Cellular Automata with Memory

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Standard CA

Standard Cellular Automata (CA) are ahistoric (memoryless): i.e., the new state of a cell depends on the neighborhood configuration only at the preceding time step.

$$\sigma_i^{(T+1)} = \phi\big(\sigma_j^{(T)} \in \mathcal{N}_i\big)$$

$\boxed{\textbf{CA with memory in cells}}$

$$\sigma_i^{(T+1)} = \phi\big(s_j^{(T)} \in \mathcal{N}_i\big)$$

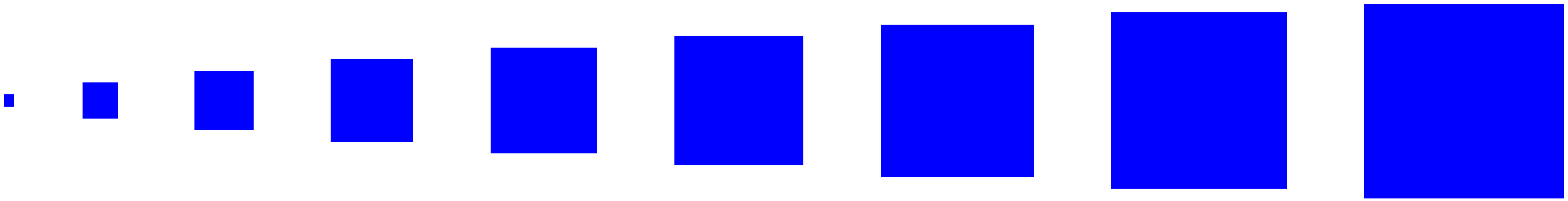
$$s_j^{(T)} = s\big(\sigma_j^{(1)}, \ldots, \sigma_j^{(T-1)}, \sigma_j^{(T)}\big)$$

We consider here an extension to the standard framework of CA by implementing memory capabilities in cells. Thus in CA with memory here: while the update rules of the CA remain unaltered, historic memory of all past iterations is retained by featuring each cell (and link) by a summary of its past states.

Example

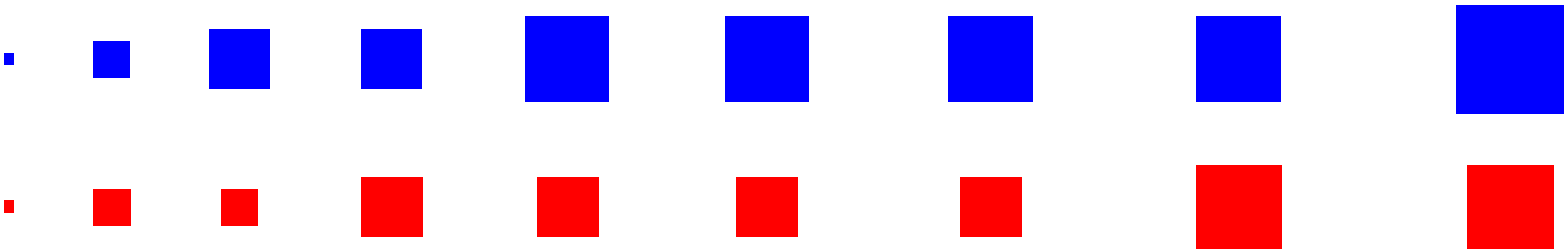
ϕ: **cell alive if any cell in its neighborhood is alive.**

Ahistoric (speed of light)



σ

Mode (most frequent) memory



σ

s

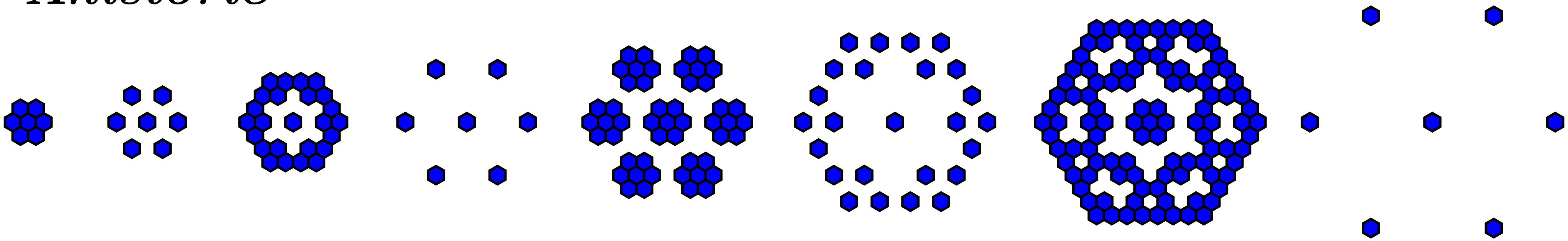
$$s_i^{(T)} = mode(\sigma_i^{(1)}, \ldots, \sigma_i^{(T)}) \qquad s_i^{(T)} = \sigma_i^{(T)} \text{ if } \#0 = \#1$$

INERTIAL EFFECT

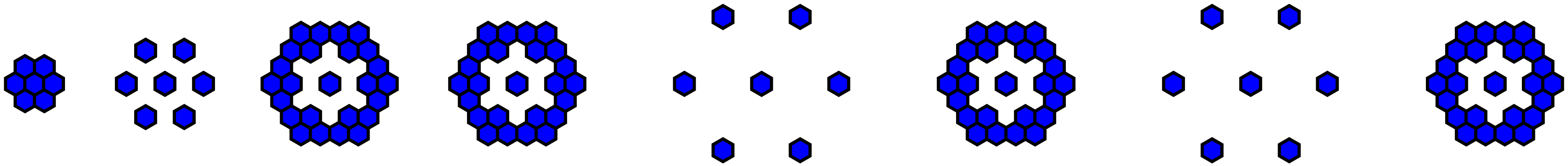
Parity rule

ϕ: cell alive iff odd number of alive cells in its neighborhood.

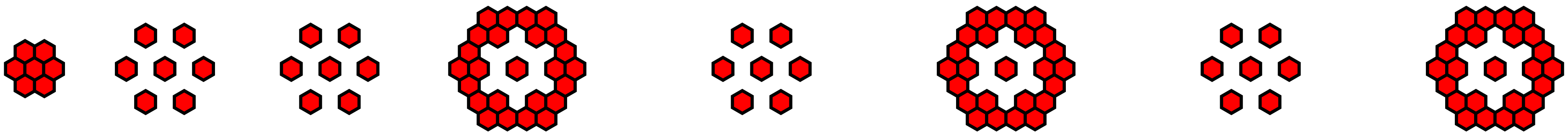
Ahistoric



***Mode** (most frequent) memory*



σ



s

$$\sigma_i^{(T+1)} = \sum_{j \in \mathcal{N}_i} \sigma_j^{(T)} \; mod \; \mathbf{2} \to \sum_{j \in \mathcal{N}_i} s_j^{(T)} \; mod \; \mathbf{2}$$

Weighted memory (unlimited trailing)

$$\boxed{\sigma_i^{(1)}} \times \alpha^{T-1}$$
$$\cdots\cdots\cdots$$
$$\boxed{\sigma_i^{(T-2)}} \times \alpha^2$$
$$\boxed{\sigma_i^{(T-1)}} \times \alpha$$
$$\boxed{\sigma_i^{(T)}} \times 1$$

$$m_i^{(T)}(\sigma_i^{(1)},\ldots,\sigma_i^{(T)}) = \frac{\sigma_i^{(T)} + \sum_{t=1}^{T-1} \alpha^{T-t} \sigma_i^{(t)}}{1 + \sum_{t=1}^{T-1} \alpha^{T-t}} \equiv \frac{\omega_i^{(T)}}{\Omega(T)}$$

The choice of the memory factor $0 \leq \alpha \leq 1$ simulates the long-term or remnant memory effect: the limit case $\alpha = 1$ corresponds memory with equally weighted records (*full* memory model, *mode* if $k = 2$), whereas $\alpha << 1$ intensifies the contribution of the most recent states and diminishes the contribution of the past ones (*short* type memory). The choice $\alpha = 0$ leads to the ahistoric model.

If $\sigma \in \{0,1\}$, the rounded weighted mean state (s) will be obtained by comparing the weighted mean (m) to 0.5, so that :

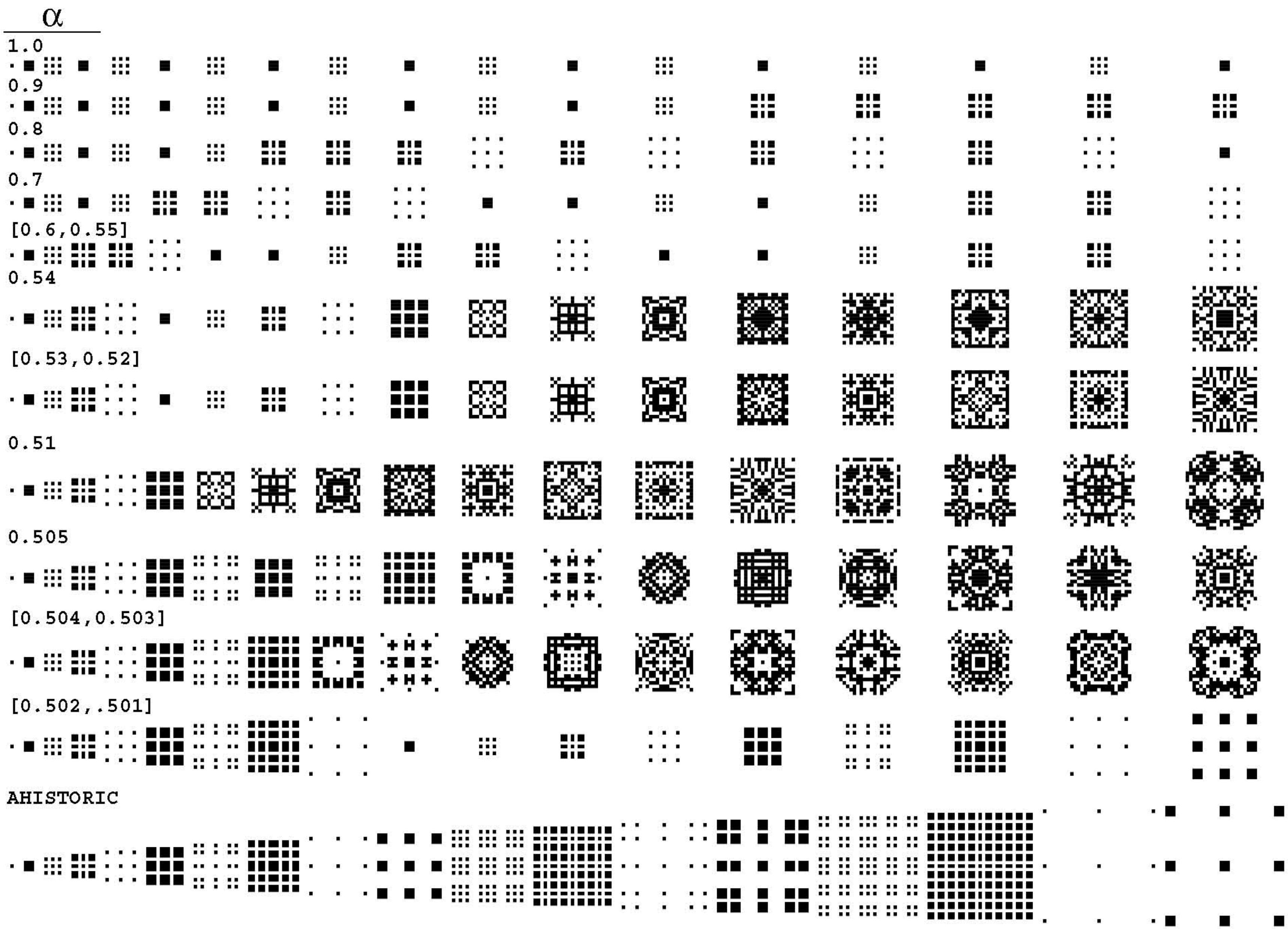
$$s_i^{(T)} = \begin{cases} 1 & if \quad m_i^{(T)} > 0.5 \\ \sigma_i^{(T)} & if \quad m_i^{(T)} = 0.5 \\ 0 & if \quad m_i^{(T)} < 0.5 \end{cases} . \qquad s_i^{(1)} = \sigma_i^{(1)} \;,\; s_i^{(2)} = \sigma_i^{(2)}$$

Implementation

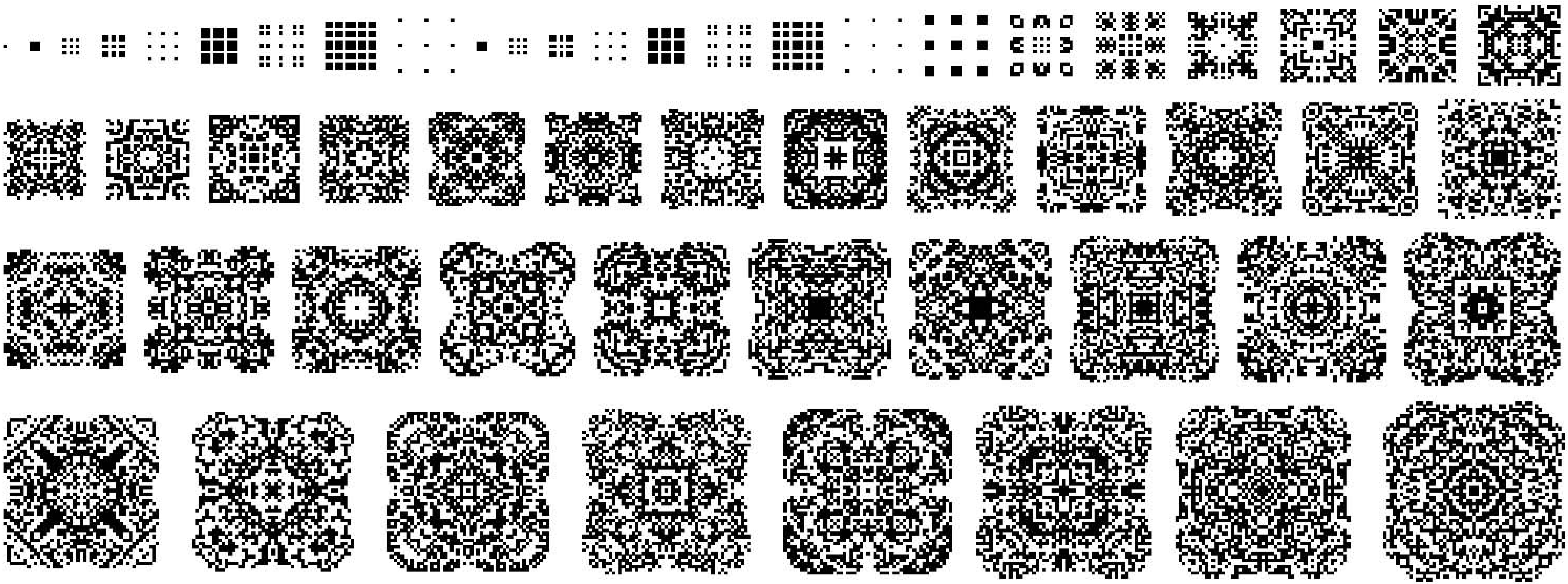
$$\omega_i^{(T)} = \alpha\omega_i^{(T-1)} + \sigma_i^{(T)} \quad \rightarrow \quad \{\sigma_i^{(t)}\} \textbf{ NO NEEDED}$$

$$k = 2 : \quad \alpha\textbf{-MEMORY EFFECTIVE if } \alpha > 0.5$$

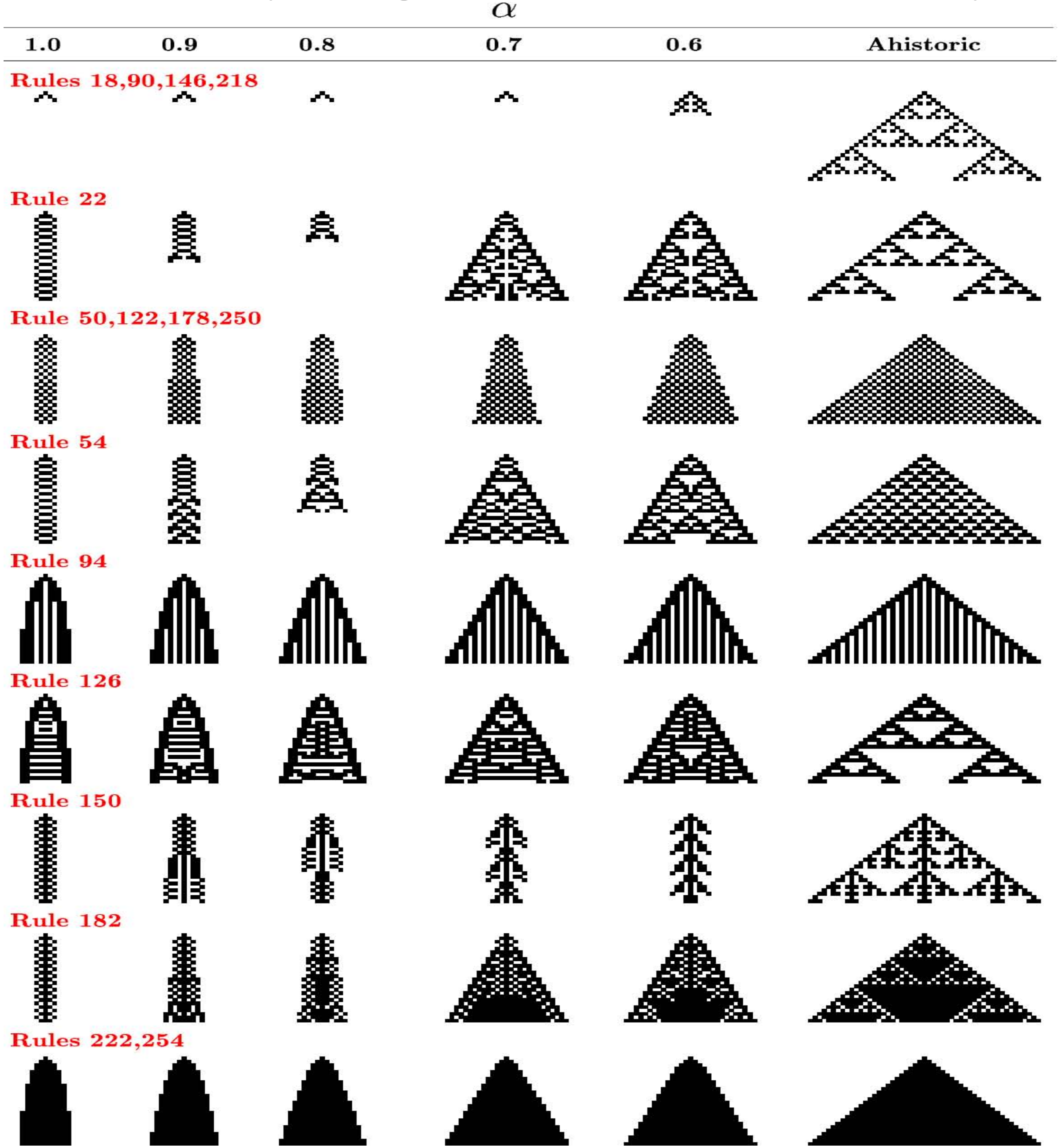
The 2D PARITY rule with Memory. Moore N. [19]



The 2D PARITY rule with minimal memory: $\alpha = 0.501$



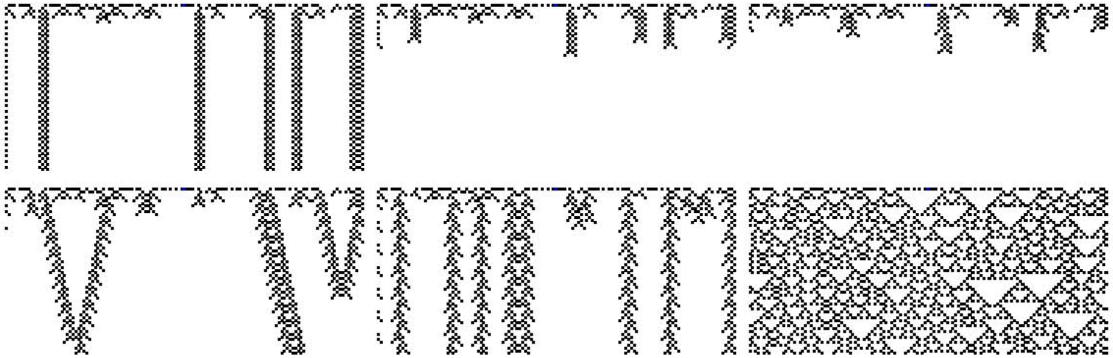
Elementary, Legal Rules with Memory [16]



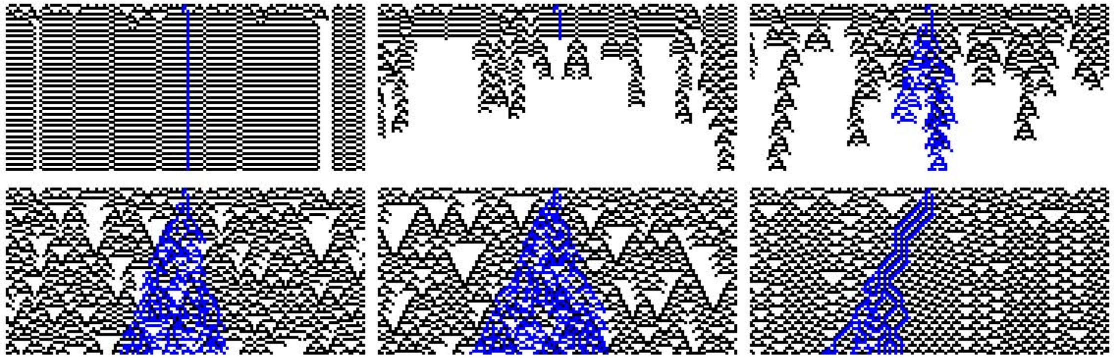
Elementary, Legal Rules with Memory [16]

α			α		
1.0	**0.9**	**0.8**	**1.0**	**0.9**	**0.8**
0.7	**0.6**	**Ahistoric**	**0.7**	**0.6**	**Ahistoric**

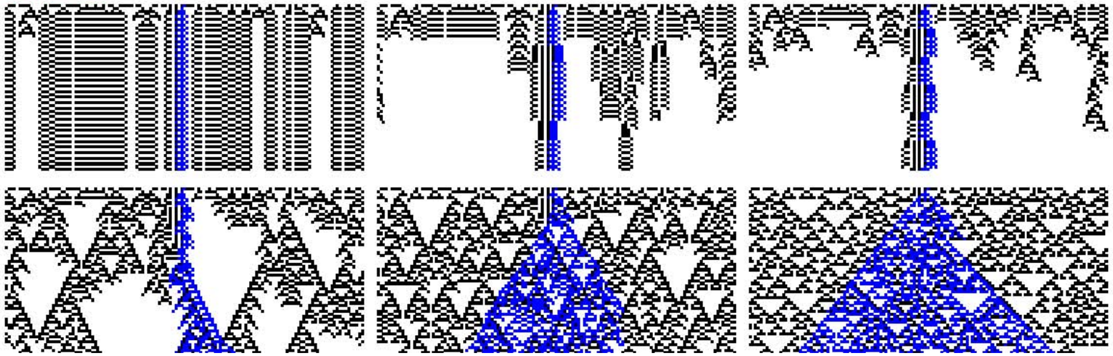
Rule 18



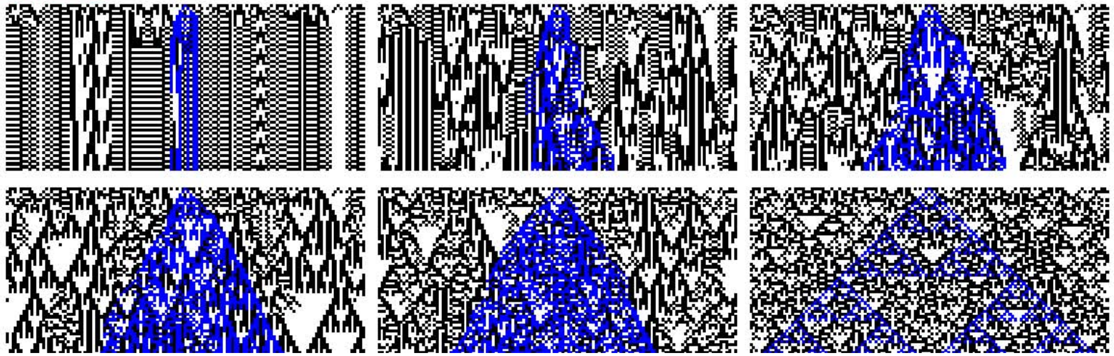
Rule 54



Rule 22

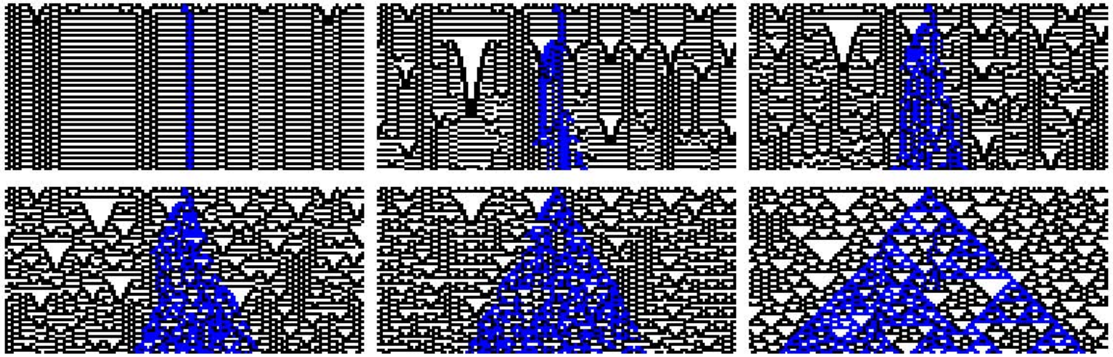


Rule 90

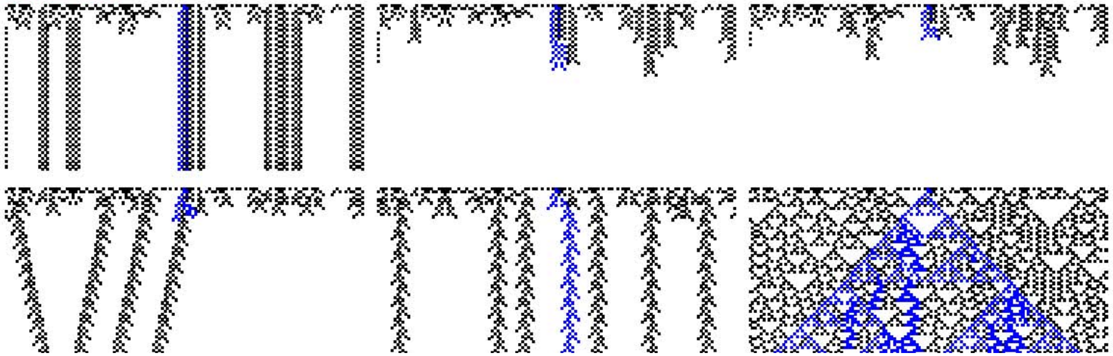


	α	
1.0	**0.9**	**0.8**
0.7	**0.6**	**Ahistoric**

Rule 126

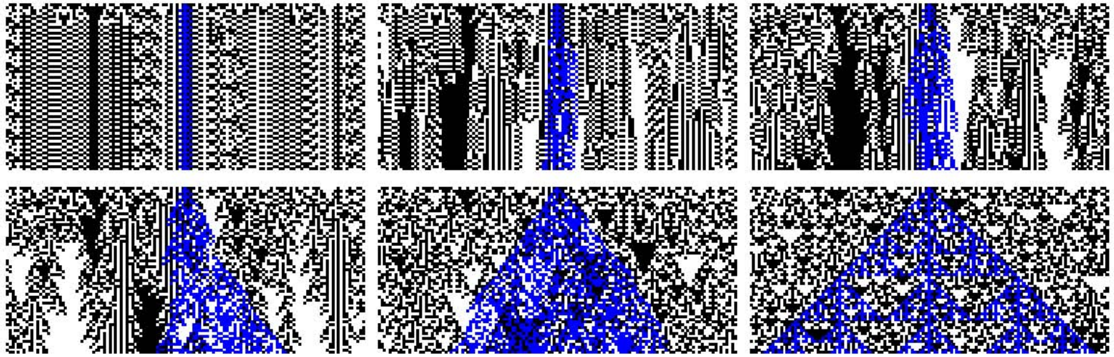


Rule 146

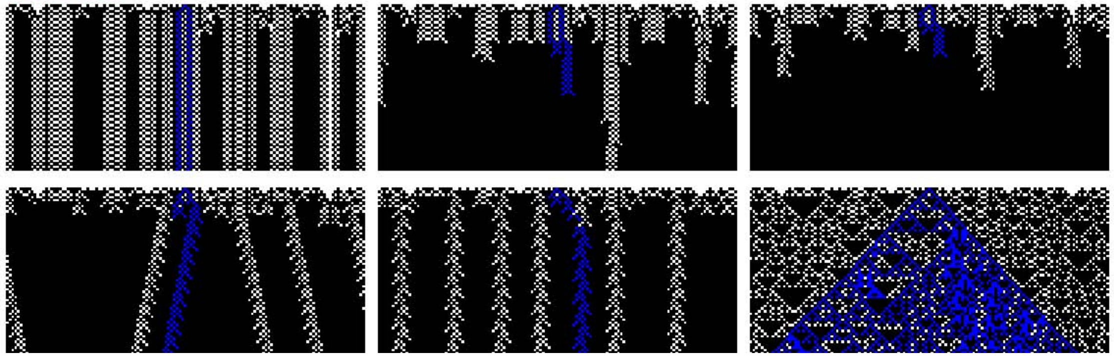


	α	
1.0	**0.9**	**0.8**
0.7	**0.6**	**Ahistoric**

Rule 150

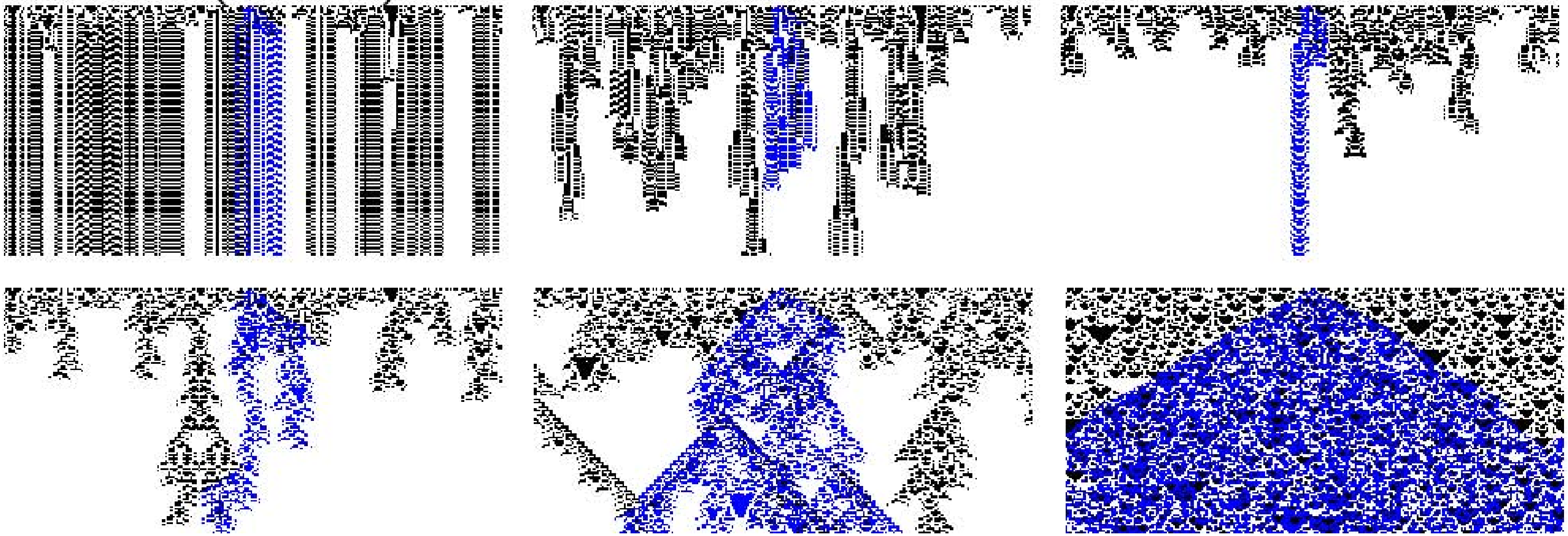


Rule 182

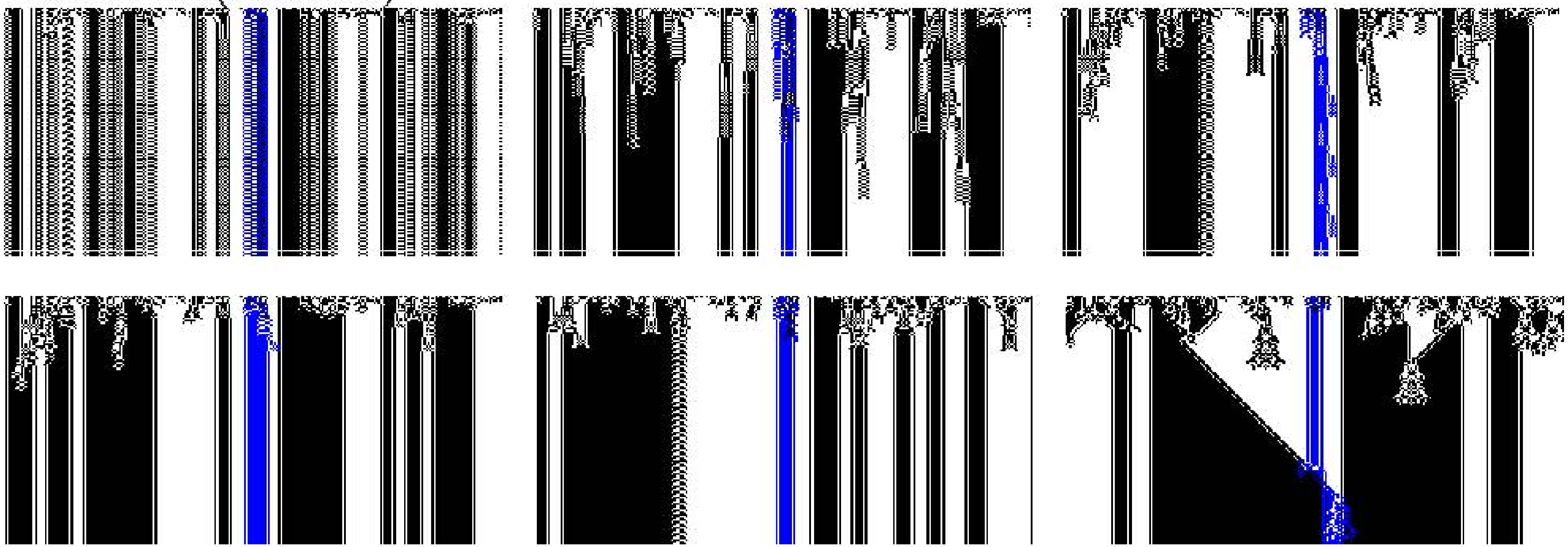


r=2

RULE 50(110010)



RULE 52(110100)



Average-like memory mechanisms

$$m_i^{(T)} = \frac{\sum_{t=1}^{T} \delta(t)\sigma_i^{(t)}}{\sum_{t=1}^{T} \delta(t)} \equiv \frac{\omega_i^{(T)}}{\Omega(T)}$$

- **exponential :** $\delta(t) = e^{-\beta(T-t)}$ $\quad \beta \in \mathbb{R}^+$

$$\alpha = e^{-\beta}$$

- **inverse:** $\delta(t) = \alpha^{t-1}$ $\quad \delta(t) > \delta(t+1)$
- **integer-based (*à la* CA):** $c \in \mathbb{N}$ [**17**]

$$\delta(t) = t^c \;\rightarrow\; \omega_i^{(T)} = \omega_i^{(T-1)} + T^c \sigma_i^{(T)}$$

$$\delta(t) = c^t \;\rightarrow\; \omega_i^{(T)} = \omega_i^{(T-1)} + c^T \sigma_i^{(T)}$$

<u>Limited trailing</u> (τ states)

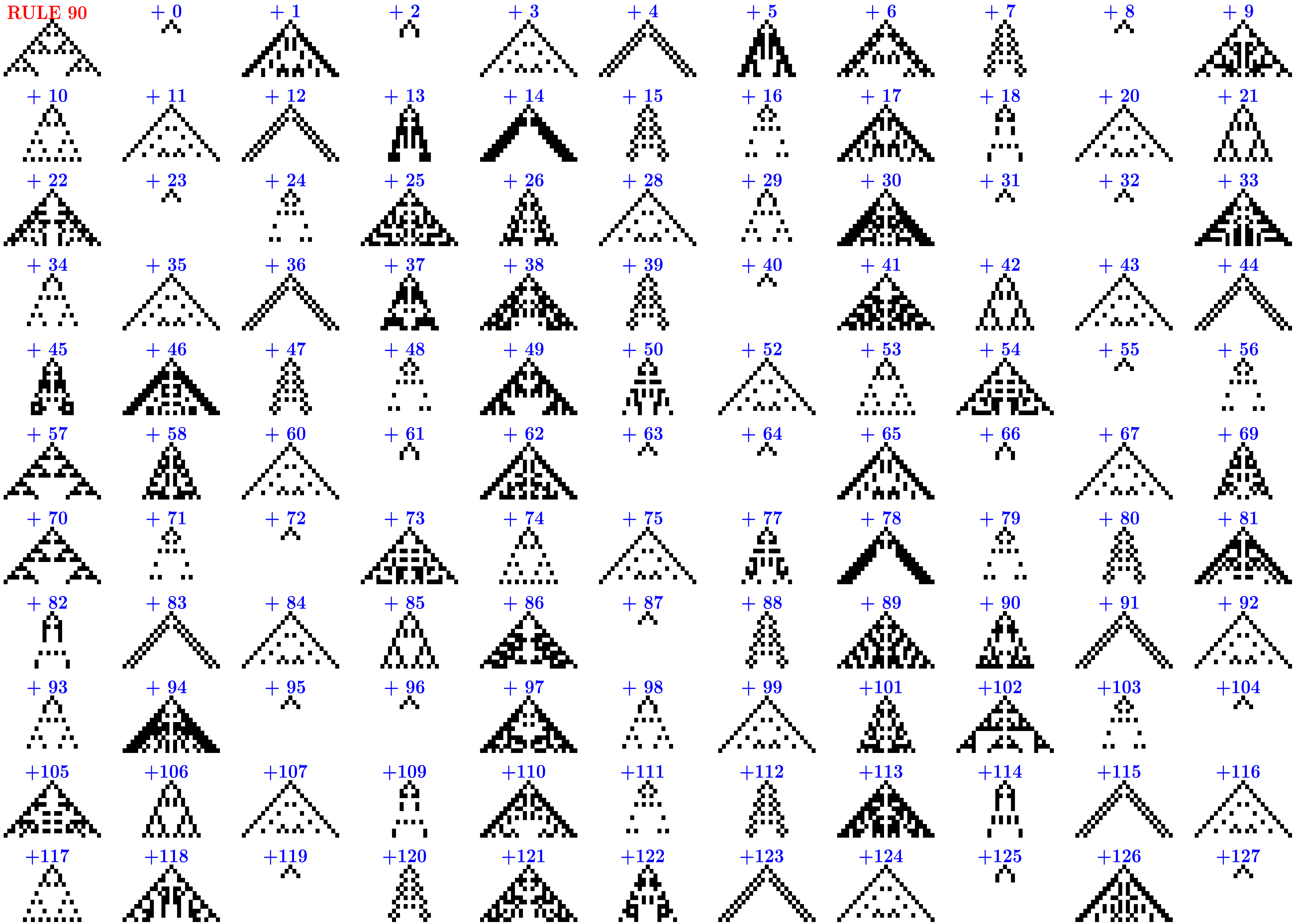
- **α-memory**

$$m_i^{(T)}(\sigma_i^{(T-\tau+1)},\ldots,\sigma_i^{(T)}) = \frac{\sum\limits_{t=\top}^{T}\delta(t)\sigma_i^{(t)}}{\sum\limits_{t=\top}^{T}\delta(t)} \qquad \top = \max(1, T-\tau+1)$$

- **Elementary Rules as Memory ($\tau = 3$):**

$$s_i^{(T)} = \phi\big(\sigma_i^{(T-2)}, \sigma_i^{(T)}, \sigma_i^{(T-1)}\big) \qquad [\mathbf{9}]$$

$$\textbf{e.g.:} \quad s_i^{(T)} = mode\big(\sigma_i^{(T-2)}, \sigma_i^{(T)}, \sigma_i^{(T-1)}\big) \equiv \textbf{ER 232} \qquad [\mathbf{11}]$$


RULE 90
+ 0
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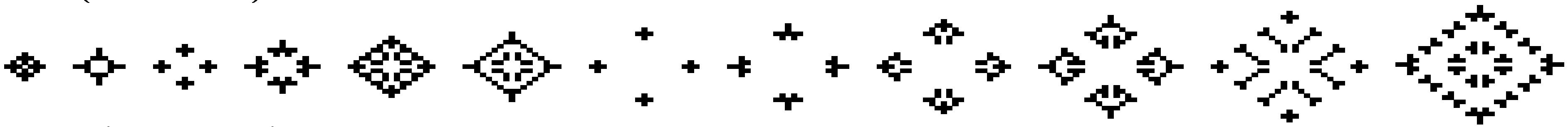
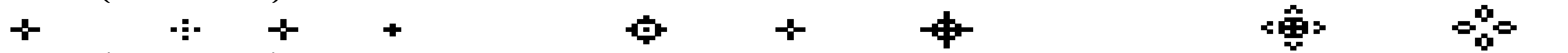
The Parity rule with Elementary Rules as Memory. T=4-15. vNN [4],[9]

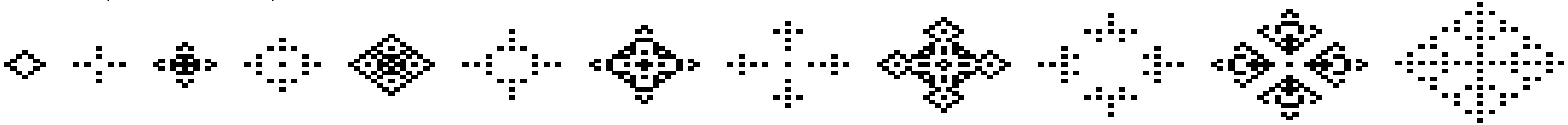
+ 4 (00000100)



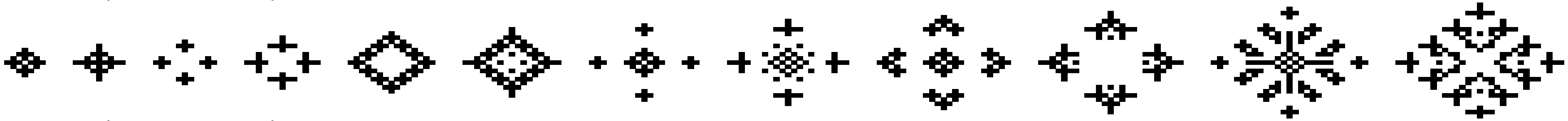
+ 18 (00010010)



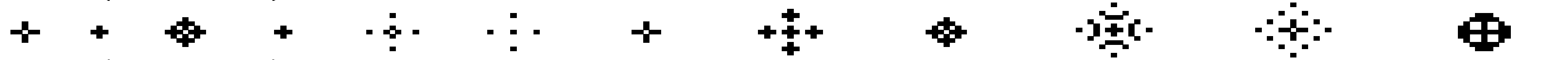
+ 22 (00010110)



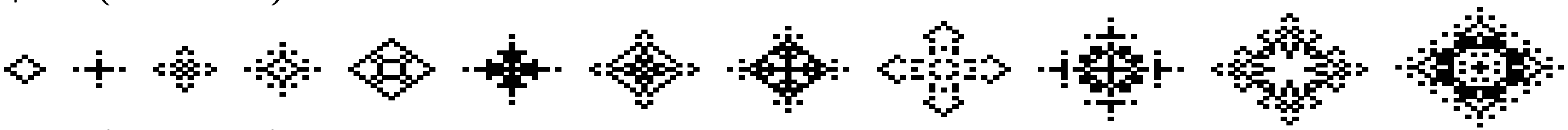
+ 36 (00100100)



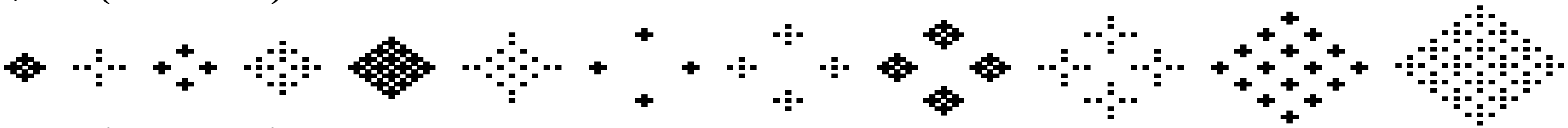
+ 50 (00110010)



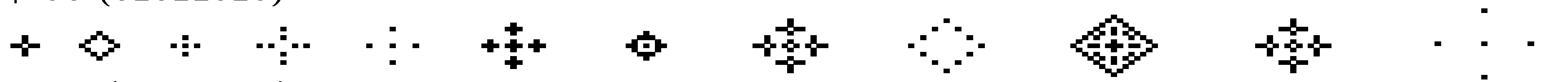
+ 54 (00110110)



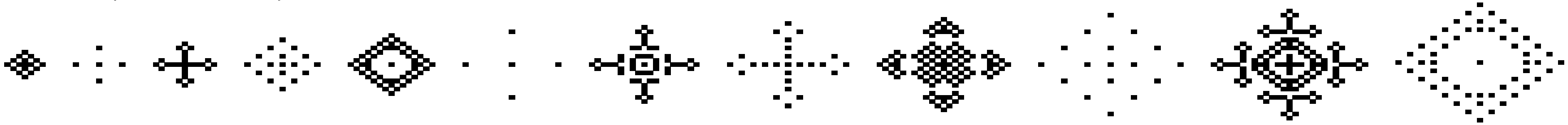
+ 76 (01001100)



+ 90 (01011010)



+150 (10010110) PARITY



THREE STATES $\{0, 1, 2\}$: [**13**]

$$s_i^{(T)} = \begin{cases} 0 & if \ m_i^{(T)} < 0.5 \\ \sigma_i^{(T)} & if \ m_i^{(T)} = 0.5 \\ 1 & if \ 0.5 < m_i^{(T)} < 1.5 \\ \sigma_i^{(T)} & if \ m_i^{(T)} = 1.5 \\ 2 & if \ m_i^{(T)} > 1.5 \end{cases}$$

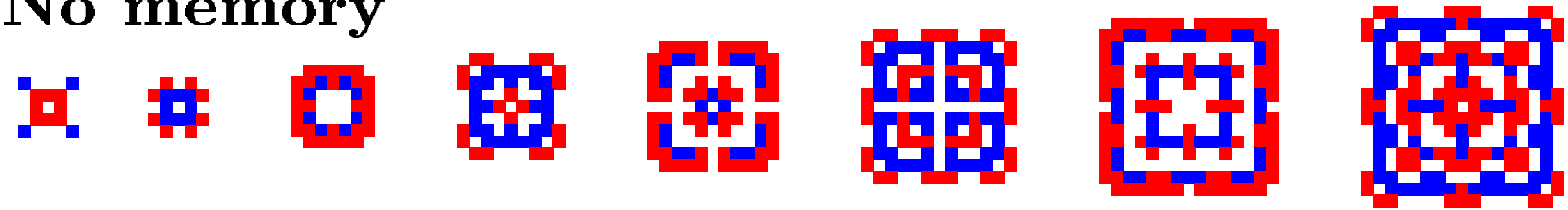
$k = 3:$ α**-MEMORY EFFECTIVE if** $\alpha > 0.25 = \frac{1}{2(k-1)}$

EXCITABLE CA [1]

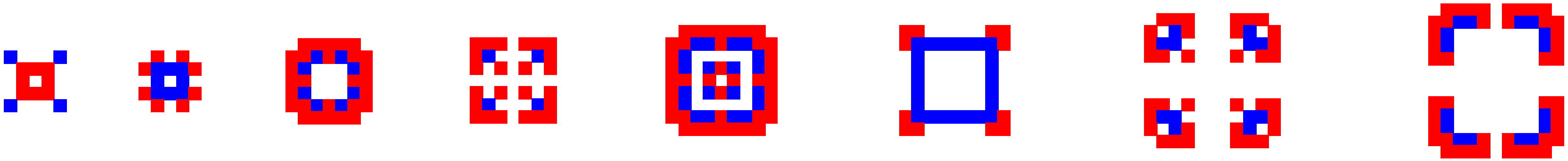
excited ■ $\rightarrow$ refractory ■ $\rightarrow$ resting *blank* $\rightarrow \phi \rightarrow$ excited

$\phi = \#$ of excited cells in neigh. 1 or 2

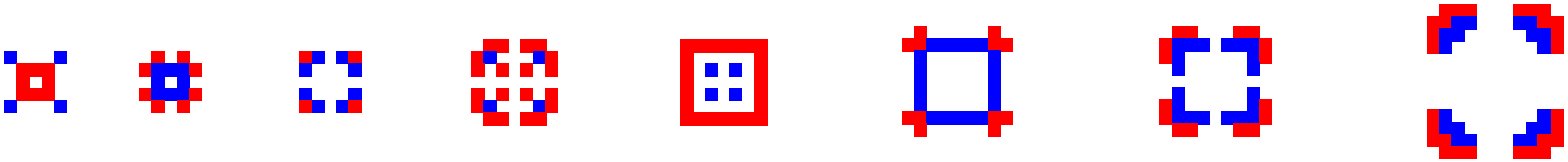
No memory



Mode memory



σ



s

$\tau = \mathbf{3}$

REVERSIBLE CA with MEMORY [15]:

$$\sigma_i^{(T+1)} = \phi\big(\sigma_j^{(T)} \in \mathcal{N}_i\big) \ominus \sigma_i^{(T-1)}$$

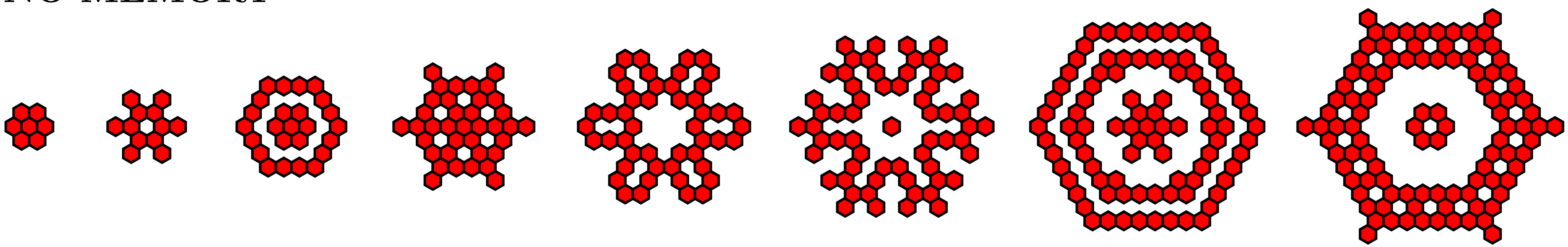
$$s_i^{(T-1)} = round\Big(\frac{\omega_i^{(T-1)}=(\omega_i^{(T)}-\sigma_i^{(T)})/\alpha}{\Omega(T-1)}\Big)$$

$$\sigma_i^{(T-1)} = \phi\big(s_j^{(T)} \in \mathcal{N}_i\big) \ominus \sigma_i^{(T+1)}$$

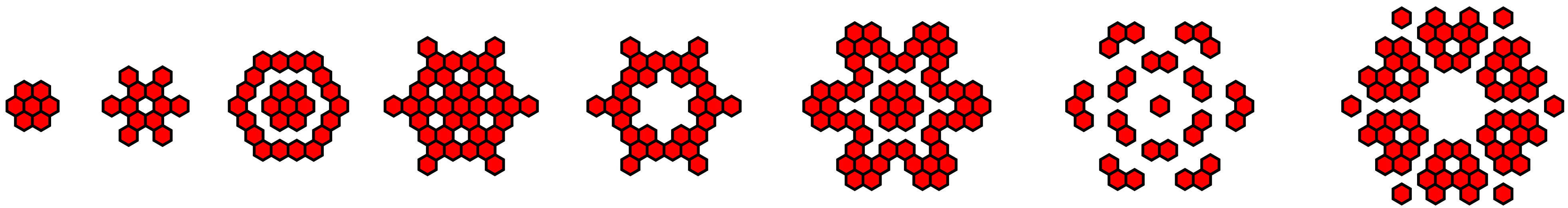
Reversion: $\sigma_i^{(T-1)} = \phi\big(\qquad\qquad\big) \ominus \sigma_i^{(T+1)}$

Reversible Parity Rule $(\{\sigma^{(0)}\} = \{\sigma^{(1)}\})$

NO MEMORY

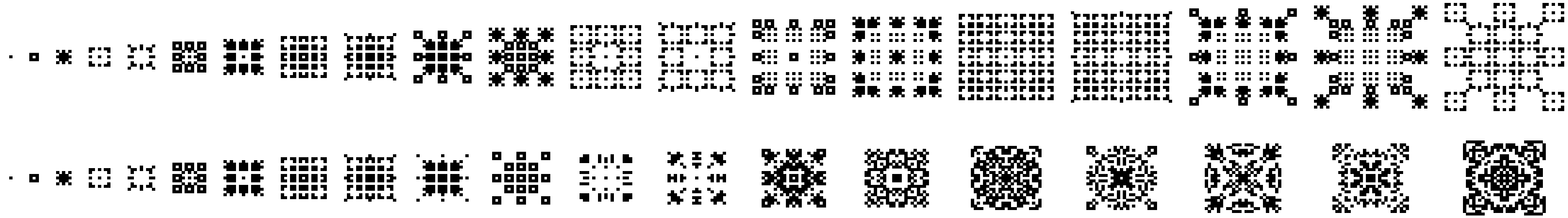


α**=0.6**



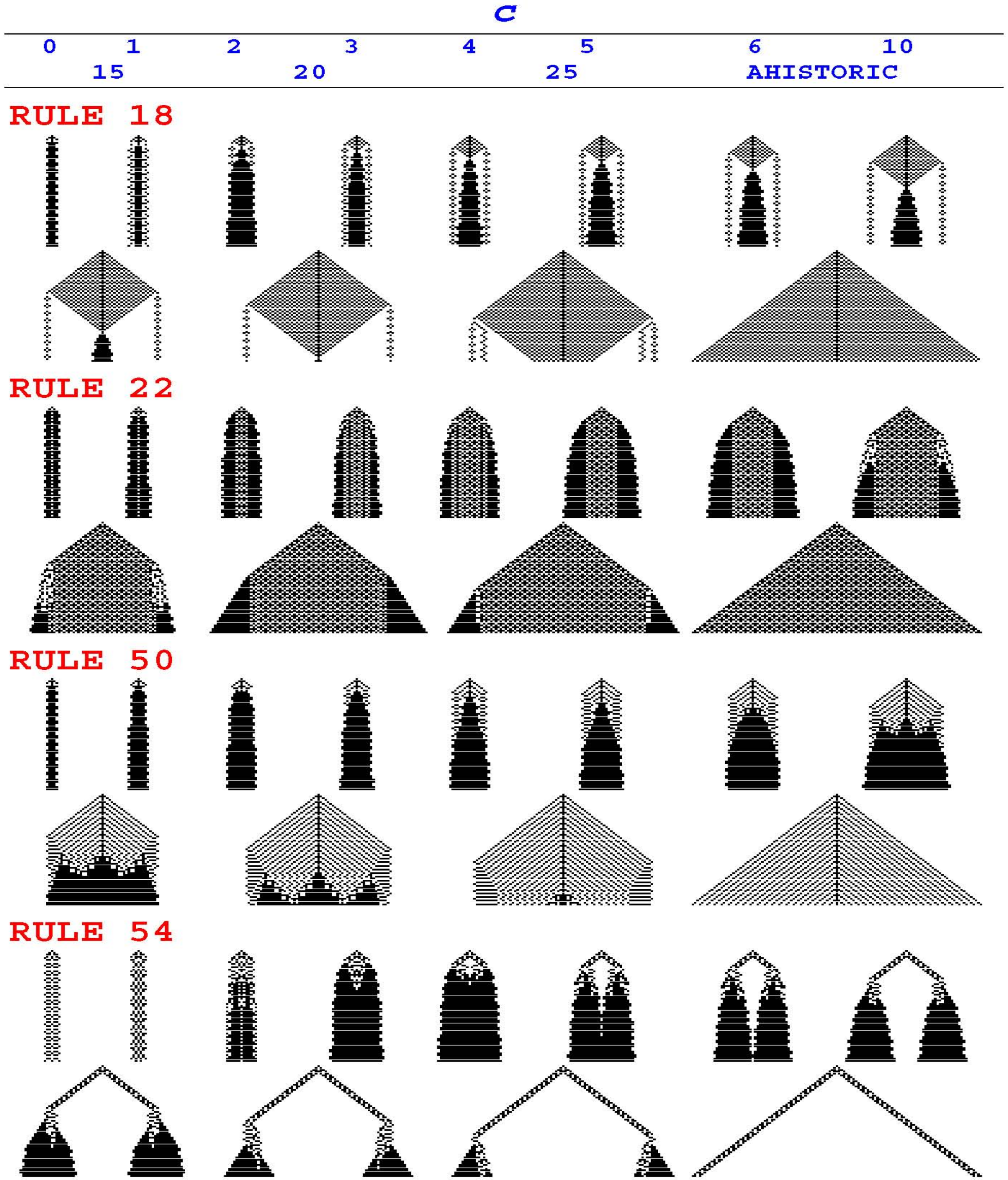
Reversible Parity Rule ($\{\sigma^{(0)}\} = \{\sigma^{(1)}\}$)

NO MEMORY



α=0.501

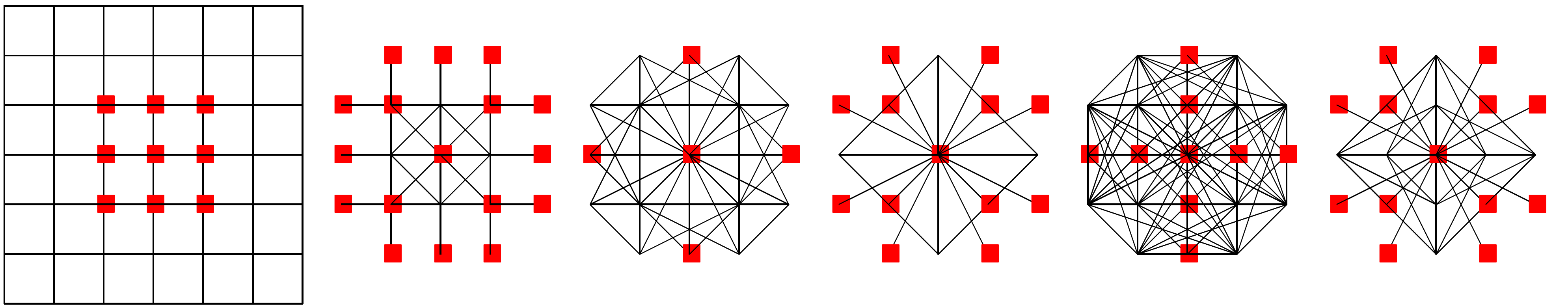
Reversible ER with $\delta = t^c$ memory [17]



STRUCTURALLY DYNAMIC CA (SDCA)

State and link config. are both dynamic, altering each other

Example



Mass Parity rule (mod **2**)

Links

Coupling

Add links between next-NN sites in which both values are 1

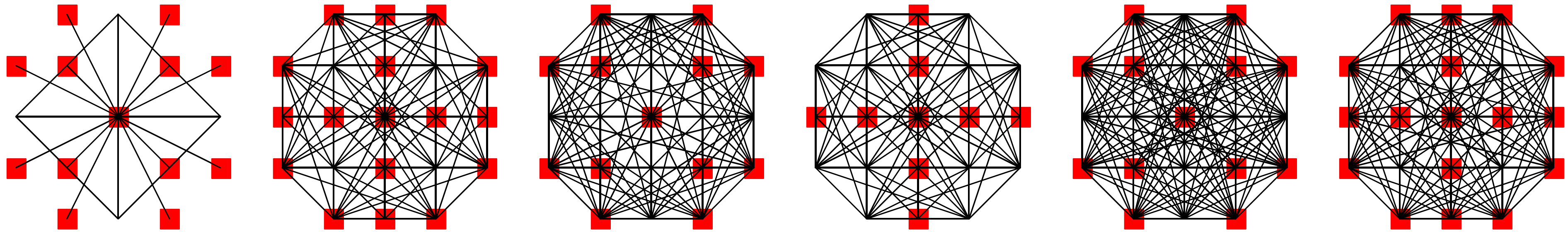
Decoupling

Remove links connected to sites in which both values are 0

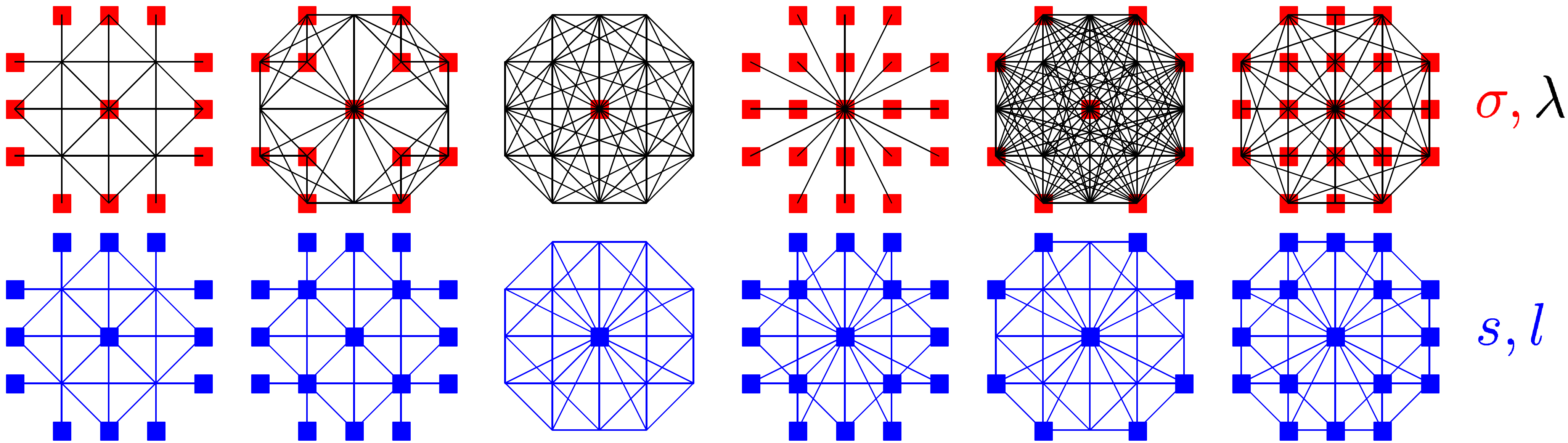
SDCA with MEMORY [4]

$$\sigma_i^{(T+1)} = \phi\big(s_j^{(T)} \in N_i^{(T)}\big) \qquad \lambda_{i,j}^{(T+1)} = \psi\big(s_i^{(T)}, s_j^{(T)}, \{l^{(T)}\}\big)$$

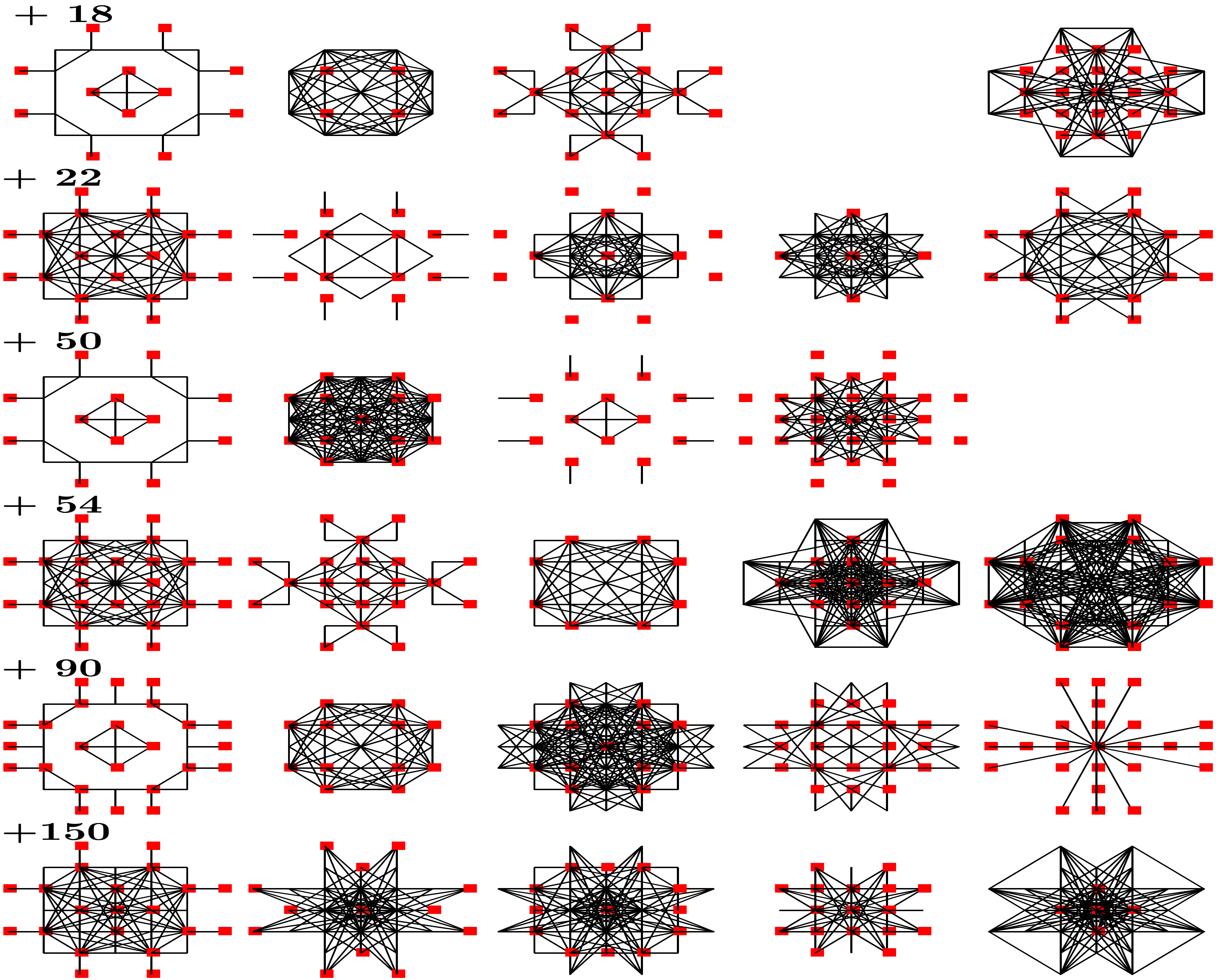
$\alpha = 0.6 \qquad T = 4 - 9$



$\alpha = 1.0 \qquad T = 4 - 9$



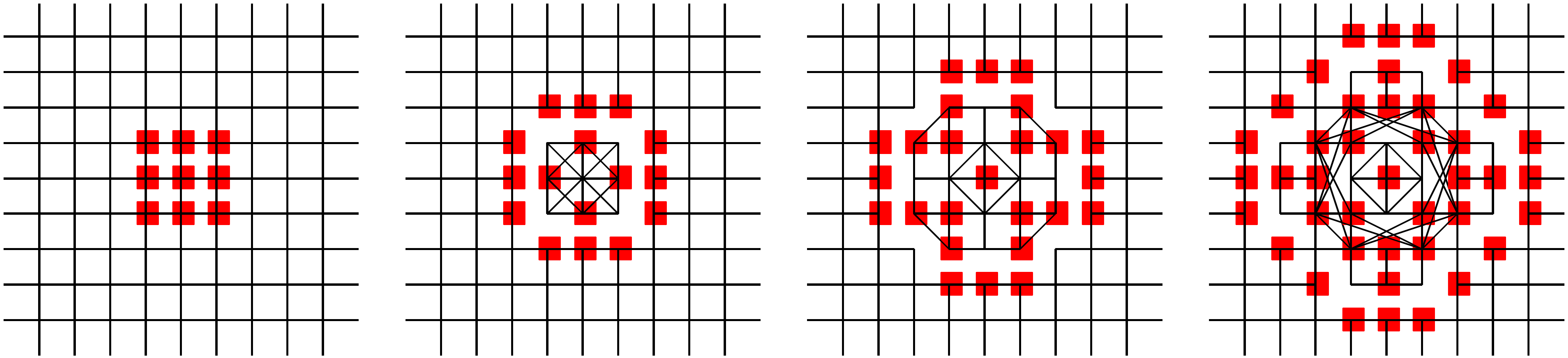
The SDCA Parity rule with Elementary Rules as Memory



REVERSIBLE SDCA [6]

$$\sigma_i^{(T+1)} = \phi\big(\sigma_j^{(T)} \in \mathcal{N}_i^{(T)}\big) \ominus \sigma_i^{(T-1)}$$

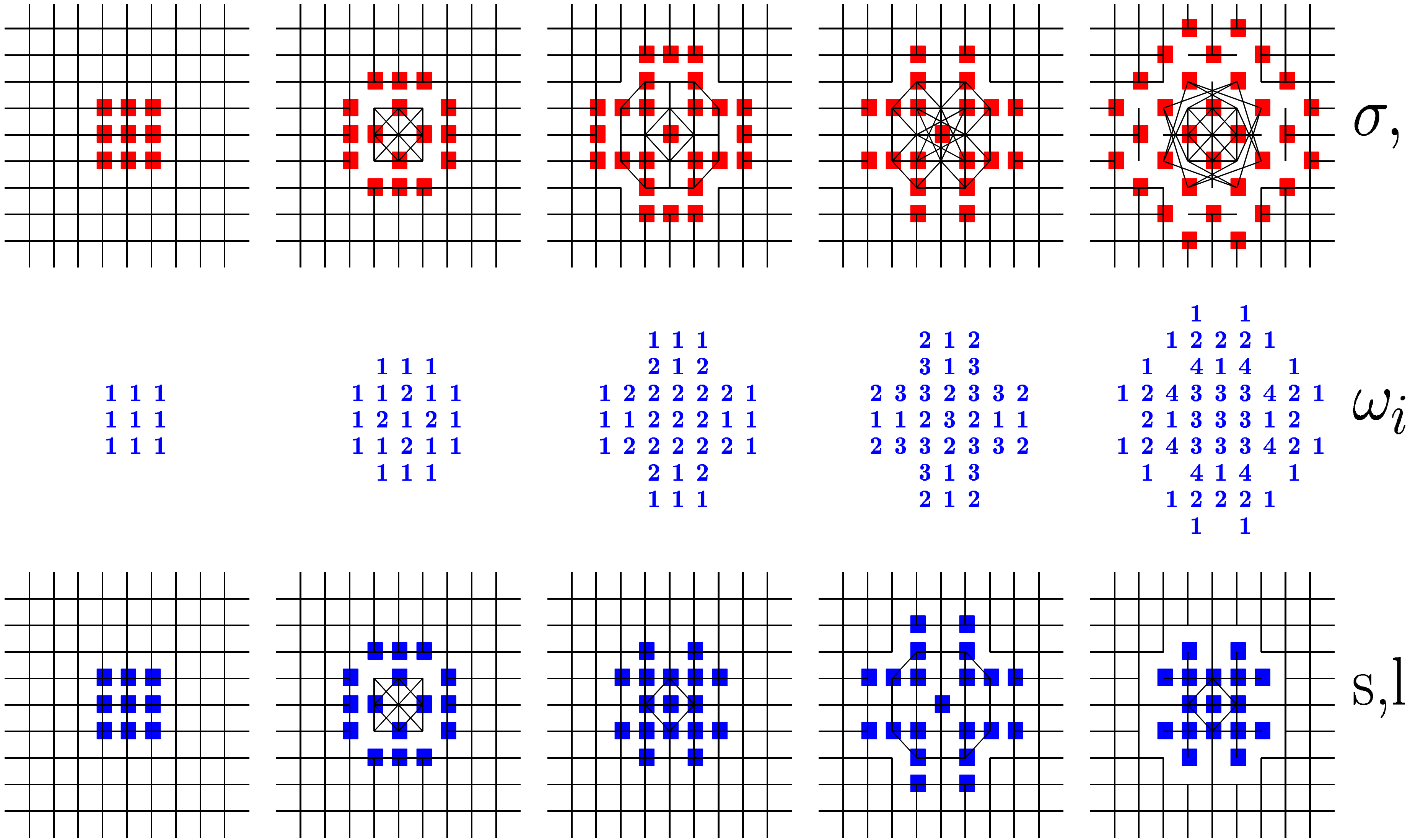
$$\lambda_{i,j}^{(T+1)} = \psi\big(\sigma_i^{(T)}, \sigma_j^{(T)}, \{\lambda^{(T)}\}\big) \ominus \lambda_{i,j}^{(T-1)}$$



REVERSIBLE SDCA with MEMORY

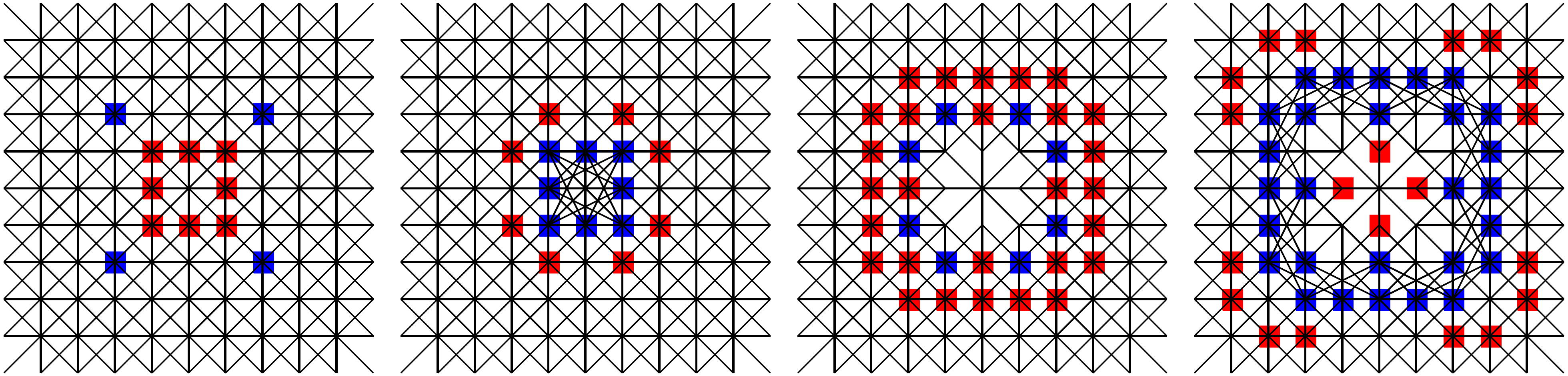
$$\sigma_i^{(T+1)} = \phi\left(s_j^{(T)} \in N_i^{(T)}\right) \ominus \sigma_i^{(T-1)} \quad , \quad \lambda_{i,j}^{(T+1)} = \psi\left(s_i^{(T)}, s_j^{(T)}, \{l^{(T)}\}\right) \ominus \lambda_{i,j}^{(T-1)}$$

FULL MEMORY ($\alpha = 1.0$)



EXCITABLE SDCA

Example



Mass

Resting cell excited if $\frac{\#\ \textbf{excited connected}}{\#\ \textbf{connected cells}} \in [\frac{1}{8}, \frac{2}{8}]$

Links

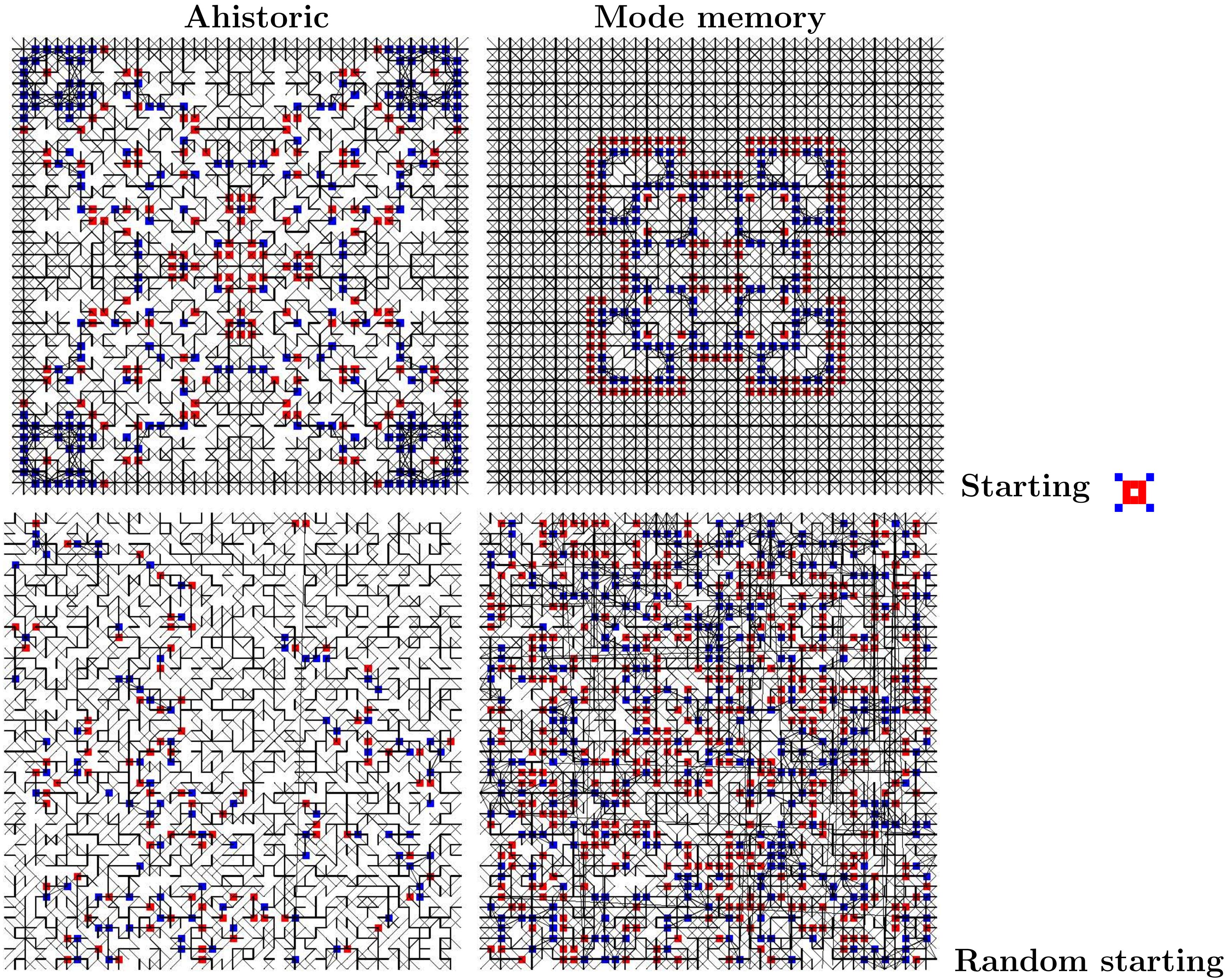
Coupling

Add links between next-NN sites in which both values are excited

Decoupling

Remove links connected to sites in which both values are refractory

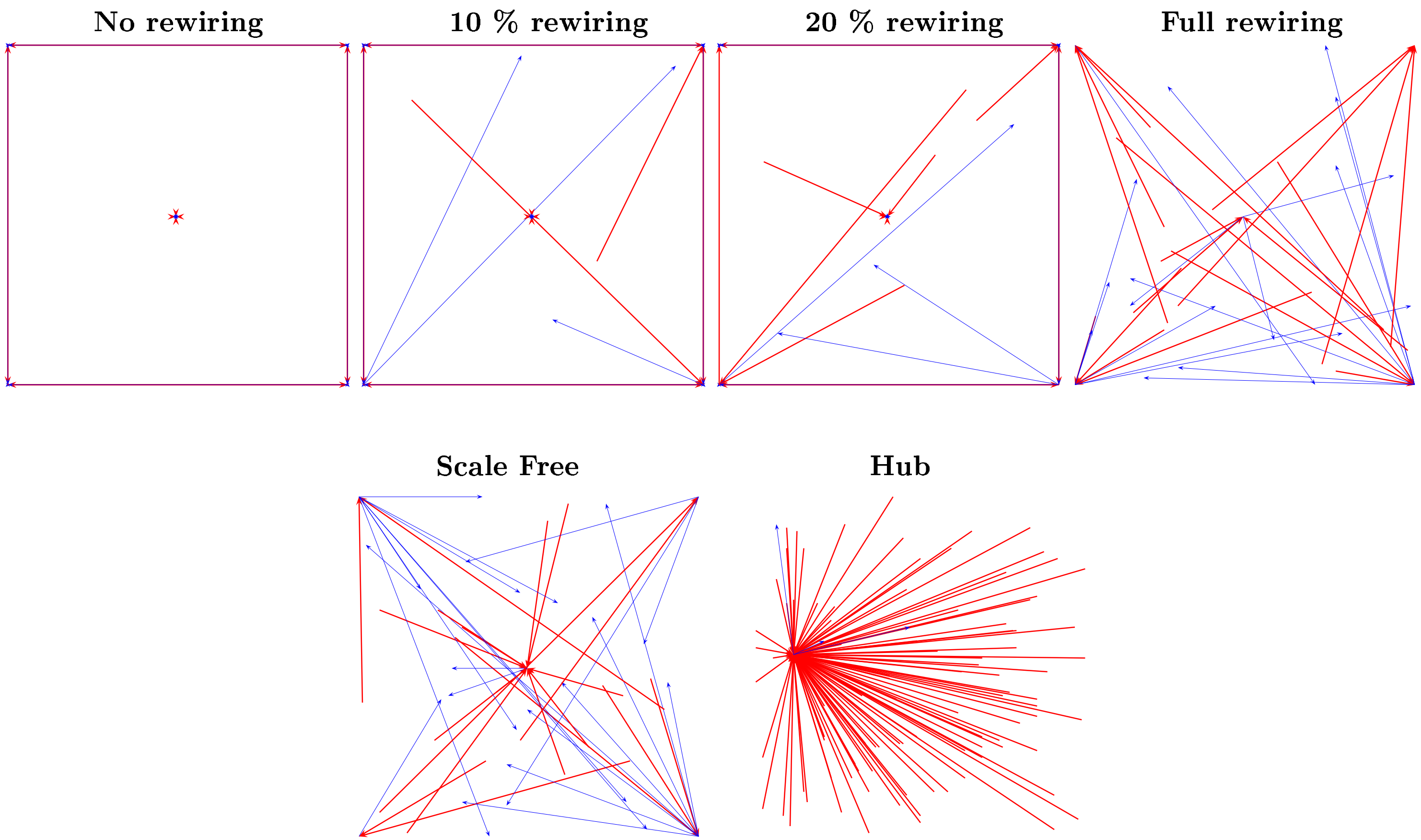
EXCITABLE SDCA at T=50

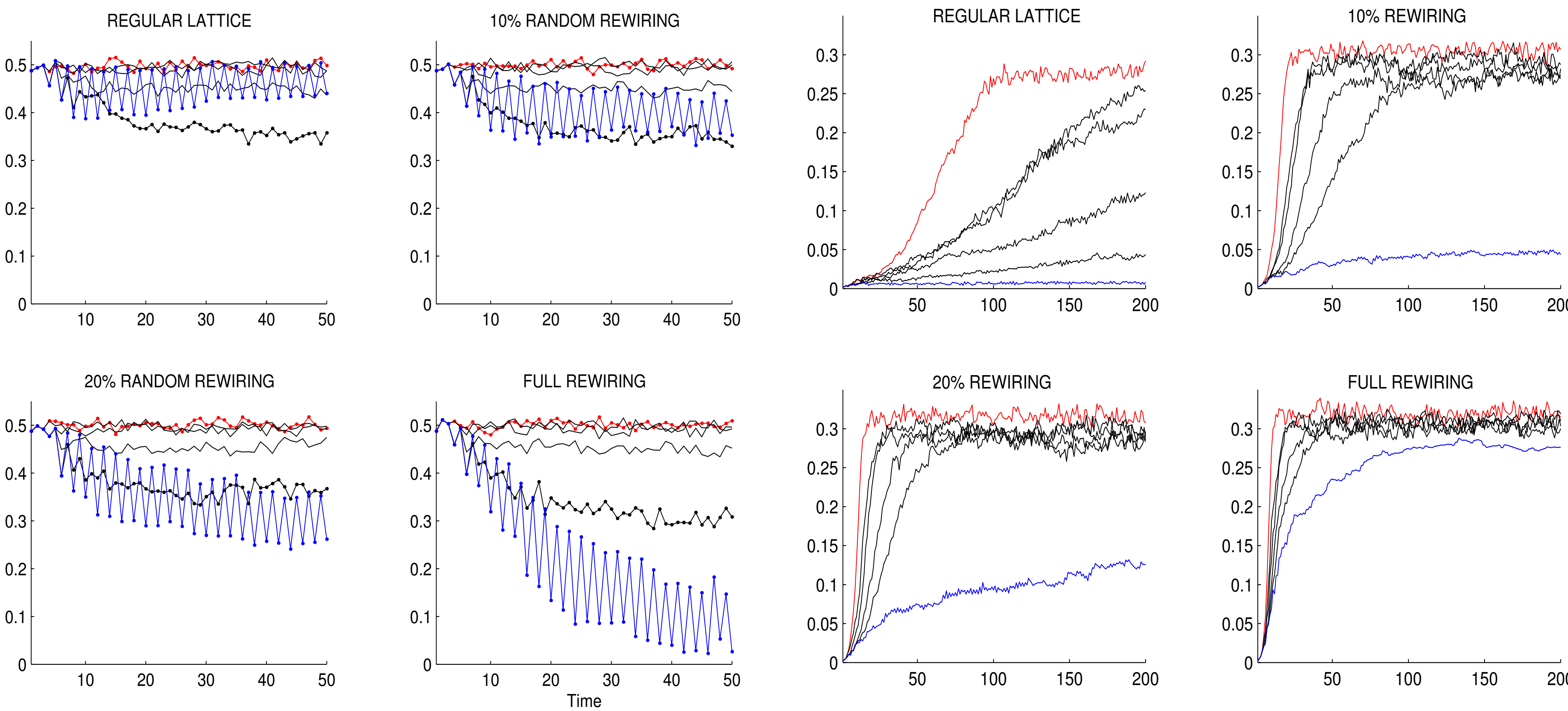


There is plenty of room *with simple memory* :
Unaltered transition rule (function of previous states)

- **Probabilistic CA** [10]: $p = P\left(\sigma_i^{(T+1)} = 1 \big/ s_{i-1}^{(T)}, s_i^{(T)}, s_{i+1}^{(T)}\right)$

- **Boolean networks**[2, 3]: $\sigma_i^{(T+1)} = \phi_i\left(s_j^{(T)} \in \mathcal{N}_i\right)$





Changing rate **Damage sepreading**

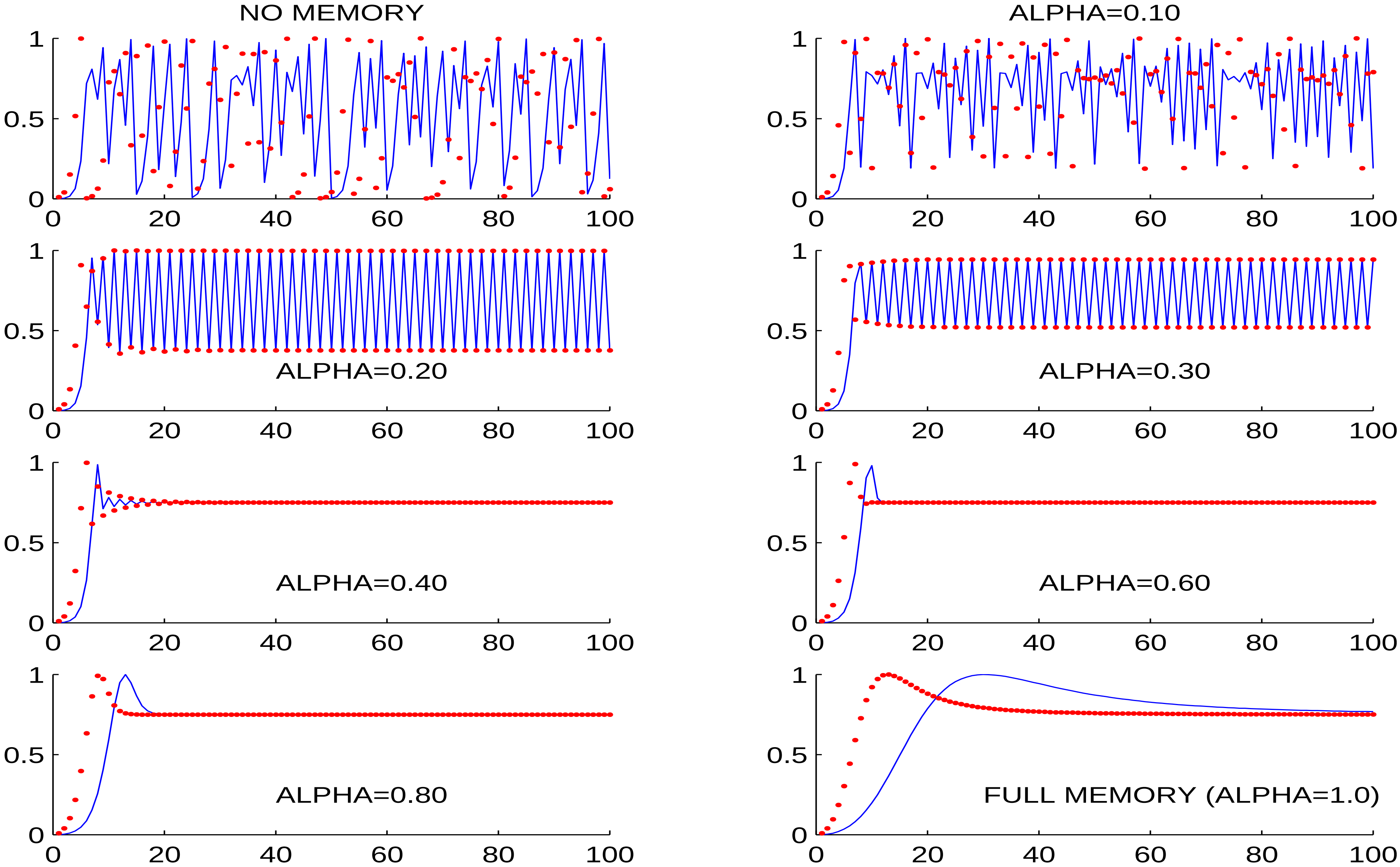
$K = 4$ **Boolean Network**

- **Continuous CA (CML) :** $\sigma_i^{(T+1)} = \varphi\big(m_j^{(T)} \in \mathcal{N}_i\big)$
- **Discrete Dynamical Systems :** $x_{T+1} = f(m_T)$

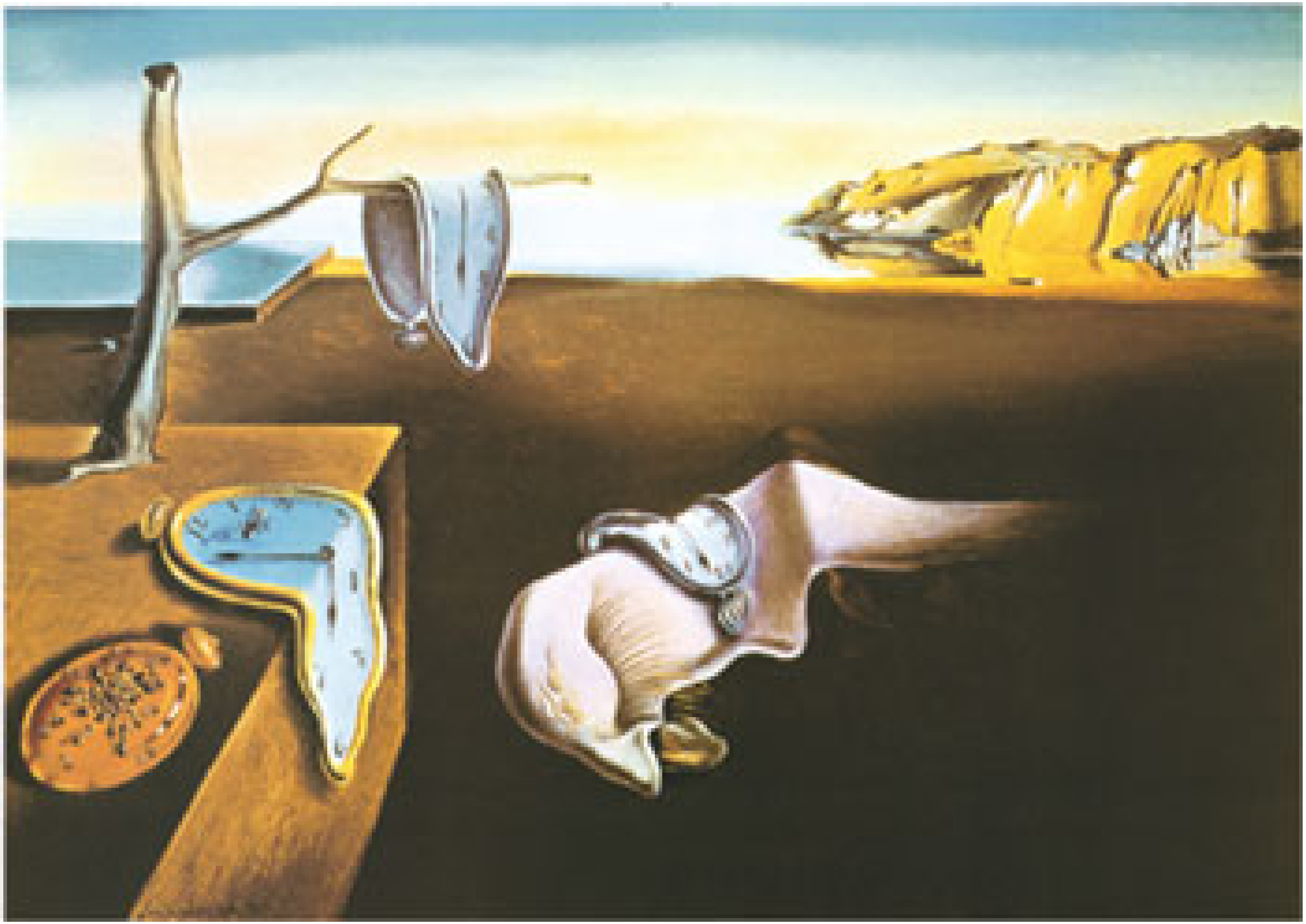
$$m_T = \frac{x_T + \displaystyle\sum_{t=1}^{T-1} \alpha^{T-t} x_t}{1 + \displaystyle\sum_{t=1}^{T-1} \alpha^{T-t}} \equiv \frac{\omega_T}{\Omega(T)}$$

The LOGISTIC map with memory [13]

$$x_{T+1} = 4m_T(1 - m_T) \quad Fixed\ point : x = 0.75$$



SALVADOR DALI



The Persistence of Memory

Disintegration of the Persistence of Memory

http://uncomp.uwe.ac.uk/alonso-sanz

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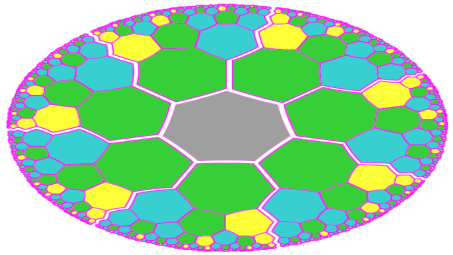

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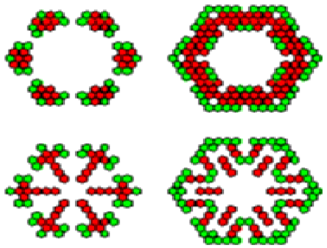

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Prisoner's Dilemma: Cooperate or Defect ?

	C	D
C	R, R	S, $\mathfrak{T}$
D	$\mathfrak{T}$, S	P, P

$\mathfrak{T}$emptation > Reward > Penalization > Sucker payoff

Spatial Prisoner's Dilemma [21]–[28]

$d^{(1)} = \delta^{(1)}$					$p^{(1)} = \pi^{(1)}$				
C	C	C	C	C	9	9	9	9	9
C	C	C	C	C	9	8	8	8	9
C	C	D	C	C	9	8	8b	8	9
C	C	C	C	C	9	8	8	8	9
C	C	C	C	C	9	9	9	9	9

$d^{(2)} = \delta^{(2)}$							$\pi^{(2)} = \alpha p^{(1)} + p^{(2)}$						
C	C	C	C	C	C	C	$9\alpha+9$	$9\alpha+9$	$9\alpha+9$	$9\alpha+9$	$9\alpha+9$	$9\alpha+9$	$9\alpha+9$
C	C	C	C	C	C	C	$9\alpha+9$	$9\alpha+8$	$9\alpha+7$	$9\alpha+6$	$9\alpha+7$	$9\alpha+8$	$9\alpha+9$
C	C	D	D	D	C	C	$9\alpha+9$	$9\alpha+9$	$8\alpha+5b$	$8\alpha+3b$	$8\alpha+5b$	$9\alpha+9$	$9\alpha+9$
C	C	D	D	D	C	C	$9\alpha+9$	$9\alpha+9$	$8\alpha+3b$	$8b\alpha$	$8\alpha+3b$	$9\alpha+6$	$9\alpha+9$
C	C	D	D	D	C	C	$9\alpha+9$	$9\alpha+9$	$8\alpha+5b$	$8\alpha+3b$	$8\alpha+5b$	$9\alpha+9$	$9\alpha+9$
C	C	C	C	C	C	C	$9\alpha+9$	$9\alpha+8$	$9\alpha+7$	$9\alpha+6$	$9\alpha+7$	$9\alpha+8$	$9\alpha+9$
C	C	C	C	C	C	C	$9\alpha+9$	$9\alpha+9$	$9\alpha+9$	$9\alpha+9$	$9\alpha+9$	$9\alpha+9$	$9\alpha+9$

	C	D
C	1, 1	0, b
D	b, 0	0, 0

$\pi^{(2)} = \mathbf{p}^{(2)}$

9	9	9	9	9	9	9
9	8	7	6	7	8	9
9	9	5b	3b	5b	9	9
9	9	3b	0	3b	6	9
9	9	5b	3b	5b	9	9
9	8	7	6	7	8	9
9	9	9	9	9	9	9

α=0

$\pi^{(2)} = \mathbf{p}^{(1)} + \mathbf{p}^{(2)}$

18	18	18	18	18	18	18
18	17	16	15	16	17	18
18	18	8+5b	8+3b	8+5b	18	18
18	18	8+3b	8b	8+3b	15	18
18	18	8+5b	8+3b	8+5b	18	18
18	17	16	15	16	17	18
18	18	18	18	18	18	18

α=1

b=1.85

5b=9.25 $>$ 9

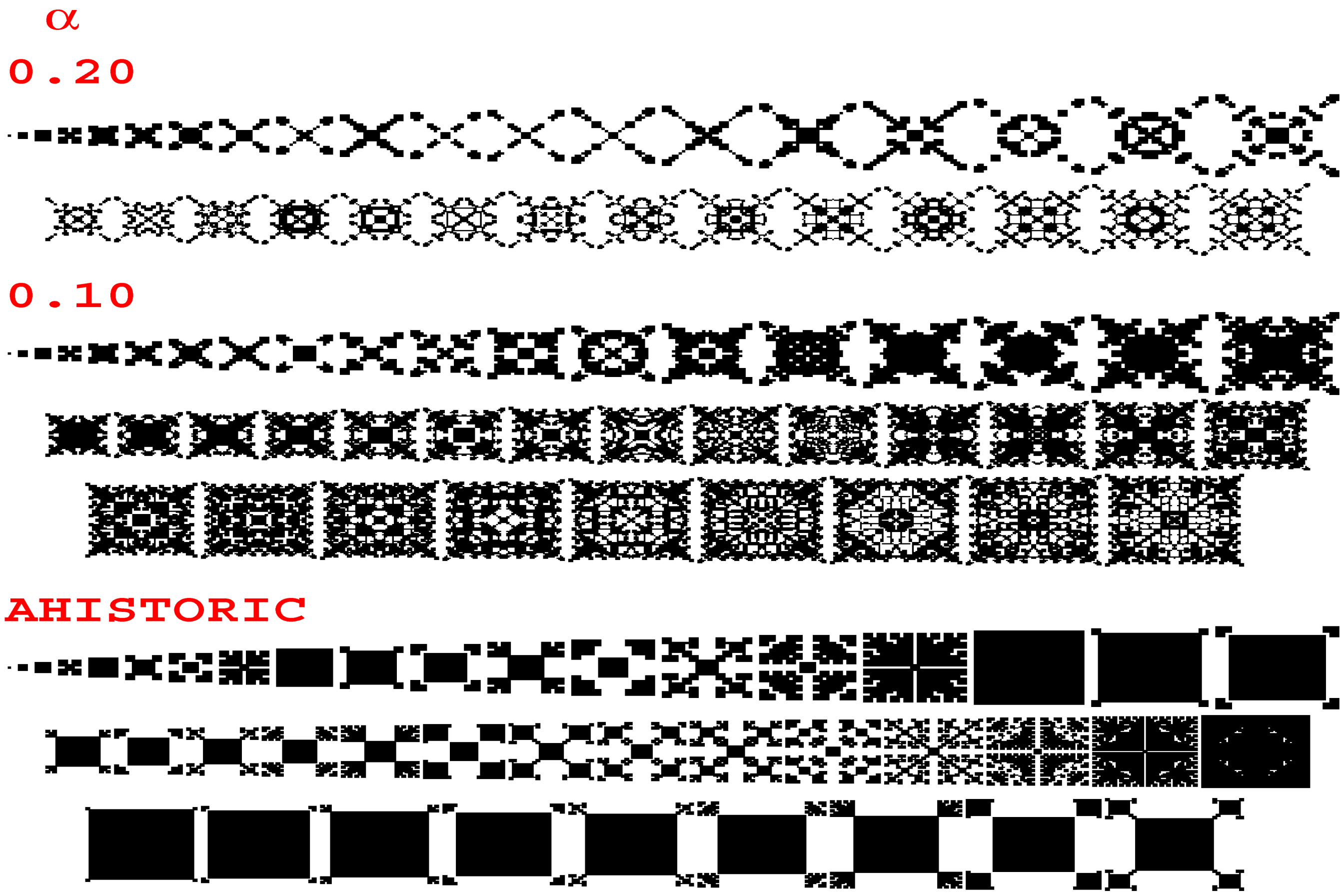
$\mathbf{d}^{(3)}$

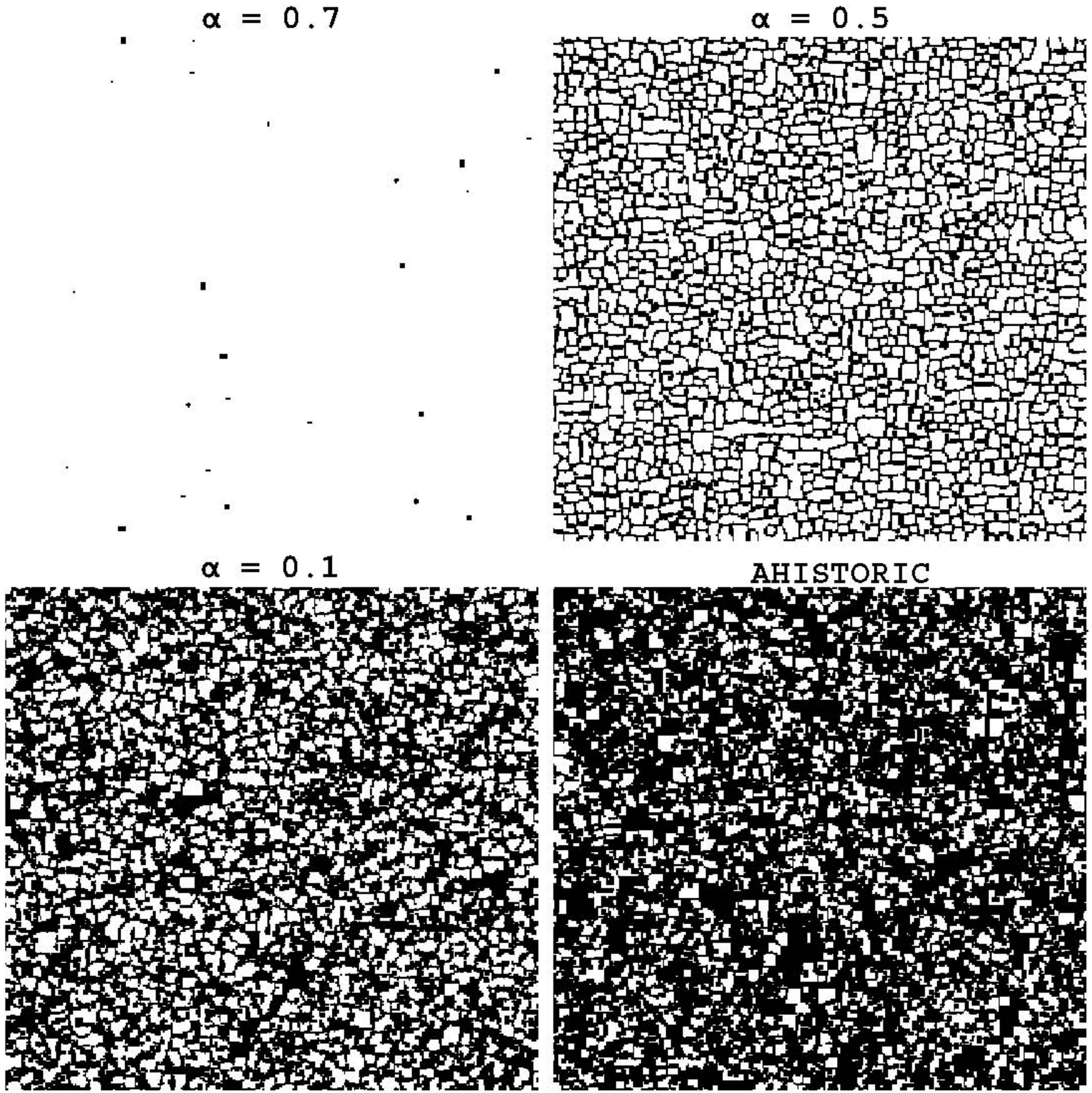
C	C	C	C	C	C	C
C	D	D	D	D	D	C
C	D	D	D	D	D	C
C	D	D	D	D	D	C
C	D	D	D	D	D	C
C	D	D	D	D	D	C
C	C	C	C	C	C	C

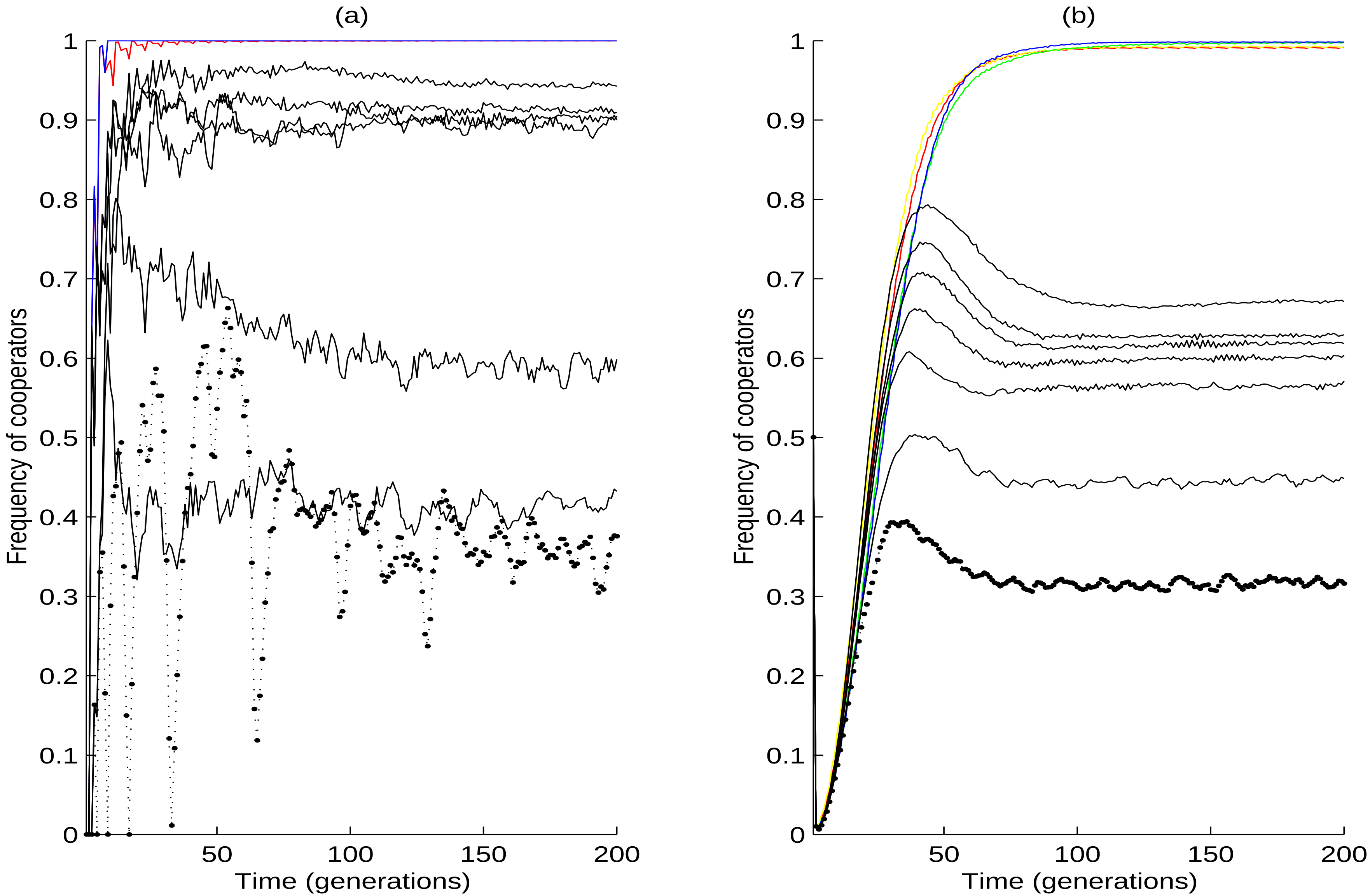
8+5b=17.25 $<$ 18

$\mathbf{d}^{(3)}$

C	C	C	C	C	C	C
C	C	C	C	C	C	C
C	C	D	D	D	C	C
C	C	D	D	D	C	C
C	C	D	D	D	C	C
C	C	C	C	C	C	C
C	C	C	C	C	C	C


α
0.20
0.10
AHISTORIC


α = 0.7
α = 0.5
α = 0.1
AHISTORIC



Freq. of C with memory of the last 3 iterations, starting (a) **from a single defector,** (b) **at random. Dotted** $\equiv$ **ahistoric, red, yellow, green, blue:** $\alpha = 1.0, 0.9, 0.8, 0.7$**. Remaining:** α **from 0.1 to 0.6 by 0.1. The higher the** α **the higher the** f **.**

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