On the influence of symmetries and neighborhoods on constructing two dimensional number-conserving cellular automata rules

Katsunobu Imai, Naonori Tanimoto

Hiroshima University

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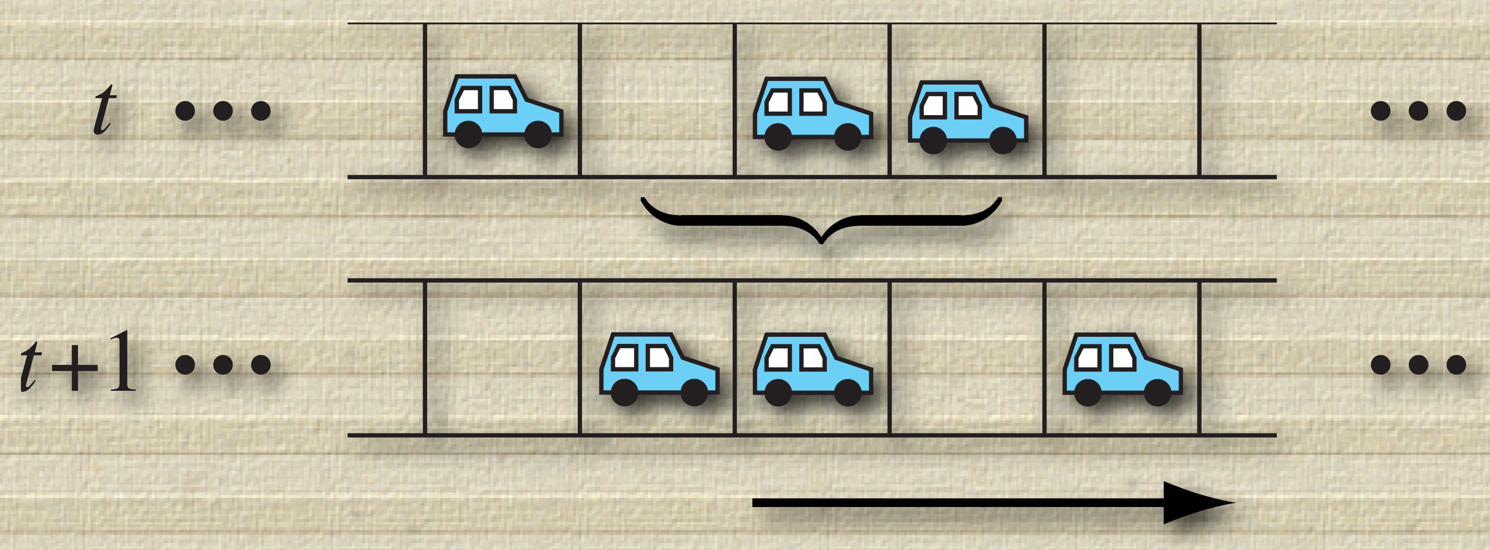
Number-conserving cellular automata (NCCA)

All states of cells are represented by integers.
The total number of its configuration is conserved.



local function

$f(0,0,0)=0$
$f(0,0,1)=0$
$f(0,1,0)=0$
$f(0,1,1)=1$
$f(1,0,0)=1$
$f(1,0,1)=1$
$f(1,1,0)=0$
$f(1,1,1)=1$

A simple car traffic rule
(Nagel, Schreckenberg 1992)

NCCA... Necessary and sufficient conditions

- 1991 Hattori, Takesue
- 2000 Boccara, Fuks
  - A necessary and sufficient condition for 1D NCCAs with cyclic configurations
- 2001 Durand, Formenti, Roka
  - 2D case, equality of several boundary conditions
- 2003 Moreira
- ...

Necessary and sufficient condition for NCCA

1D case (Boccara, Fuks 2001)

neighborhood size *n*

$$f(x_1, \cdots, x_n) =$$
$$x_1 + \sum_{k=1}^{n-1} (f(0, \cdots, 0, x_2, \cdots, x_{k+1}) - f(0, \cdots, 0, x_1, \cdots, x_k))$$

radius 1

i.e. 3 neighbor or von Neumann

Motion representation

Any transition of NCCA can be characterized by movement of particles.

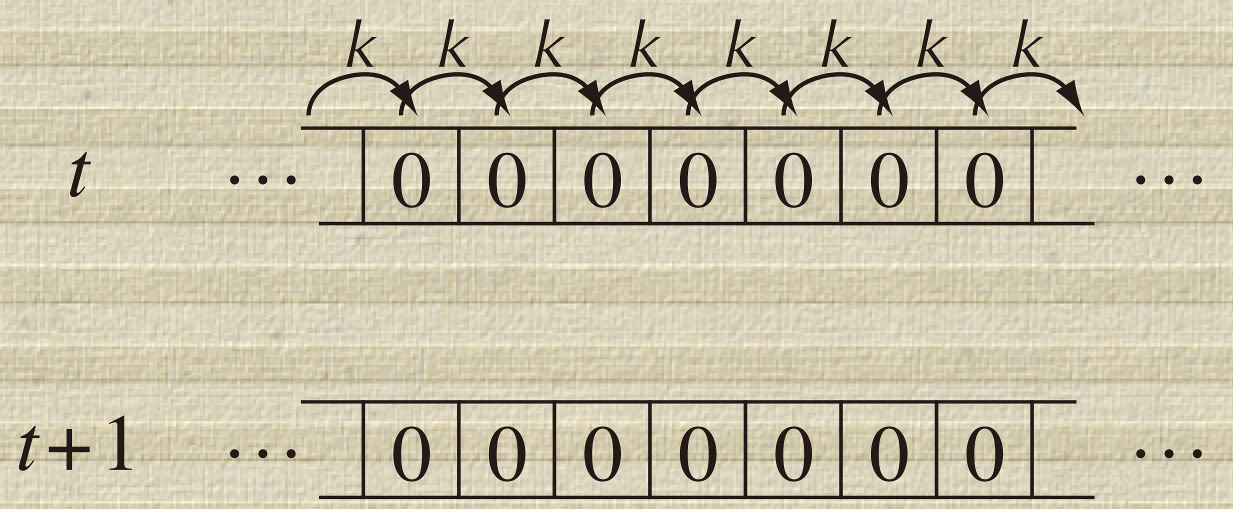
t	0	1	1	1	1	1	1	$\cdots$	1	1	1	0	0	0
$t+1$	0	0	1	1	1	1	1	$\cdots$	1	1	1	1	0	0

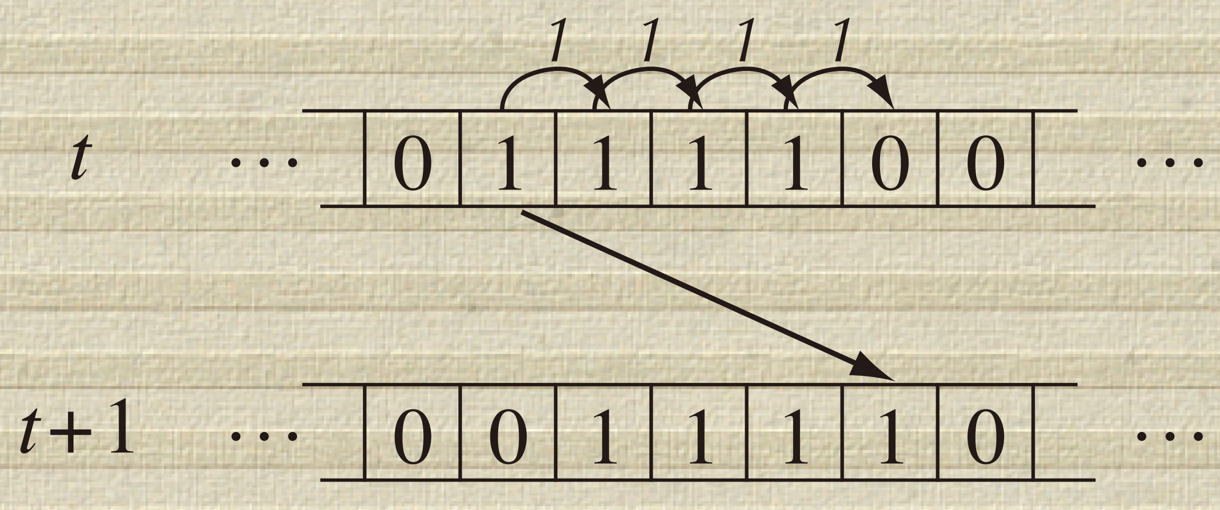
$$\{\overset{\overset{1}{\curvearrowright}}{10}, \overset{\overset{1}{\curvearrowright}}{11}\} \Rightarrow$$

$$f(0,0,0)=0$$
$$f(0,0,1)=0$$
$$f(0,1,0)=0$$
$$f(0,1,1)=0$$
$$f(1,0,0)=1$$
$$f(1,0,1)=1$$
$$f(1,1,0)=1$$
$$f(1,1,1)=1$$

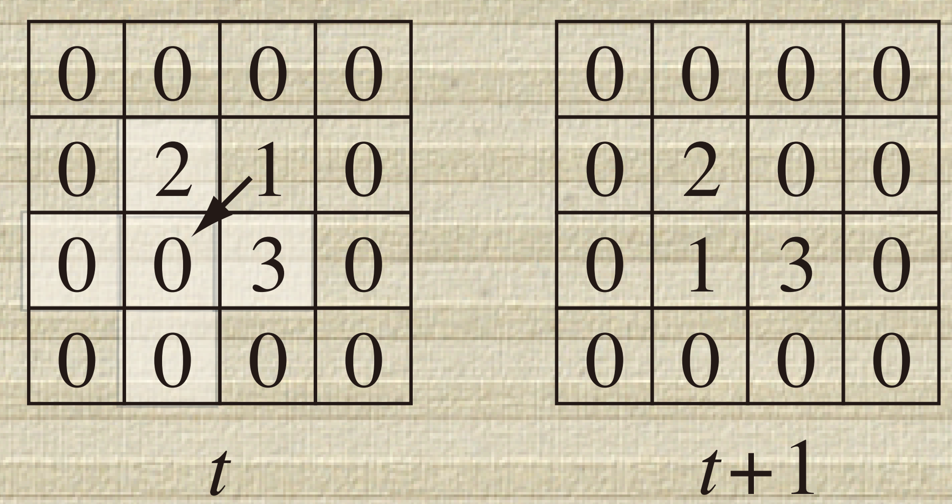
(cf. Fuks 2000, Pivato 2002)

Motion representation for a CA is not unique





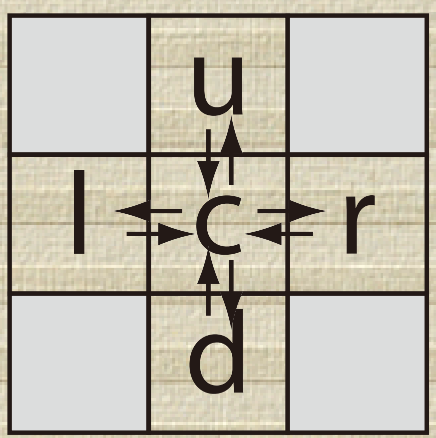
Combination of implicit movements case a remote movement of numbers



How to play with 2D NCCA programming?

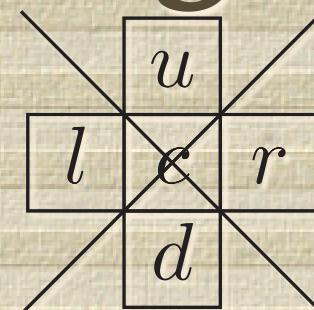
To find interesting patters, build self-reproducing or computationally universal models.

The game of life, The Langton's loop,...



How to assign $f(c,u,r,d,l)$?

A Necessary and Sufficient Condition for von Neumann neighbor NCCA



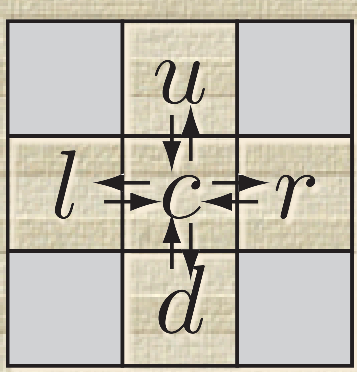
$$f(c,u,r,d,l) = f(c,r,u,l,d) = f(c,l,d,r,u)$$

± 45-degree reflection symmetric von Neumann neigobor CA is number-conserving iff

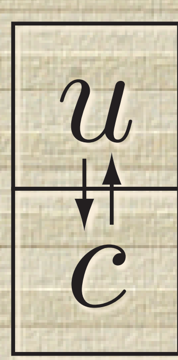
$$\exists g : Q^2 \to Q, \quad \forall c,u,r,d,l \in Q,$$

$$f(c,u,r,d,l) = c + g(c,u) + g(c,r) + g(c,d) + g(c,l)$$

$$g(c,u) = -g(u,c)$$



$g(c,u)$



$g(u,c)$

(Imai, Fujita, Iwamoto, and Morita 2001)

To design an NCCA...

You have only to design a binary function g

(flow function).

$$\exists g : Q^2 \to Q, \quad \forall c, u, r, d, l \in Q,$$

$$f(c, u, r, d, l) = c + g(c, u) + g(c, r) + g(c, d) + g(c, l)$$

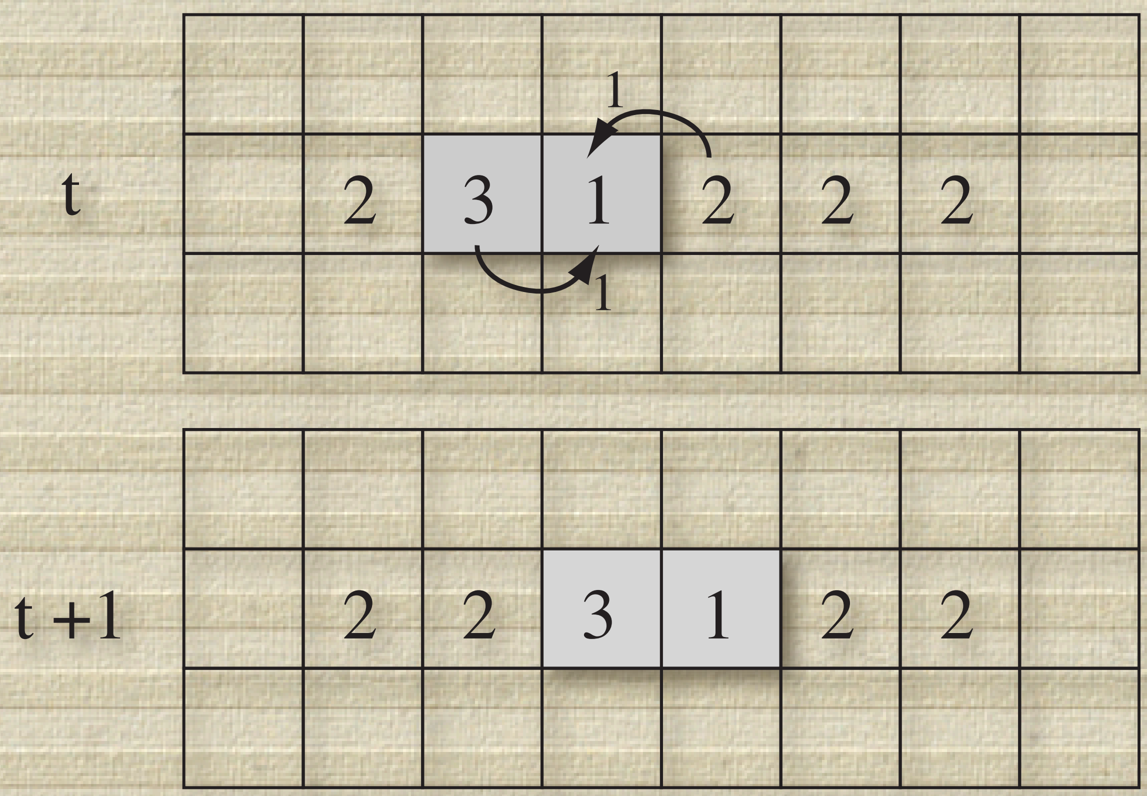
$$\Downarrow \qquad g(c, u) = -g(u, c)$$

CA becomes *permutation-symmetric*.

$f(c,u,r,d,l)$ is independent of the order of u,r,d,l.

cf. totalistic

An example: unit wire



$$\tilde{Q} = \{0, 1, 2, 3\}$$

$$g(1, 2) = 1$$

$$g(1, 3) = 1$$

otherwise 0

A number conserving Loop

based on the Langton's loop

$A = (\mathbf{Z}, N_{395}, f, 235)$ (Imai, Fujita, Iwamoto, and Morita 2001)

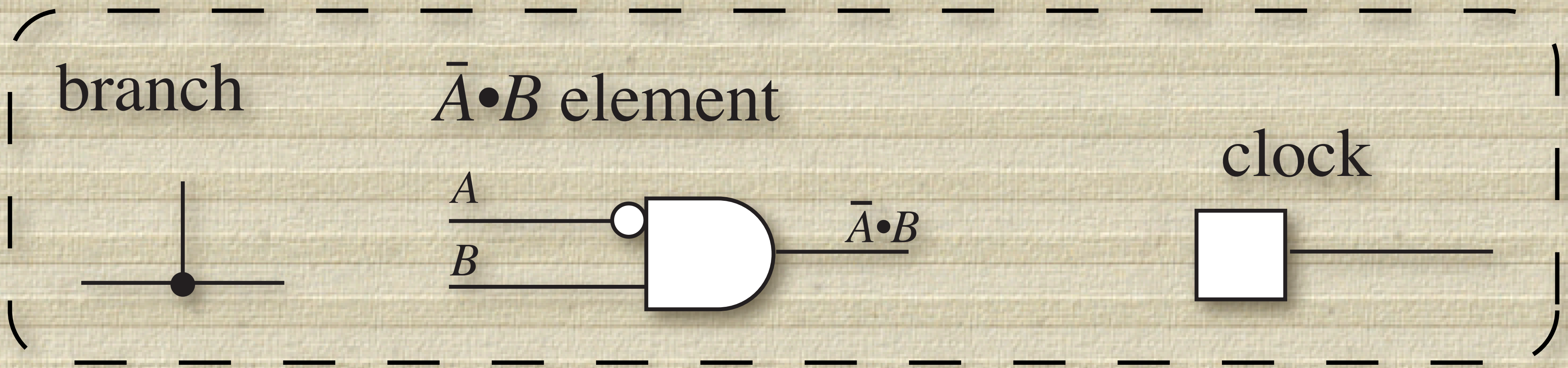
180	195	195	195	195	195	195	195	210	180					
210	275	355	315	275	355	315	285	345	195					
195	315	195	195	195	195	195	195	315	195					
195	355	195					195	315	195					
195	275	195					195	315	195					
195	315	195					195	315	195					
195	355	195					195	315	195					
195	275	195	195	195	195	195	195	315	195	195	195	195	195	203
195	315	355	275	315	355	275	315	315	315	315	315	315	315	184
180	210	195	195	195	195	195	195	195	195	195	195	195	195	198

(blank cells are 235)

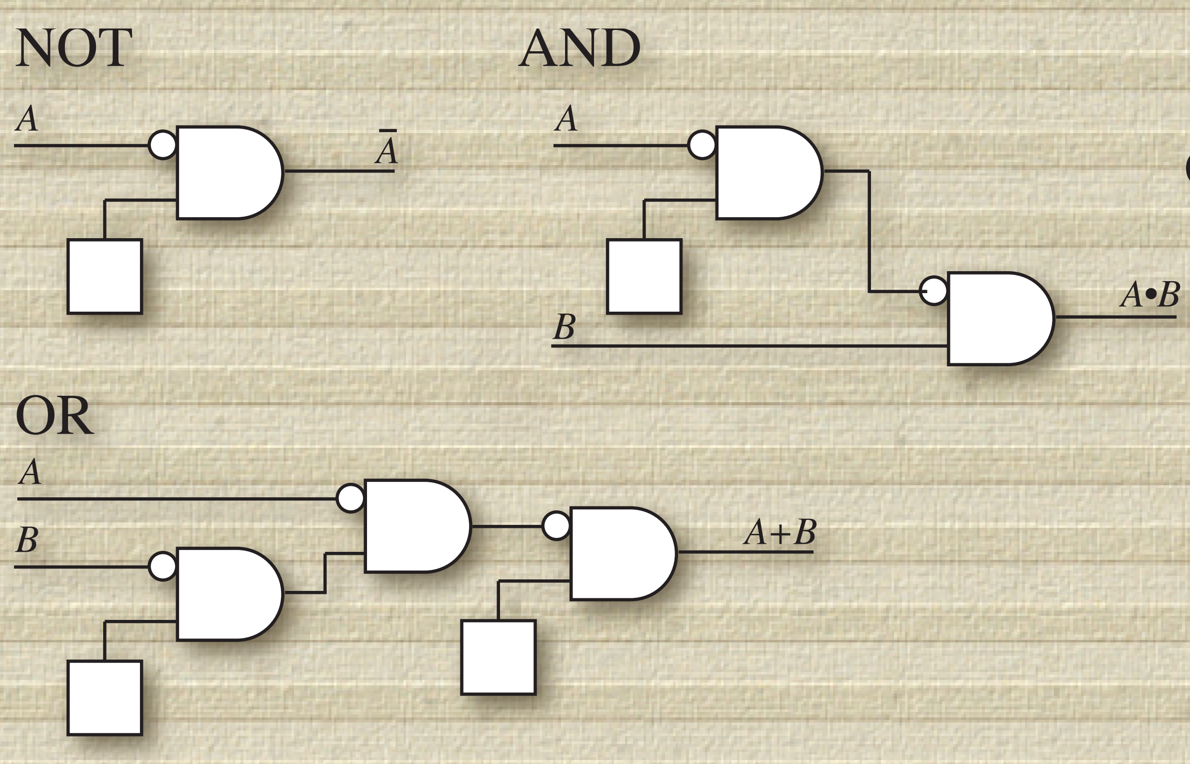
Logically universal CA

Banks 1970

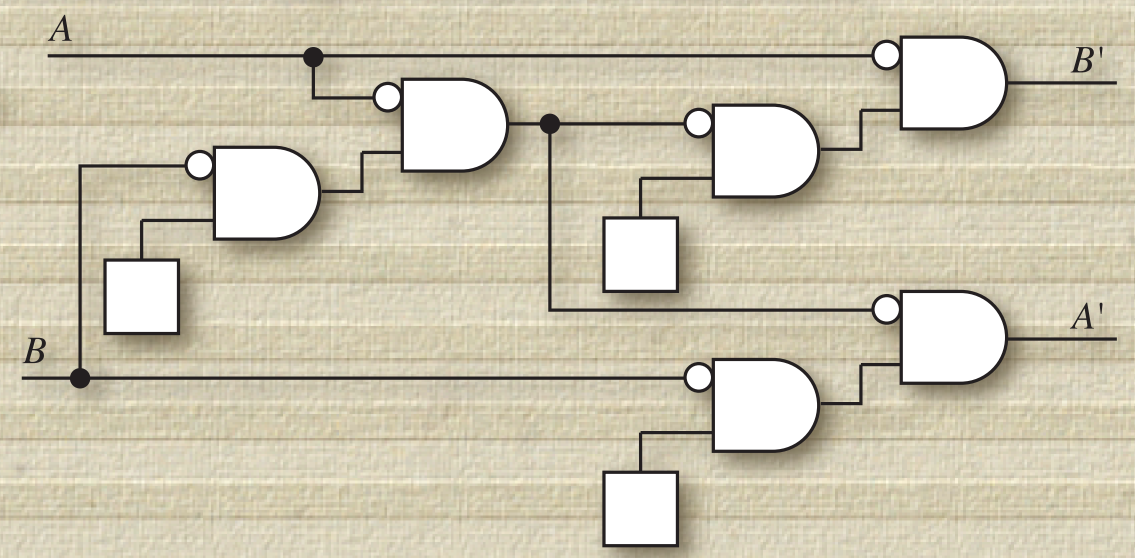
2-state von Neumann neighbor CA



Construction of basic logical elements



Crossover (input A=B=1 is inhibited)



A logically universal NCCA

$$\tilde{Q}_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$$

$$\begin{aligned} &\boxed{g(1,2) = 1, g(1,3) = 1}, g(1,4) = 1, g(1,6) = 5, \\ &g(3,5) = 1, g(3,6) = 1, g(3,7) = 3, g(4,3) = 1, \\ &g(5,4) = 1 \end{aligned}$$

$$\text{otherwise} \quad 0$$

(Imai, Furuta, Iwamoto, and Morita 2006)

Branch

				2					
				2					
				2					
	3	1	2	2	2	2	2		

t

				2					
				2					
				2					
	2	3	1	2	2	2	2		

t +1

				2					
				2					
				2					
	2	2	3	1	2	2	2		

t +2

				2					
				2					
				1					
	2	2	2	4	1	2	2		

t +3

				2					
				1					
				3					
	2	2	2	2	3	1	2		

t +4

				1					
				3					
				2					
	2	2	2	2	2	3	1		

t +5

A clock is built by a closed loop.

NOT A AND B (A=0,B=1)

A

B

0	0	0	0	2	0	0	0
0	0	0	0	2	0	0	0
0	0	0	0	4	0	0	0
2	2	3	1	6	2	2	2
0	0	0	0	0	0	0	0

$\overline{A}$ AND B

t

0	0	0	0	2	0	0	0
0	0	0	0	2	0	0	0
0	0	0	0	4	0	0	0
2	2	3	7	1	2	2	2
0	0	0	0	0	0	0	0

t+1

0	0	0	0	2	0	0	0
0	0	0	0	2	0	0	0
0	0	0	0	3	0	0	0
2	2	3	7	3	1	2	2
0	0	0	0	0	0	0	0

t+2

0	0	0	0	2	0	0	0
0	0	0	0	2	0	0	0
0	0	0	0	3	0	0	0
2	2	3	4	5	3	1	2
0	0	0	0	0	0	0	0

t+3

0	0	0	0	2	0	0	0
0	0	0	0	2	0	0	0
0	0	0	0	4	0	0	0
2	2	2	3	4	3	3	1
0	0	0	0	0	0	0	0

t+4

0	0	0	0	2	0	0	0
0	0	0	0	2	0	0	0
0	0	0	0	4	0	0	0
2	2	2	2	6	2	2	3
0	0	0	0	0	0	0	0

t+5

NOT A AND B (A=1,B=0)

A

B

0	0	0	0	3	0	0	0
0	0	0	0	1	0	0	0
0	0	0	0	4	0	0	0
2	2	2	2	6	2	2	2
0	0	0	0	0	0	0	0

$\bar{A}$ AND B

t

0	0	0	0	2	0	0	0
0	0	0	0	3	0	0	0
0	0	0	0	3	0	0	0
2	2	2	2	6	2	2	2
0	0	0	0	0	0	0	0

t+1

0	0	0	0	2	0	0	0
0	0	0	0	3	0	0	0
0	0	0	0	4	0	0	0
2	2	2	2	5	2	2	2
0	0	0	0	0	0	0	0

t+2

0	0	0	0	2	0	0	0
0	0	0	0	2	0	0	0
0	0	0	0	4	0	0	0
2	2	2	2	6	2	2	2
0	0	0	0	0	0	0	0

t+3

NOT A AND B (A=B=1)

A

B

0	0	0	0	3	0	0	0
0	0	0	0	1	0	0	0
0	0	0	0	4	0	0	0
3	1	2	2	6	2	2	2
0	0	0	0	0	0	0	0

$\bar{A}$ AND B

t

0	0	0	0	2	0	0	0
0	0	0	0	3	0	0	0
0	0	0	0	3	0	0	0
2	3	1	2	6	2	2	2
0	0	0	0	0	0	0	0

t+1

0	0	0	0	2	0	0	0
0	0	0	0	3	0	0	0
0	0	0	0	4	0	0	0
2	2	3	1	5	2	2	2
0	0	0	0	0	0	0	0

t+2

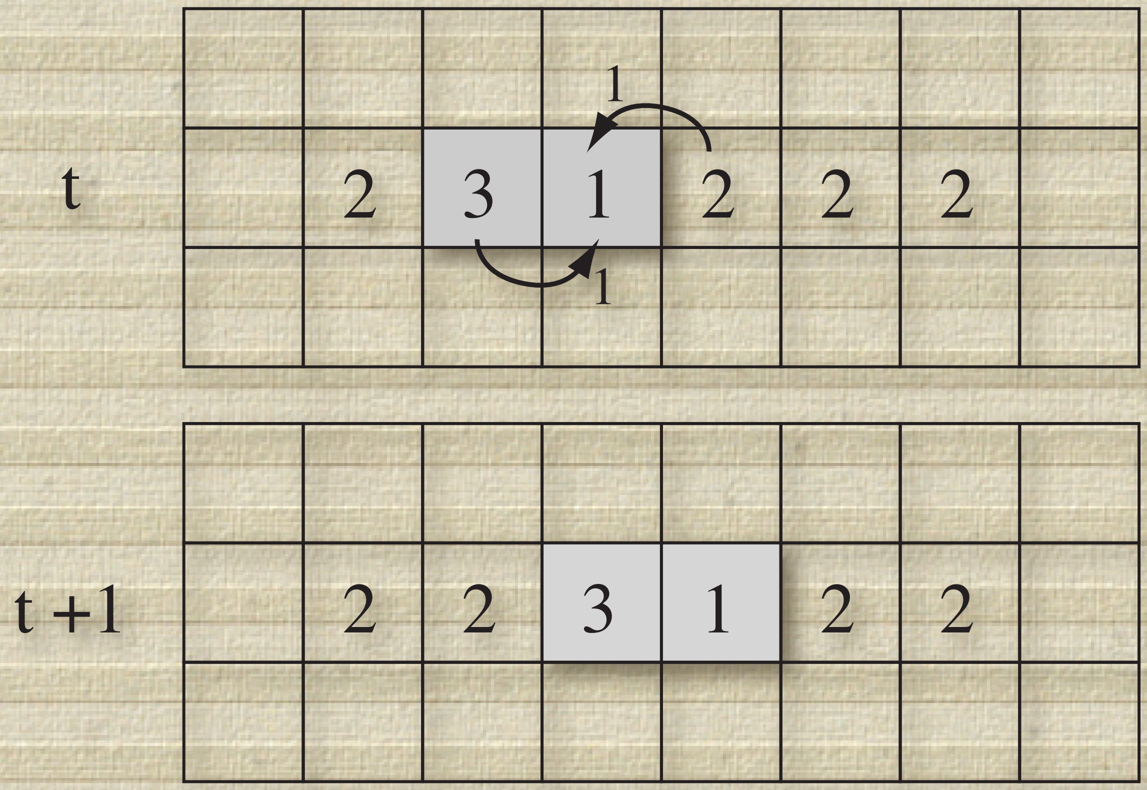
0	0	0	0	2	0	0	0
0	0	0	0	2	0	0	0
0	0	0	0	4	0	0	0
2	2	2	2	6	2	2	2
0	0	0	0	0	0	0	0

t+3

The number of states apt to be large

and moreover...

An example: unit wire



$$\tilde{Q} = \{0, 1, 2, 3\}$$

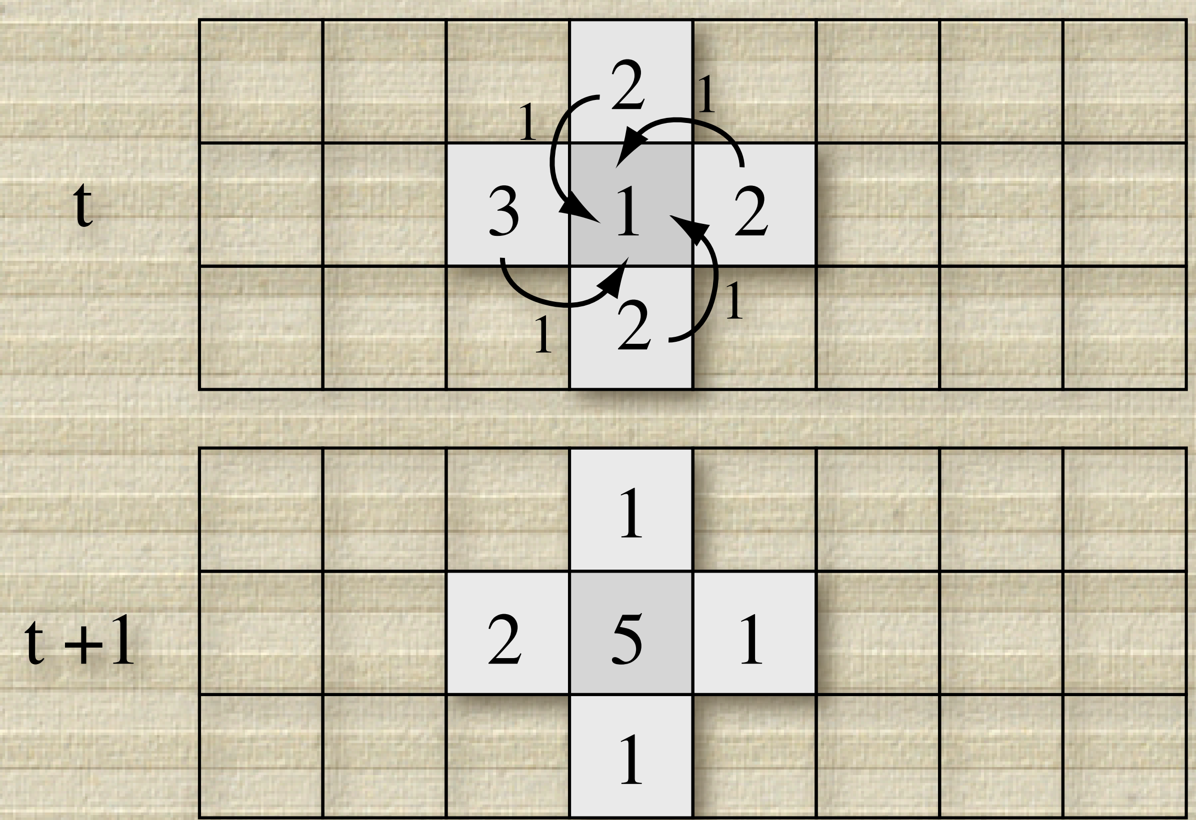
$$g(1, 2) = 1$$

$$g(1, 3) = 1$$

otherwise 0

only 4 states used, but...

Resolving the state set for this NCCA...



$$\tilde{Q} = \{0, 1, 2, 3\}$$

$$g(1, 2) = 1$$

$$g(1, 3) = 1$$

otherwise 0

State set turned out to be

$$Q = \{-2, -1, 0, 1, 2, 3, 4, 5\}$$

This NCCA can be regarded as an 8-state CA.

A logically universal NCCA

$$\tilde{Q}_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$$

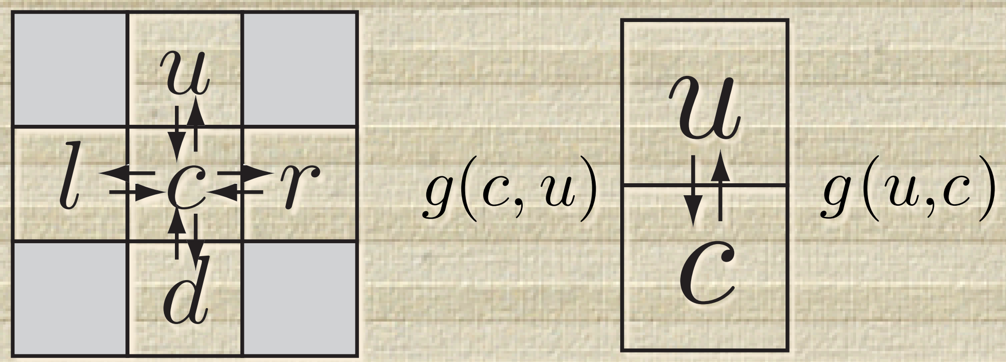
$$g(1,2) = 1, g(1,3) = 1, g(1,4) = 1, \underline{g(1,6) = \boxed{5,}}$$
$$g(3,5) = 1, g(3,6) = 1, g(3,7) = 3, g(4,3) = 1,$$
$$g(5,4) = 1$$

$$\text{otherwise} \quad 0$$

$$Q = \tilde{Q}_8 \cup \{-14, -9, -5, -4, -2, -1, 8, 9, 11, 12, 15, 16, 21\}$$

This NCCA can be regarded as an 21-state CA.

How to reduce the number of states?



Is there any way to take three or more cells into account?

General case

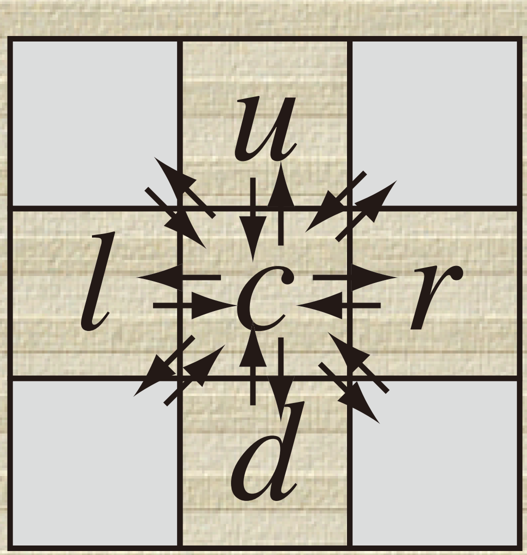
A von Neumann neigobor CA $A = (\mathbf{Z}^2, Q, f, q)$
is number-conserving iff

$$\exists g_U, g_R, g_D, g_L, h_{UR}, h_{RD}, h_{DL}, h_{LU} : Q^2 \to \mathbf{Z}, \forall c, u, r, d, l \in Q,$$

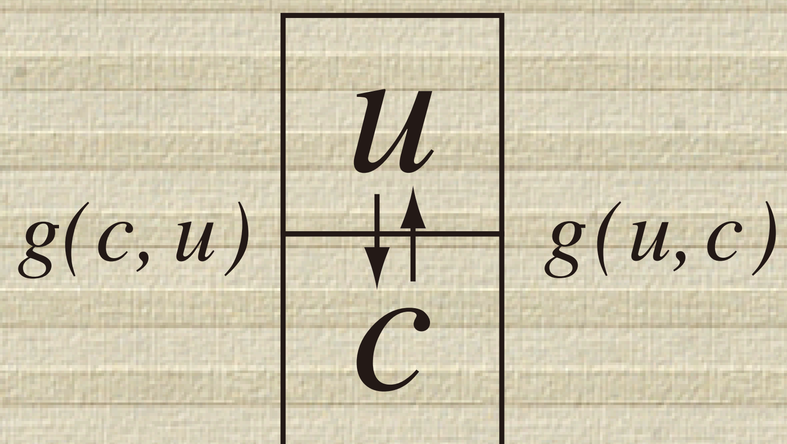
$$f(c, u, r, d, l) = c + g_U(c, u) + g_R(c, r) + g_D(c, d) + g_L(c, l)$$
$$+ h_{UR}(u, r) + h_{RD}(r, d) + h_{DL}(d, l) + h_{LU}(l, u),$$

$$g_U(c, u) = -g_D(u, c), g_R(c, r) = -g_L(r, c),$$

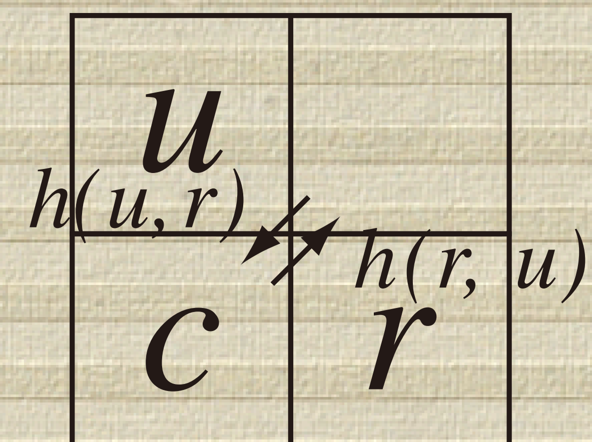
$$h_{UR}(u, r) = -h_{DL}(r, u), h_{RD}(r, d) = -h_{LU}(d, r).$$



horizontal/vertical flow



diagonal flow



Rotation-symmetric case

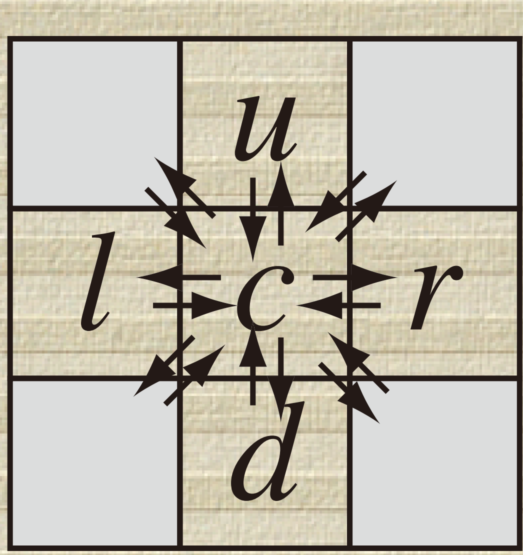
A rotation-symmetric von Neumann neighbor CA $A = (\mathbf{Z}^2, Q, f, q)$ is number-conserving iff

$$\exists g, h : Q^2 \to \mathbf{Z}, \forall c, u, r, d, l \in Q,$$

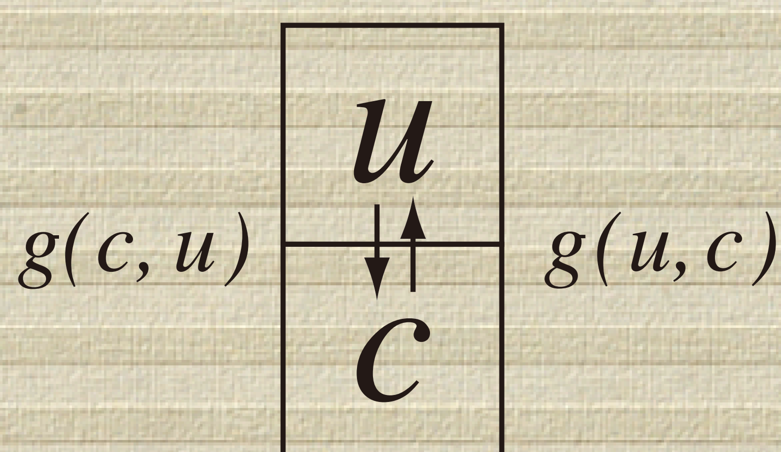
$$f(c,u,r,d,l) = c + g(c,u) + g(c,r) + g(c,d) + g(c,l)$$
$$+h(u,r) + h(r,d) + h(d,l) + h(l,u),$$

$$g(c,u) = -g(u,c),$$

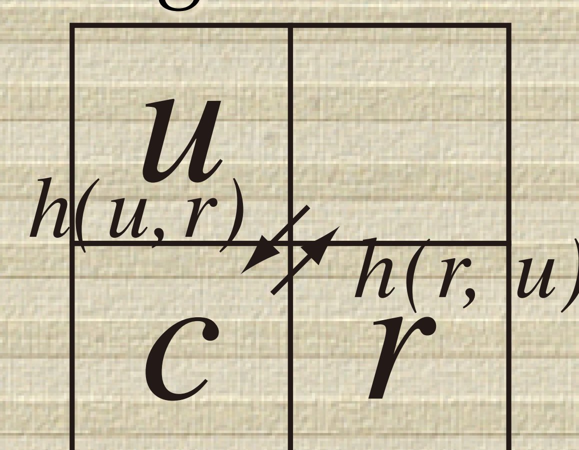
$$h(u,r) = -h(r,u).$$



horizontal/vertical flow



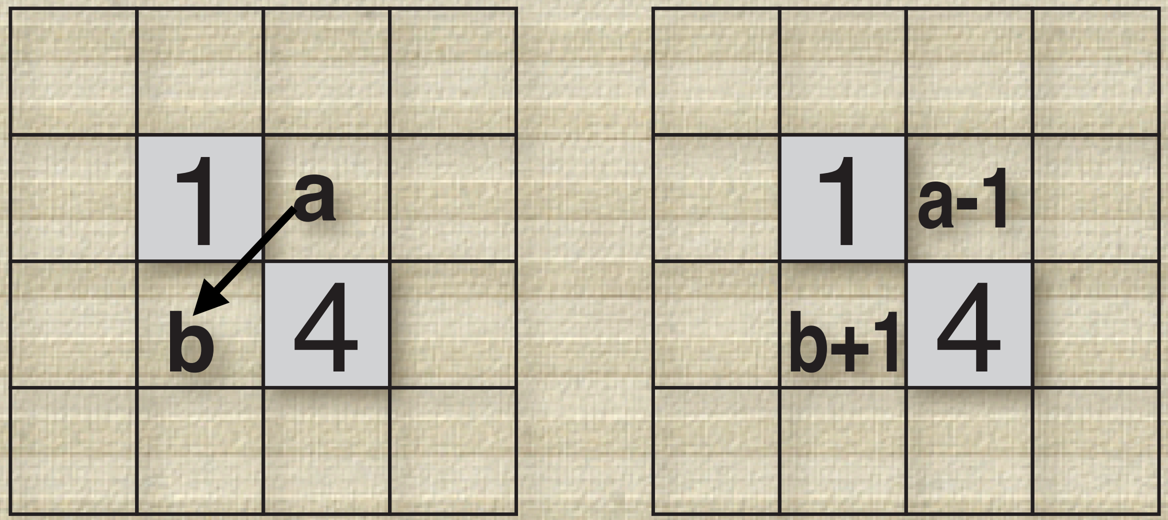
diagonal flow



Diagonal flow *h*

$$h(1,4) = 1$$

"virtual"



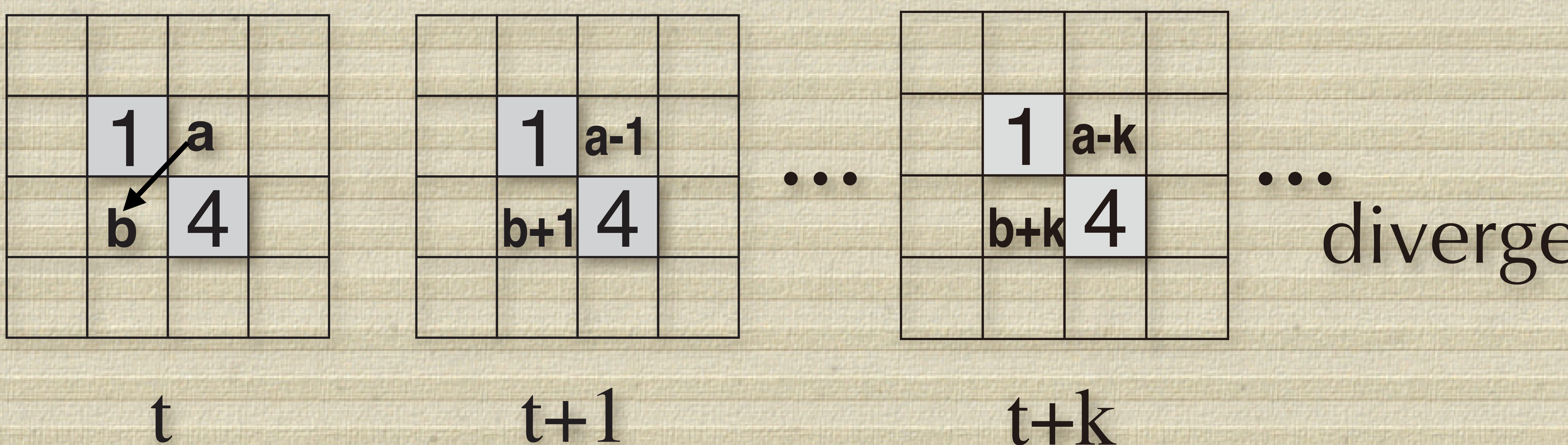
t t+1

Asymmetric diagonal movement of number is possible.

(Rotaion-symmetric but not permutarion-symmetric.)

Divergence of Diagonal flow

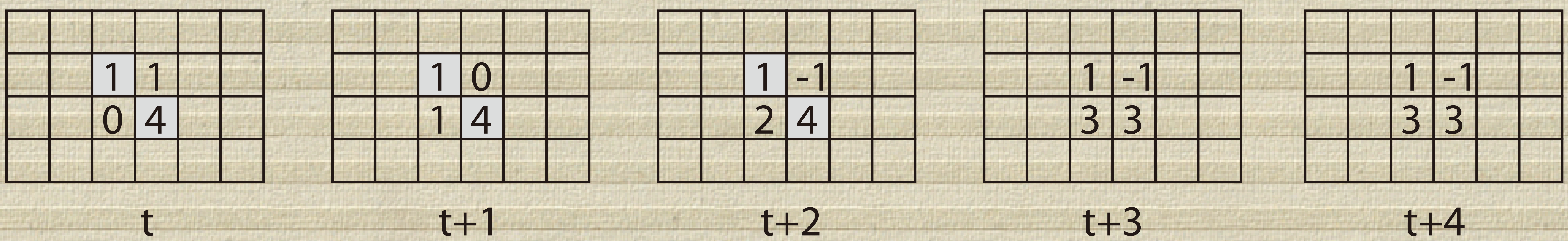
$$h(1,4) = 1$$



Not all combined function be a local function of CA.

Prevent divergence by assigning values to *g*

$\tilde{Q} = \{0, 1, 4\}$ $\quad h(1,4) = 1$



by adding (extra states) and values without destroying the embedded function.

$$g(-1,4) = 1, g(5,1) = -1$$

$$-1, 5 \notin \tilde{Q}$$

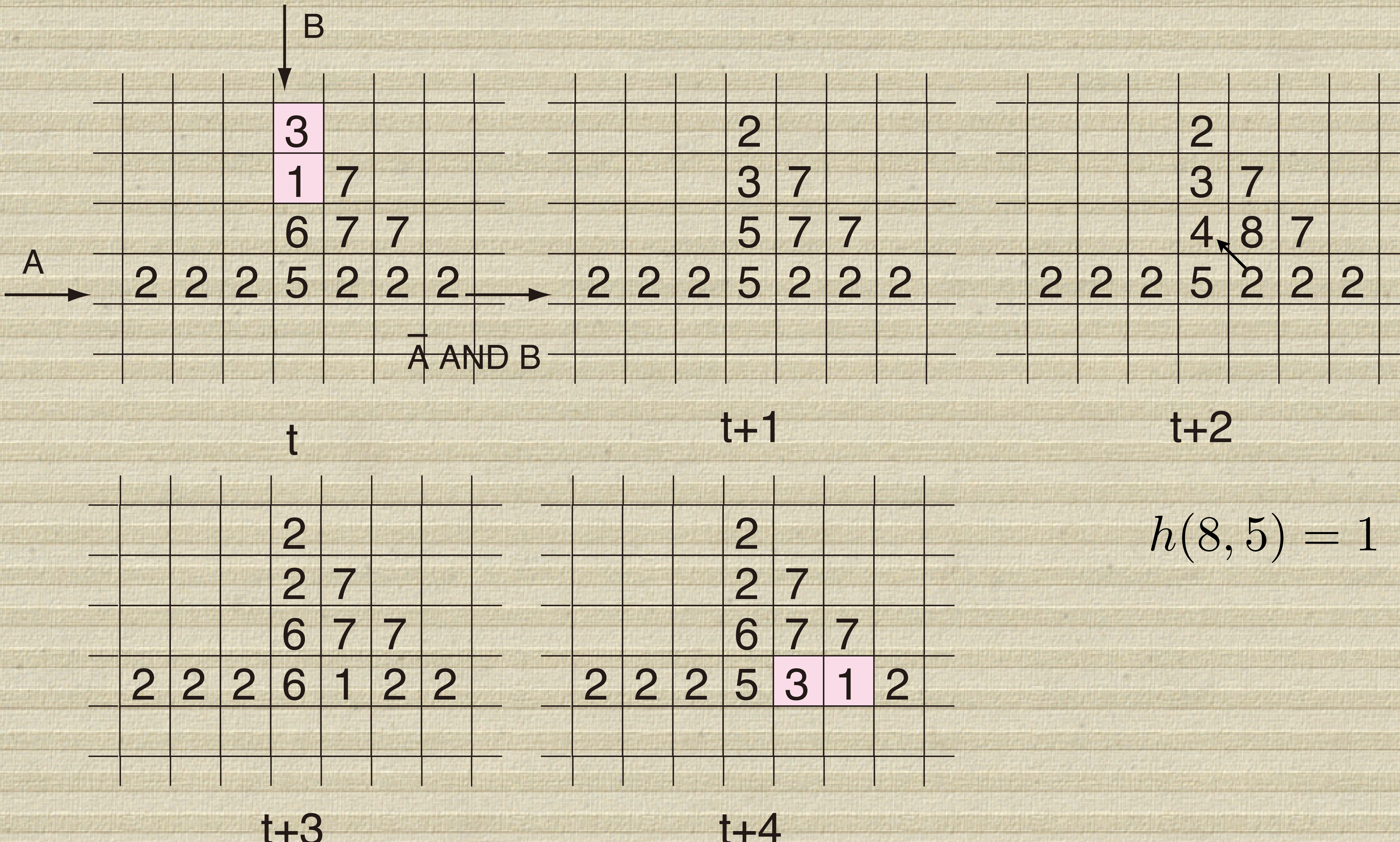
A 14 state logically universal NCCA

$$Q = \{-2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$$

$$g(1,2) = 1, g(1,3) = 1, g(1,4) = 1, g(1,6) = 1,$$
$$g(7,5) = 1, g(4,3) = 1, g(5,4) = 1, g(4,8) = 1,$$
$$g(1,5) = 1, g(5,11) = 1, g(-2,8) = 1,$$
$$\boxed{h(8,5) = 1}$$

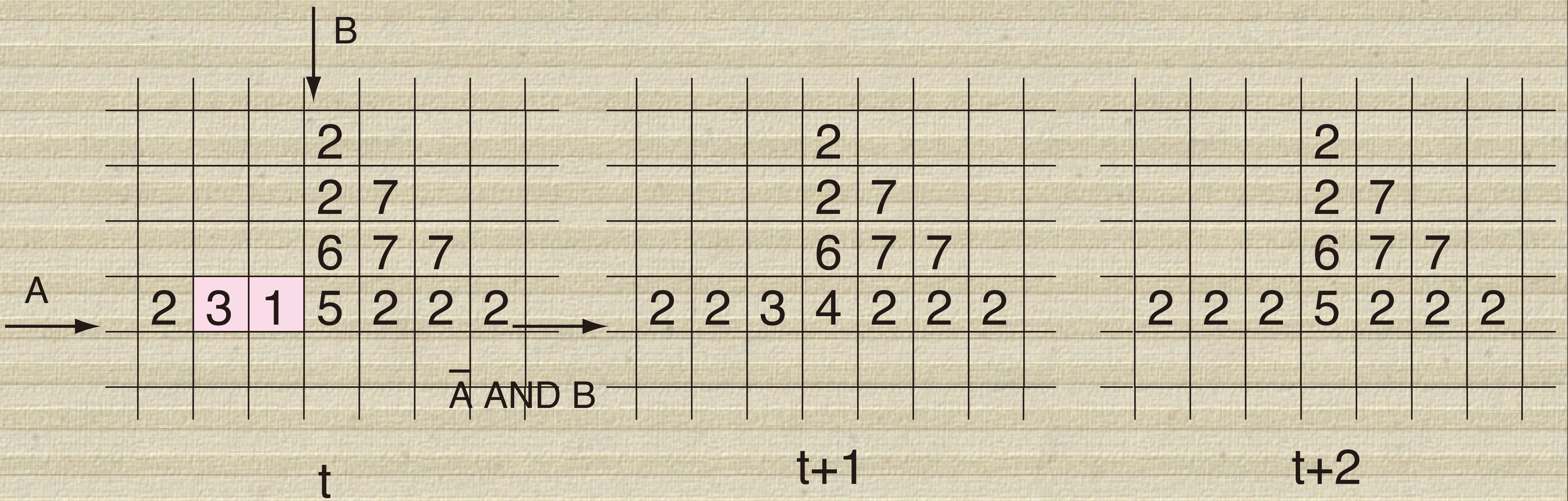
$$\text{otherwise} \quad 0$$

NOT A AND B (A=0,B=1)

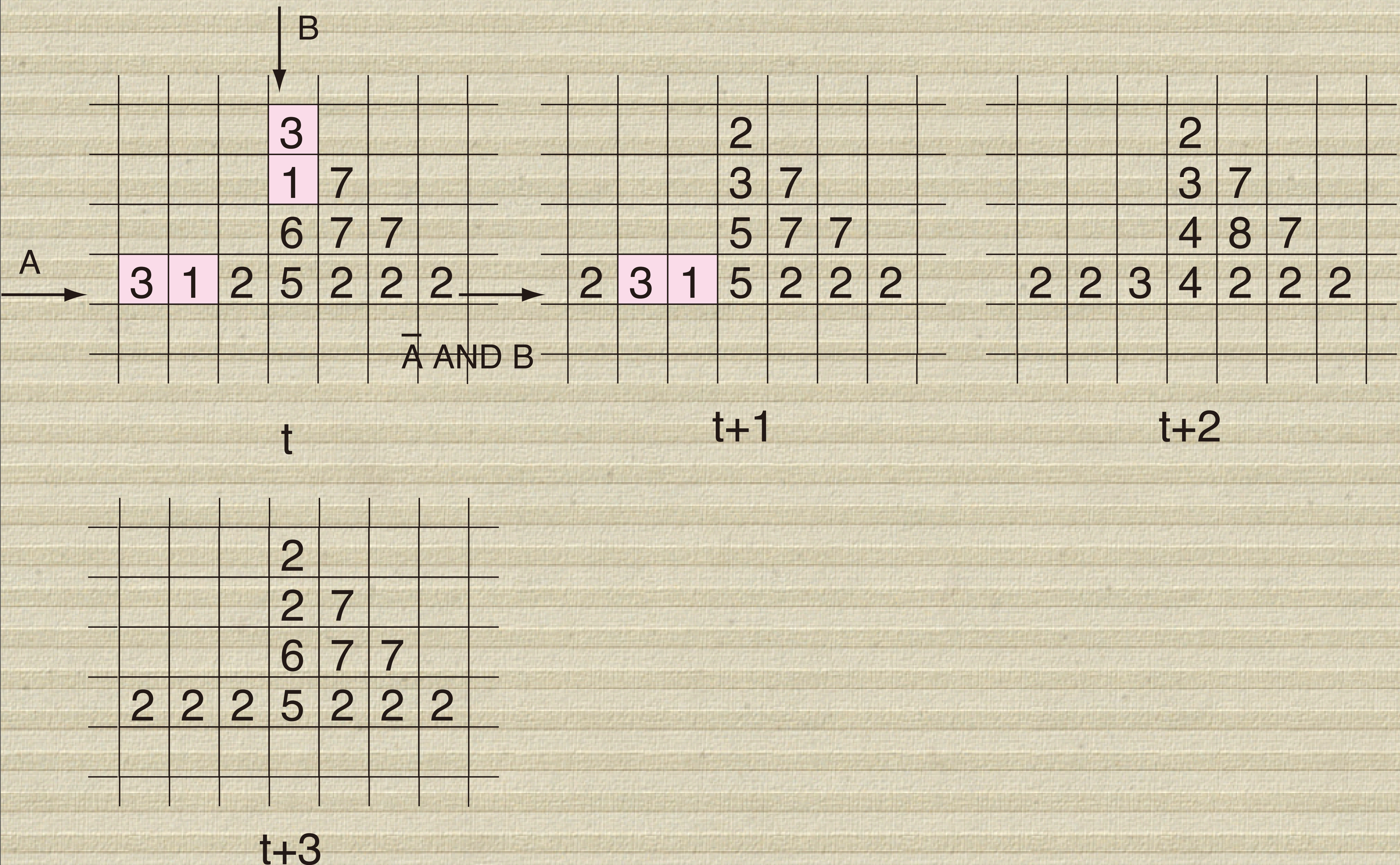


$$h(8,5) = 1$$

NOT A AND B (A=1,B=0)



NOT A AND B (A=B=1)



Conclusion

- Two binary functions characterize any rotation-symmetric von Neumann neighbor NCCA.
- A 14 state universal von Neuman neighbor NCCA is shown.
- What is the smallest state to be universal?