k(w)]$. The diagram

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P_{k-1} \arrow[r, "{[\mathbf{F}_Y, \boldsymbol{\sigma}_{k-1}(w)]}"] \arrow[d, "F_{w(k)}"'] & P_{k-1} \arrow[d, "F_{w(k)}"] \\
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$$|\mathrm{Per}[\mathbf{F}_Y, w]| \le |\mathrm{Per}[\mathbf{F}_Y, \boldsymbol{\sigma}_1(w)]| \le \cdots \le |\mathrm{Per}[\mathbf{F}_Y, \boldsymbol{\sigma}_{m-1}(w)]| \le |\mathrm{Per}[\mathbf{F}_Y, w]| \,.$$

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Proposition

Let $K = \mathbb{F}_2$ and $P = \mathrm{Per}[\mathbf{F}_Y, w] \subset \mathbb{F}_2^n$. Then $([\mathbf{F}_Y, w]|_P)^{-1} = [\mathbf{F}_Y, \boldsymbol{\rho}(w)]|_P$.

Enumeration of Cycle Equivalence Classes

Let $C(Y)$ and $D(Y)$ be the undirected graphs defined by

$$\mathrm{v}[C(Y)] = \{[\pi]_Y \mid \pi \in S_Y\}, \qquad \mathrm{e}[C(Y)] = \big\{\{[\pi]_Y, [\boldsymbol{\sigma}_1(\pi)]_Y\} | \pi \in S_Y\big\} ,$$
$$\mathrm{v}[D(Y)] = \{[\pi]_Y \mid \pi \in S_Y\}, \qquad \mathrm{e}[D(Y)] = \big\{\{[\pi]_Y, [\boldsymbol{\rho}(\pi)]_Y\} \ |\pi \in S_Y\big\} \cup \mathrm{e}[C(Y)] .$$

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Let $C(Y)$ and $D(Y)$ be the undirected graphs defined by

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- The components in $C(Y)$ are the click equivalence classes in $\mathrm{Acyc}(Y)$.
- (Extended) Click-equivalence of acyclic orientations is an equivalence relation.

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▶ Can therefore analyze cycle equivalence and enumeration over $\mathrm{Acyc}(Y)$.

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Let Y be a tree. Then $\kappa(Y) = 1$.

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All permutation SDS maps $[Fy, \pi]$ for fixed $\mathbf{F}_Y$ are cycle equivalent when Y is a tree.

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Corollary

All permutation SDS over trees induced by parity functions are dynamically equivalent.

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Proposition

If Circ_n *is a tree pruning of a graph* Y *then* $\kappa(Y) = n - 1$ *and* $\delta(Y) = \lfloor n/2 \rfloor$*.*

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Proof.

Part I: By Shi in the context enumeration of conjugacy classes of Coxeter elements. Part II follows from ρ being an involution. $\square$

Enumeration: Special Graph Classes

Proposition

Let $Z = Y \oplus v$ (vertex join). Then $\kappa(Z) = 2\delta(Z) = a(Y)$.

Proof.

Show that each (κ) equivalence class contains a unique acyclic orientation $v \longrightarrow Y$. $\square$

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Corollary

$\kappa(Wheel_n) = 2^n - 2$, $\delta(Wheel_n) = 2^{n-1} - 1$, $\kappa(K_n) = (n-1)!$.

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Proposition

If Y has an odd cycle then $\delta(Y) = \frac{1}{2}\kappa(Y)$.

Further Research and Open Questions

- Questions/topic:
  - More properties/structure of $C(Y)$ and $D(Y)$.
  - Are the bounds $\kappa(Y)$ and $\delta(Y)$ sharp? Will Nor-SDS suffice to prove sharpness?
  - Connection to Coxeter theory and Coxeter elements barely explored. What more?
  - Computational and algorithmic questions: What is the complexity of deciding if two SDS are cycle equivalent?

- Results from e.g:

  - C. M. Reidys: *Acyclic Orientations of Random Graphs*, Adv. Appl. Math., 21(2):181–192, 1998.
  - H. S. Mortveit and C. M. Reidys: *Discrete, sequential dynamical systems*, Discrete Mathematics, 226:281–295, 2001.
  - J. Y. Shi: *Conjugacy Relation on Coxeter Elements*, Adv. Math., 161:1–19, 2001.
  - M. Macauley and H. S. Mortveit: *Cycle Equivalence of Graph Dynamical Systems*. Preprint.

SDS – Collaborators & Information

Joint work with: Matt Macauley

Collaborators: Christian M. Reidys, Chris L. Barrett, Reinhard Laubenbacher, Bodo Pareigis, Anders Å. Hansson, Madhav Marathe, Jon McCammond.

SDS **course web page with link to papers:**

Web: http://www.math.vt.edu/people/hmortvei/class_home/4984_15748.html

NDSSL:

Web: http://ndssl.vbi.vt.edu