Surjectivity an surjunctivity of cellular automata in Besicovitch topology

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1D cellular automata

Definition

$\mathcal{A} = \langle Q, \mathcal{N}, f \rangle$, Q finite, $\mathcal{N} = [-r, \ldots, +r]$, $f : Q^{2r+1} \to Q$.

$$(F_{\mathcal{A}}(c))(x) = f(c(x-r), \ldots, c(x+r)) \ .$$

A problem

Translation: action of $\mathbb{Z}$ on $Q^{\mathbb{Z}}$ defined by

$$c^x(y) = c(x+y) \ .$$

No distance on $Q^{\mathbb{Z}}$ invariant by translation can induce the product topology. (Formenti, 1998)

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The Besicovitch topology on $Q^{\mathbb{Z}}$

Definition (Formenti et al.)

For $c_1, c_2 \in Q^{\mathbb{Z}}$, put

$$d_B(c_1, c_2) = \limsup_{n \to \infty} \frac{|\{x \in \{-n, \ldots, n\} : c_1(x) \neq c_2(x)\}|}{2n+1}$$

d_B is a pseudodistance.

$c_1 \sim_B c_2$ iff $d_B(c_1, c_2) = 0$, is an equivalence relation.

d_B is a distance on the quotient space.

A possible solution to our problem

d_B is invariant by translation.

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Besicovitch topology and 1D CA

Comparison of topologies

$Q^{\mathbb{Z}}$	$Q^{\mathbb{Z}} / \sim_B$
uncountable	uncountable
perfect	perfect
compact	not locally compact
totally disconnected	arcwise connected
zero-dimensional	infinite-dimensional

CA in the new topology (Formenti et al.)

Any 1D CA $\mathcal{A}$ induces a continuous transformation F of $Q^{\mathbb{Z}} / \sim_B$.
Many properties of $\mathcal{A}$ can be inferred from those of F.
In particular, $\mathcal{A}$ is surjective iff F is.

Can we generalize this? and how?

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Finitely generated groups

Definitions

- Set of generators: S s.t. the graph $(G, \mathcal{E}_S)$ with
$$\mathcal{E}_S = \{(x, xz) : z \in S \cup S^{-1}\}$$
is connected.
- Length w.r.t. (finite) S: distance $\|x\|_S$ from 1_G in $(G, \mathcal{E}_S)$.
- Disk of radius n: $D_{n,S} = \{x \in G : \|x\|_S \leq n\}$.
- Growth rate: the function $n \mapsto |D_{n,S}|$.
Well defined for f.g. groups, up to an equivalence.
- Boundary: $\partial_E X = \{g \in G : gE \cap X \neq \emptyset \neq gE \setminus X\}$
- Translation: action of G on Q^G defined by $c^x(y) = c(xy)$.

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Exhaustive and amenable sequences

Exhaustive sequence

$\{X_n\}_{n\in\mathbb{N}} \subseteq \mathcal{P}(G)$ such that $X_n \nearrow G$. Example: disks.

Amenable sequence

Exhaustive sequence s.t. for all finite E,

$$\lim_{n\to\infty} \frac{|\partial_E X_n|}{|X_n|} = 0$$

E.g., the von Neumann (or Moore) neighborhoods of range $n \geq 0$.

Growth rate and amenable sequences

1. G of polynomial growth $\Rightarrow \{D_{n,S}\}$ amenable.
2. G of subexponential growth $\Rightarrow \{D_{n,S}\}$ has amenable $\{D_{n_k,S}\}$

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Besicovitch topologies induced by exhaustive sequences

General definition

$\{X_n\}$ exhaustive. Besicovitch distance induced by $\{X_n\}$:

$$d_{B,\{X_n\}}(c_1, c_2) = \limsup_{n \to \infty} \frac{|\{x : c_1(x) \neq c_2(x)\} \cap X_n|}{|X_n|} .$$

$d_{B,\{X_n\}}$ is a pseudodistance.
$c_1 \sim_{B,\{X_n\}} c_2$ iff $d_{B,\{X_n\}}(c_1, c_2) = 0$, is an equivalence relation.
$d_{B,\{X_n\}}$ is a distance on the quotient space.
A priori, dependent on $\{X_n\}$.

Proposition 1

If $G = \mathbb{Z}^d$ and S, S' are finite set of generators, then

$$d_{B,\{D_{n,S}\}}(c_1, c_2) = 0 \Leftrightarrow d_{B,\{D_{n,S'}\}}(c_1, c_2) = 0 .$$

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Invariant by translation?

Not always!

Let G be the free group on $S = \{a, b\}$.
The graph $(G, \mathcal{E}_S)$ is the joining of 4 infinite 3-ary trees.
Let $c(x) = 1$ iff x is in the right subtree, 0 otherwise.
Then $c^a(x) = 0$ iff x is in the left subtree, 1 otherwise.
Thus

$$d_{B,\{D_{n,S}\}}(\mathbf{0}, c) = \frac{1}{4} \quad \text{but} \quad d_{B,\{D_{n,S}\}}(\mathbf{0}^a, c^a) = \frac{3}{4} \, .$$

Proposition 2

- $\{X_n^{-1}\}$ amenable $\Rightarrow$ $d_{B,\{X_n\}}$ translation invariant.
- In particular, $\{D_{n,S}\}$ amenable $\Rightarrow$ $d_{B,\{D_{n,S}\}}$ translation invariant.

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CA over finitely generated groups

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$\mathcal{A} = \langle Q, \mathcal{N}, f \rangle$, Q finite, $\mathcal{N} = \{n_0, \ldots, n_{|\mathcal{N}|-1}\} \subseteq G$, $f : Q^{|\mathcal{N}|} \to Q$.

$$(F_{\mathcal{A}}(c))(x) = f\left(c(x \cdot n_0), \ldots, c(x \cdot n_{|\mathcal{N}|-1})\right) .$$

Some basic facts remain true

- $\mathcal{A}$ surjective $\Leftrightarrow$ no Garden-of-Eden patterns. (Fiorenzi 2000)
- $\mathcal{A}$ preinjective $\Leftrightarrow$ no mutually erasable patterns. (Fiorenzi 2000)
- $\exists\{X_n\}$ amenable $\Rightarrow$ $\mathcal{A}$ surjective iff $\mathcal{A}$ preinjective (Ceccherini–Silberstein, Machì, Scarabotti 1999)

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Induced maps

Induced map

Let $F : Q^G \to Q^G$.
If $d_{B,\{X_n\}}(c_1, c_2) = 0$ implies $d_{B,\{X_n\}}(F(c_1), F(c_2)) = 0$, then

$$F([c]_{\sim_{B,\{X_n\}}}) = [F(c)]_{\sim_{B,\{X_n\}}}$$

is well defined.

Proposition 3

Let $\mathcal{A}$ be a CA. If either

- $\{X_n\}$ is amenable, or
- $X_n = D_{n,S}$ for all n and some S,

then $d_{B,\{X_n\}}(c_1, c_2) = 0$ implies $d_{B,\{X_n\}}(F_{\mathcal{A}}(c_1), F_{\mathcal{A}}(c_2)) = 0$.
Moreover, the induced map is Lipschitz continuous.

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For $F : Q^G \to Q^G$, we define

- $(B, \{X_n\})$-surjectivity:
  $\forall c \, \exists c' : d_{B,\{X_n\}}(c, F(c')) = 0$;
- $(B, \{X_n\})$-injectivity:
  $d_{B,\{X_n\}}(c_1, c_2) > 0 \Rightarrow d_{B,\{X_n\}}(F(c_1), F(c_2)) > 0$.

Observe that

- Not required F well defined modulo $\sim_{B,\{X_n\}}$.
- $F_\mathcal{A}(c') \neq c \; \forall c' \quad \not\Rightarrow \quad d_{B,\{X_n\}}(c, F_\mathcal{A}(c')) > 0 \; \forall c'$.
  (Take any c'; replace one part of $c = F_\mathcal{A}(c')$ with GoE.)

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Surjectivity

Theorem 1

Suppose that $\{X_n\}$ contains an amenable subsequence.
Then $\mathcal{A}$ is surjective iff it is $(B, \{X_n\})$-surjective.

Corollary 1

If G has subexponential growth, the following are equivalent:

1. $\mathcal{A}$ is $(B, \{D_{n,S}\})$-surjective for some S;
2. $\mathcal{A}$ is $(B, \{D_{n,S}\})$-surjective for every S;
3. $\mathcal{A}$ is surjective.

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Surjunctivity

Theorem 2

Suppose that $\{X_n\}$ contains an amenable subsequence.
If $\mathcal{A}$ is $(B, \{X_n\})$-injective, then it is preinjective.

Corollary 2

If G has subexponential growth, and $\mathcal{A}$ is $(B, \{D_{n,S}\})$-injective, then $\mathcal{A}$ is $(B, \{D_{n,S}\})$-surjective.

Surjunctivity

Theorem 2

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(U, W)-nets

Definition

$N \subseteq G$ is a (U, W)-net $(U, W \subseteq G)$ if

1. $xU \cap yU = \emptyset$ for $x, y \in N$, $x \neq y$, and
2. $\bigcup_{x \in N} xW = G$.

For every $U \neq \emptyset$, an (U, UU^{-1})-net exists by Zorn's lemma.
In particular, for every $R \geq 0$ and S set of generators, a $(D_{R,S}, D_{2R,S})$-net exists.

Lemma 1

If W is finite and $\{X_n\}$ is amenable, then

$$\liminf_{n \to \infty} \frac{|N \cap X_n|}{|X_n|} \geq \frac{1}{|W|}$$

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For every $U \neq \emptyset$, an (U, UU^{-1})-net exists by Zorn's lemma. In particular, for every $R \geq 0$ and S set of generators, a $(D_{R,S}, D_{2R,S})$-net exists.

Lemma 1

If W is finite and $\{X_n\}$ is amenable, then

$$\liminf_{n \to \infty} \frac{|N \cap X_n|}{|X_n|} \geq \frac{1}{|W|}$$

Proof of Theorem 2

Suppose $\mathcal{A}$ is not preinjective...

Let $p_1, p_2 : D_M \to Q$ be m.e. patterns with $p_1(1_G) \neq p_2(1_G)$.
Let $\{X_{n_k}\}$ be amenable. Let $R \geq M + r$ where $\mathcal{N} \subseteq D_r$.
Let N be a (D_R, D_{2R})-net.
Let c_j coincide with p_j on xD_M for all $x \in N$, and have fixed value q otherwise.
Then $F_{\mathcal{A}}(c_1) = F_{\mathcal{A}}(c_2)$, and $d_{B,\{X_n\}}(F_{\mathcal{A}}(c_1), F_{\mathcal{A}}(c_2)) = 0$.
But by construction, $c_1(x) \neq c_2(x)$ for all $x \in N$.
Then, by Lemma 1,

$$d_{B,\{X_n\}}(c_1, c_2) \geq \limsup_{n\to\infty} \frac{|N \cap X_n|}{|X_n|} \geq \liminf_{k\to\infty} \frac{|N \cap X_{n_k}|}{|X_{n_k}|} \geq \frac{1}{|D_{2R}|}.$$

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Conclusions

Let $\mathcal{A}$ be a CA on a group G of subexponential growth.
Let S be a finite set of generators for G.

- $\mathcal{A}$ induces a continuous transformation F of $Q^G / \sim_{B,\{D_{n,S}\}}$.
- F is surjective iff $\mathcal{A}$ is surjective.
- If F is injective, then F is surjective.

Amenable sequences seems to play a key role in all this.
Conjecture: In the hypotheses of Theorems 1 and 2, $(B, \{X_n\})$-injectivity is equivalent to preinjectivity.

Thank you for attention!

Any questions?

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