Parallel and Serial Dynamics in Boolean Networks

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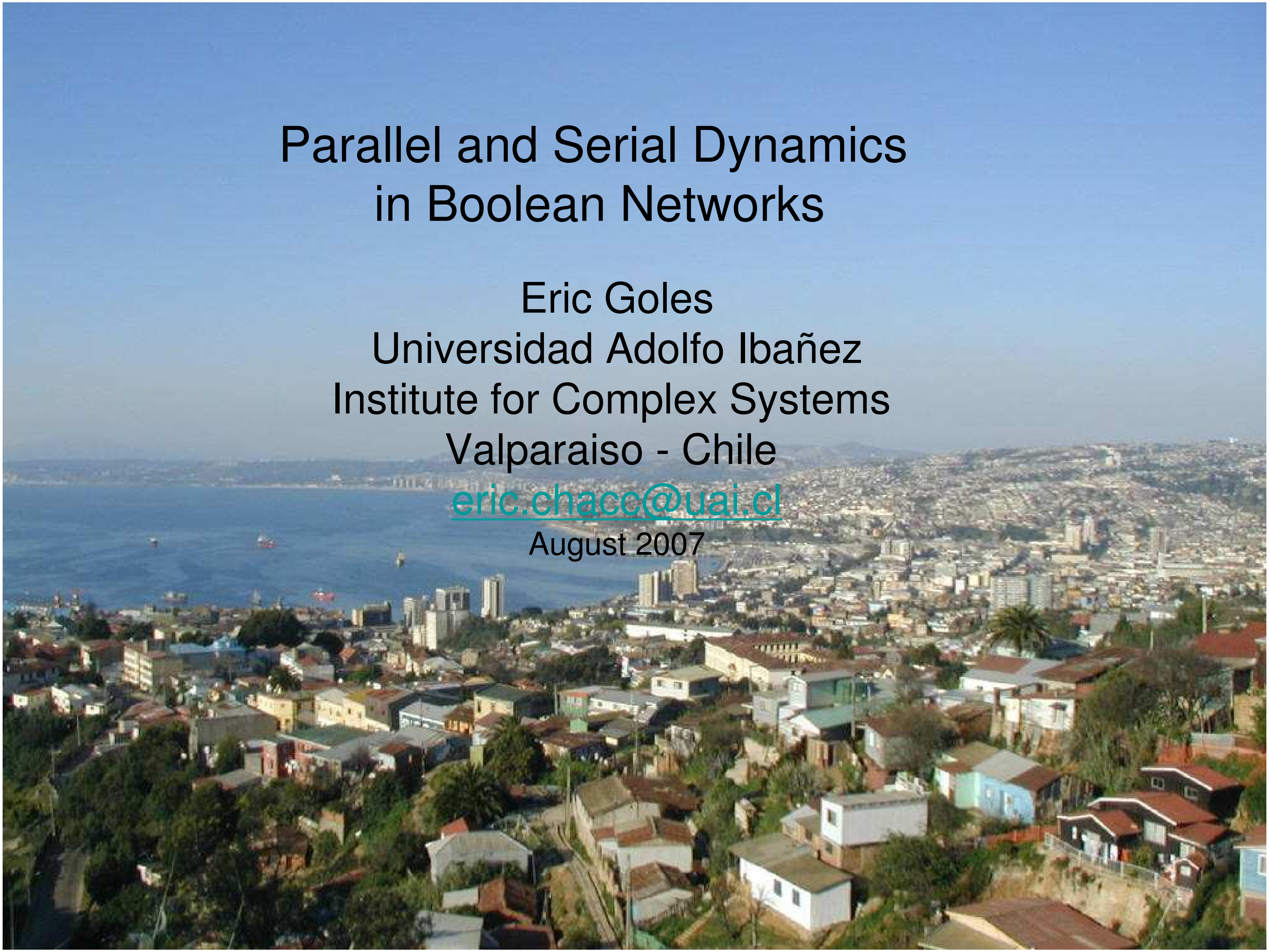


Given a boolean function

$$F:\{0,1\}^n \to \{0,1\}^n$$

$$F(x_1,x_2,\ldots,x_n)=(f_1(x_1,x_2,\ldots,x_n),\ldots,f_n(x_1,x_2,\ldots,x_n))$$

Parallel Iteration: To update each function synchronously.

$$x_1(t+1)=f(x_1(t),\ldots,x_i(t),\ldots,x_n(t))$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x_n(t+1)=f(x_1(t),\ldots,x_i(t),\ldots,x_n(t))$$

The serial iteration: {1}, {2}, … , {n}

$$x_1(t+1) = f(x_1(t),..,x_i(t),..,x_n(t))$$

$$x_2(t+1) = f(x_1(t+1),..,x_i(t),..,x_n(t))$$

....

$$x_i(t+1) = f(x_1(t+1),..,x_{i-1}(t+1),x_i(t),..,x_n(t))$$

....

$$x_n(t+1) = f(x_1(t),..,x_i(t),..,x_n(t))$$

Block sequential iterations

The set: $I_1, ..., I_k$ Is an orderer partition of {1, … , n} if and only if:

$$I_1 = \{1, 2, ..., n_1\}$$

$$I_2 = \{n_1 + 1, ..., n_2\}$$

....

$$I_k = \{n_{k-1} + 1, ..., n_k = n\}$$

Now, given F we update in parallel inside each block
and serial between differents blocks

$$x_1(t+1) = f(x_1(t), .., x_i(t), .., x_n(t))$$

$$x_2(t+1) = f(x_1(t), .., x_i(t), .., x_n(t))$$

....

$$x_{n_1}(t+1) = f(x_1(t), .., x_i(t), .., x_n(t))$$

$$x_{n_1+1}(t+1) = f(x_1(t+1), .., x_{n_1}(t+1), x_{n_1+1}(t) .., x_n(t))$$

....

$$x_{n_2}(t+1) = f(x_1(t+1), .., x_{n_1}(t+1), x_{n_1+1}(t) .., x_n(t))$$

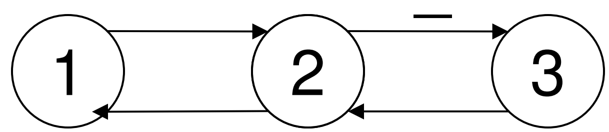
....

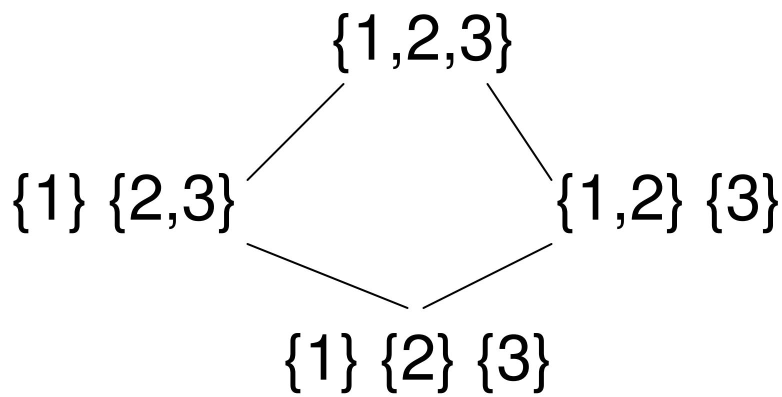
$$x_{n_{k-1}+1}(t+1) = f(x_1(t+1), .., x_{n_{k-1}}(t+1), x_{n_{k-1}+1}(t) .., x_n(t))$$

....

$$x_n(t+1) = f(x_1(t+1), .., x_{n_{k-1}}(t+1), x_{n_{k-1}+1}(t) .., x_n(t))$$

Fact: the set of fixed point is invariant under any
Block-Sequential iteration





Block Sequential partitions for three elements

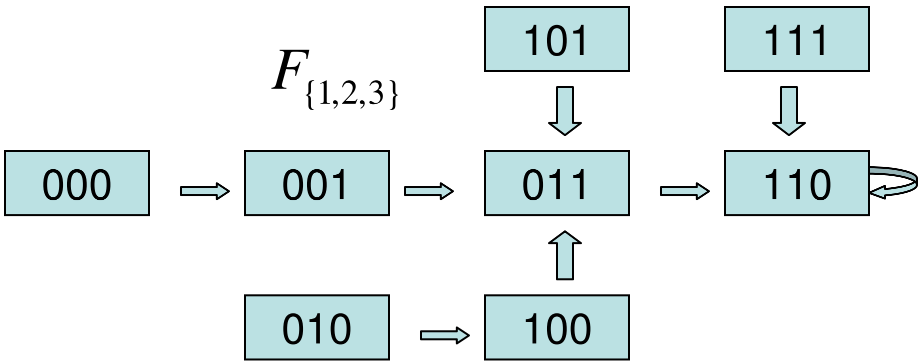
$$F_{\{1,2,3\}}(x_1,x_2,x_3)=(x_2,x_1+x_3,\neg x_2)$$

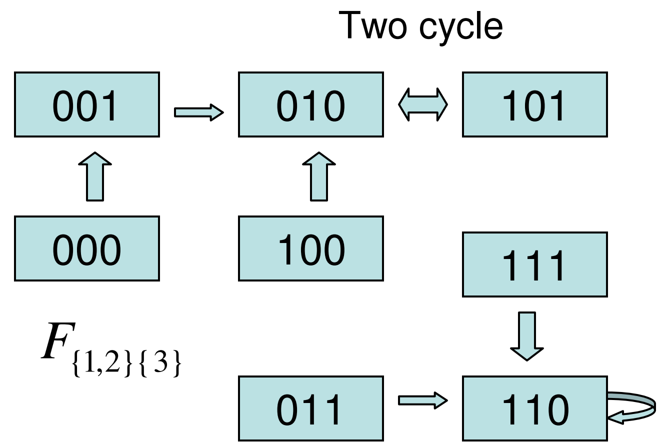
$$F_{\{1,2\}\{3\}}(x_1,x_2,x_3)=(x_2,x_1+x_3,(\neg x_1)(\neg x_3))$$

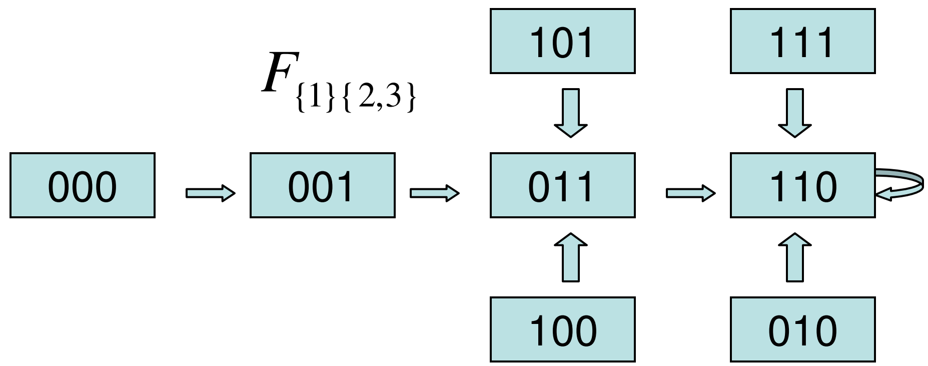
$$F_{\{1\}\{2,3\}}(x_1,x_2,x_3)=(x_2,x_2+x_3,\neg x_2)$$

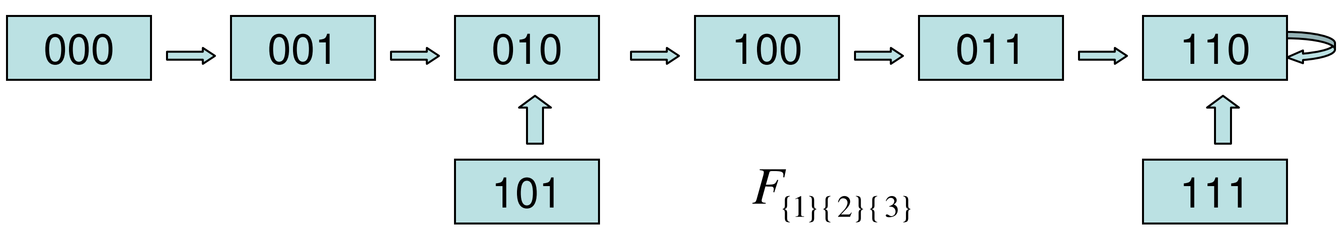
$$F_{\{1\}\{2,3\}}(x_1,x_2,x_3)=(x_2,x_2+x_3,(\neg x_2)(\neg x_3))$$

Block sequential diagrams





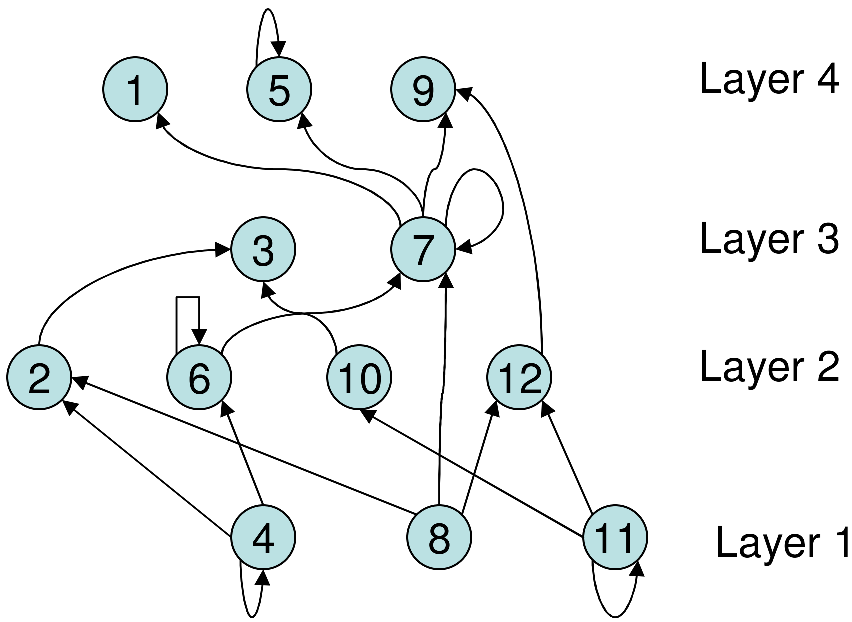




1. IF the graph of the network does not have circuits of length ≥ 2 then the cycles for parallel and serial iteration have period:

$$2^p$$

If it does not exist negative loops the atracctors are only fixed points



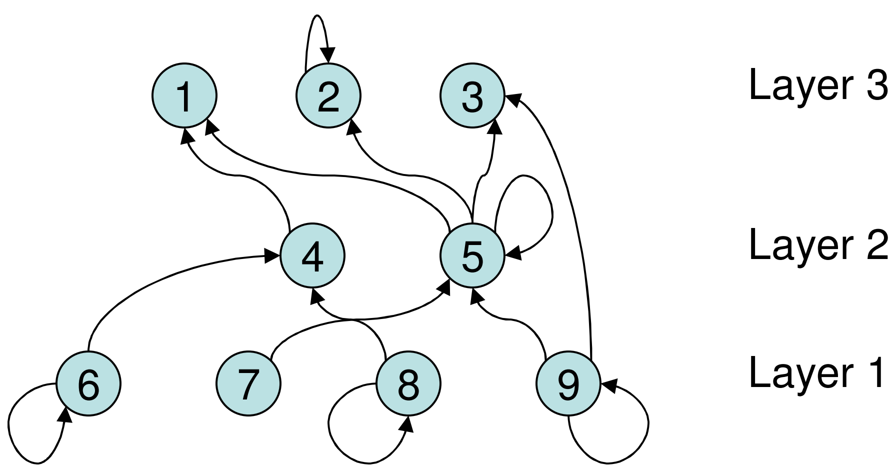
A particular case: let I(i) the set of indexes of variables such that the i-th function depends. I.e:

$$k \in I(i) \Leftrightarrow f_i(x_1,..,x_k = 0,..,x_n) \neq f_i(x_1,..,\neg x_k = 1,..,x_n)$$

then

$$\forall i \in \{1,...,n\}, I(i) \subseteq \{i,...,n\} \Rightarrow$$

The serial and the parallel dynamic are identical



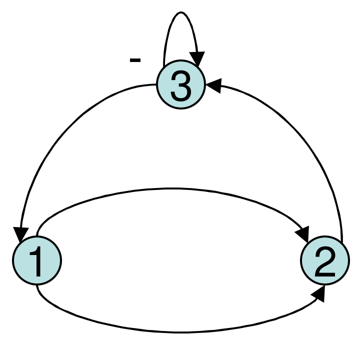
Cycles in Parallel and Serial Iterations

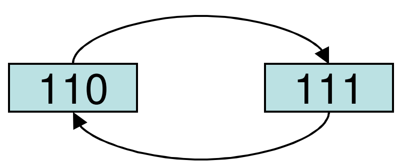
Consider a network with non-negative loops then the Cycles with period≥2, if there exists, are different for parallel and serial iteration.

i.e both iterations can not share non trivial cycles

The hypothesis about loops is neccesary. Consider:

$$F : \{0,1\}^3 \rightarrow \{0,1\}^3$$
$$f_1(x_1, x_2, x_3) = x_2$$
$$f_2(x_1, x_2, x_3) = x_1$$
$$f_3(x_1, x_2, x_3) = x_1 x_2 (\neg x_3)$$





It is a cycle for both iterations

FILTERS

A filter *G* associated to a boolean network F corresponds to the recursive application of an iteration mode, S, to F:

$$G = \lim_{p \to \infty} S^p(F)$$

We will consider S the serial update:

S = {1} {2} …. {n}

Since F is finite S converges to a network G which we call the filter.

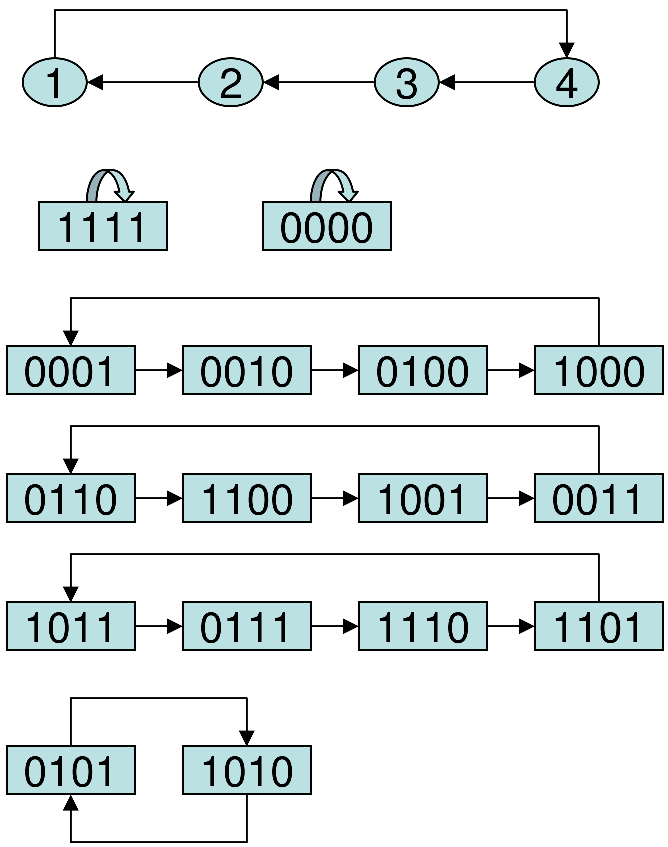
- Example: consider the function $F(x_1,x_2,x_3,x_4)=(x_2,x_3,x_4,x_1)$

  and the serial update S = {1} {2} {3} {4}

$$F^0 = F(x_1,x_2,x_3,x_4) = (x_2,x_3,x_4,x_1)$$
$$F^1 = S(F^0) = (x_2,x_3,x_4,x_2)$$
$$F^2 = S(F^1) = S^2(F^0) = (x_2,x_3,x_4,x_3)$$
$$F^3 = S(F^2) = S^3(F^0) = (x_2,x_3,x_4,x_4)$$
$$G = F^4 = S(F^3) = S^4(F^0) = (x_2,x_3,x_4,x_4)$$

The function G is the filter and fixed point of the procedure

F^0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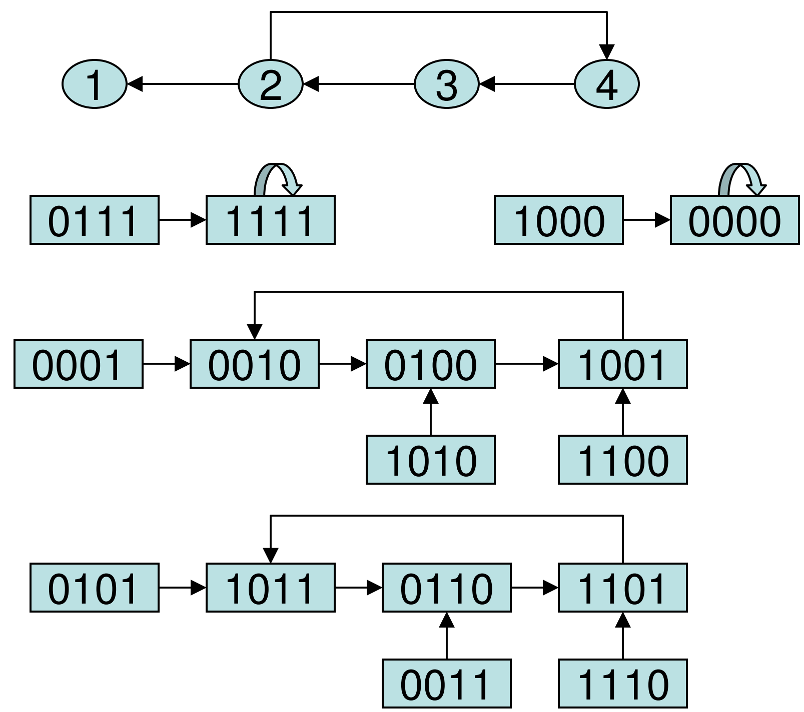
F^1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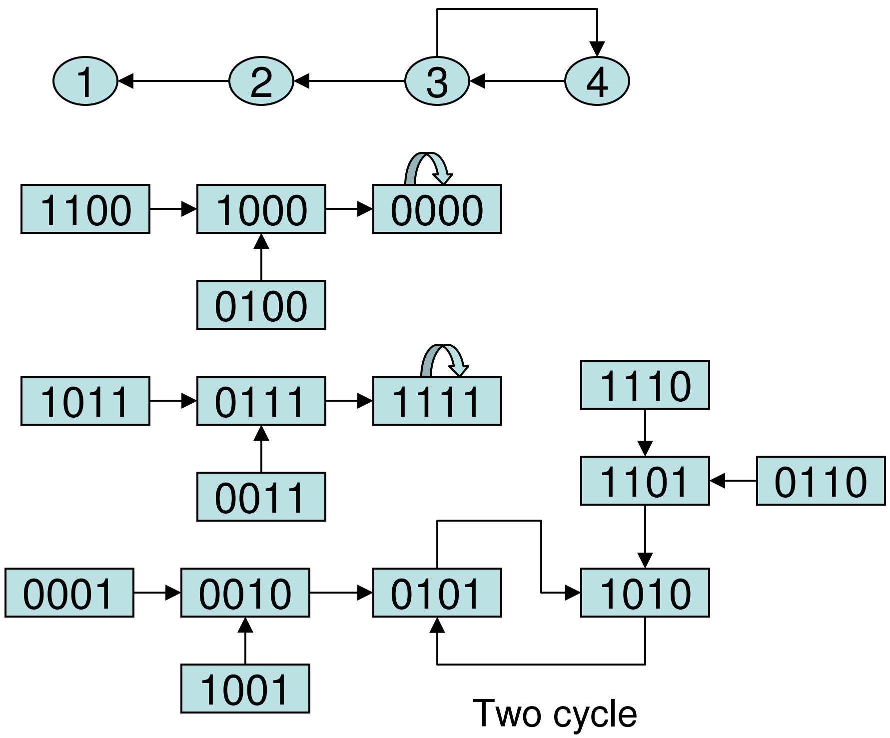
$$F^2$$



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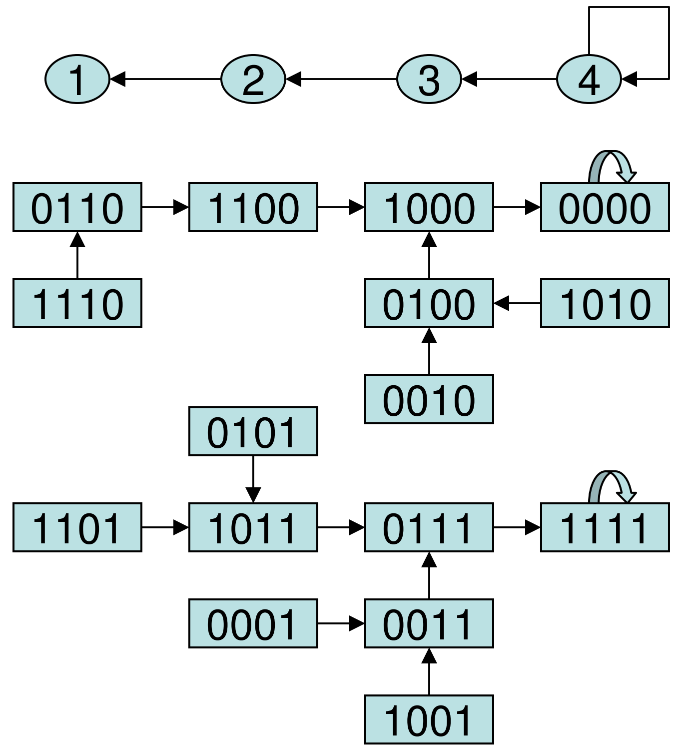
Two cycle

$$F^3 = F^4 = G$$



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THEOREM

- Given a monotone boolean network

then the serial filter converges in at most $o(n^2)$ to a network G without cycles in its dynamics

This result can be extended to networks such that its circuits are non-negative.