Capacity, Carleson Measures, and Exceptional Sets

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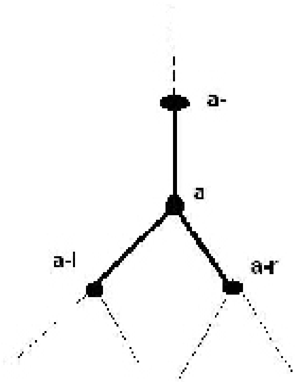
January 2008

The Dirichlet Space

The Dirichlet space, $\mathcal{D}$:

$$f \in \mathrm{Hol}(\mathbb{D}), \quad \|f\|_{\mathcal{D}}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'|^2 < \infty.$$

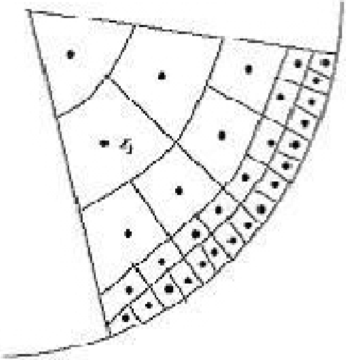
The tree, T, a rooted loopless graph, every point other than the root has three nearest neighbors.

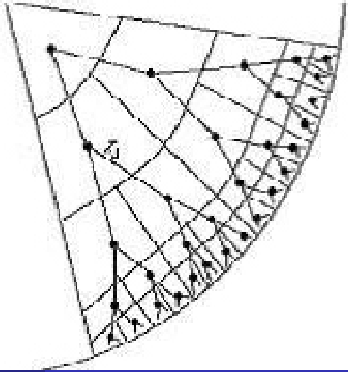


A point of ∂T is an equivalence class of geodesics infinite in one direction; a a path to the abstract boundary,

The Dyadic Dirichlet Space
The Tree in the Disk

Think of T as sitting inside $\mathbb{D}$ with the root o at 0.
Think of ∂T as being the same as $\mathbb{T}$.





The Dyadic Dirichlet Space

Operations on functions and measures on T

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- A measure μ on $\overline{\mathbb{D}}$ induces a measure μ_T on $\overline{T} = T \cup \partial T$ by......

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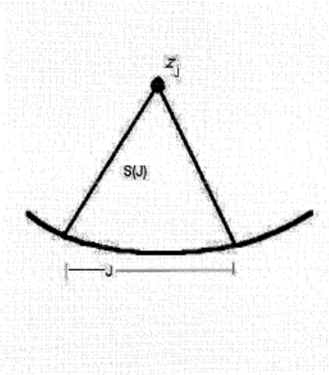
- More generally, function spaces on T model function spaces on $\mathbb{D}$; Besov spaces, spaces of $p-$harmonic functions,...

The tent over J, the shadow region below z_J.



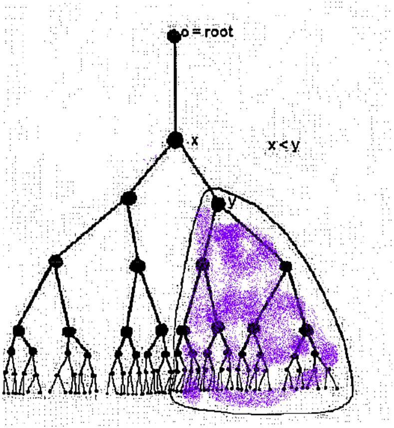
$$S(z) = S(J)$$

and set $\partial S(\alpha) = \overline{S(\alpha)} \cap \partial T$

Tents

Tents in the Tree

Here $\mathbf{S}(y)$ is the shadow region under y, etc.



Carleson Measures

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- In either case $\|\mu\|$ is defined to be the smallest C such that works.

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- In fact $\mathrm{Cap}(E) = 0$ iff $\mathrm{Cap}_T(E) = 0$ which is one of the tools in pulling results on $\mathcal{D}_d$ back to results on $\mathcal{D}$.

Characterization of Carleson Measures

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3. μ_T *satisfies a tree capacity condition. There is a constant* C *so that for all sets* $E = \cup_j \partial S(x_j)$ *in* ∂T,

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4. μ_T *satisfies the tree condition. There is a constant* C *so that* $\forall \alpha \in T$
$$I^*(I^*\mu_T)^2(\alpha) \leq C I^* \mu_T(\alpha)$$

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2. This establishes equivalence of two classes of null sets and gives an alternative approach to exceptional sets for boundary convergence.

Carleson Measures and Capacity

Theorem (ARS '07)

For E, a compact subset of $\partial\overline{T}$,

$$\mathrm{Cap}_T(E) = \sup_{\mathrm{Supp}\,(\mu)\subset E} \frac{\mu(E)}{\|\mu\|_{CM(T)}}.$$

Corollary

E has capacity zero if and only if E is $\mu-$null set for every Carleson measure μ.

It is sometimes easier to show a set carries no nontrivial Carleson measures than to estimate its capacity directly.

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- Hence, roughly, the tree condition compares the energy of the measure with its mass.

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  - For approach regions with contact between algebraic and fully exponential there is convergence with an exceptional set of capacity* zero; capacity* is defined with respect to a different space.

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- The approach focuses on tree geometry and tree combinatorics; in terms of the boundary values the approach is doing bookkeeping according to scale and position.

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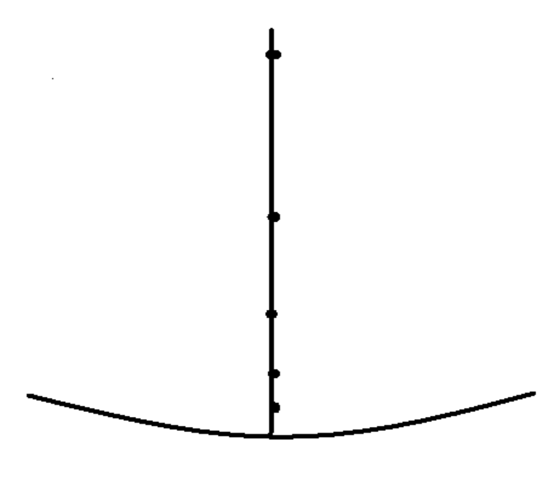
5. and thus finite radial limits off a set of capacity zero.

□

Extending this analysis to more complicated geometries.

Radial Approach



We control F on the radius using $|DF|$ at the nodes.

NonTangential Approach

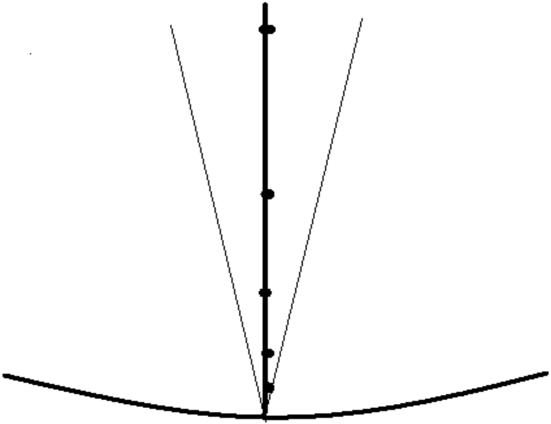


Figure: For control in a wedge we need to use $|DF|$ at more points.

NonTangential Approach—Constructing a Majorant

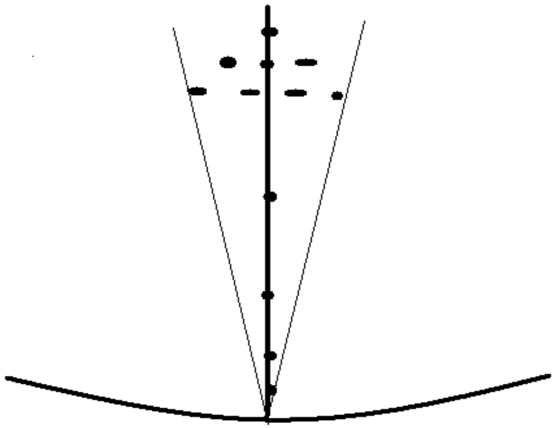


Figure: Each point on the radius looks after finitely many other points.and then the previous proof works.

Approach with Algebraic Contact

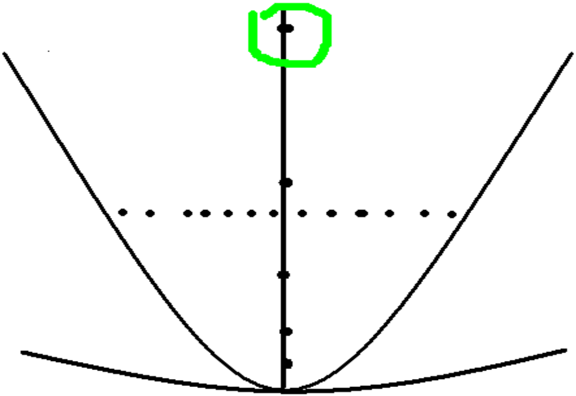


Figure: With algebraic contact there are more loci of variation to control. We use one point to control several long rows.

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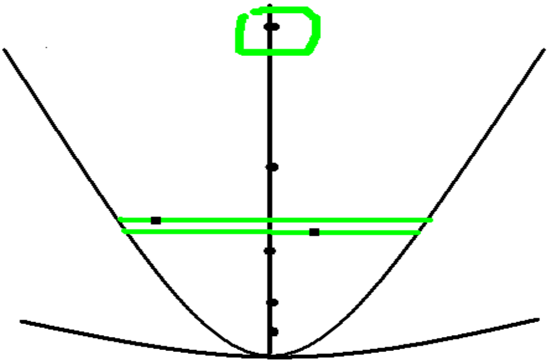
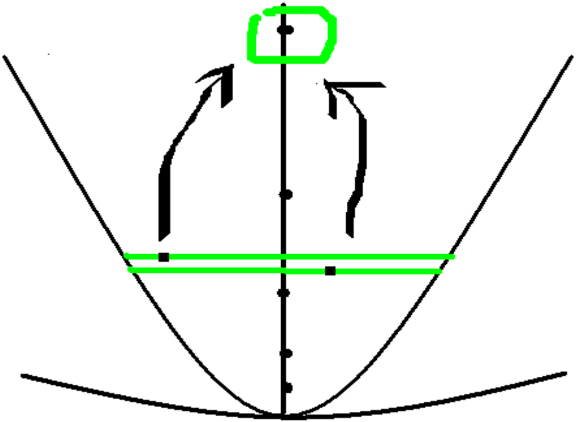


Figure: Use $\sum \max \{|DF| \text{ on a row}\}$.

The values of the majorant at the circled point controls the variation down two the green rows.

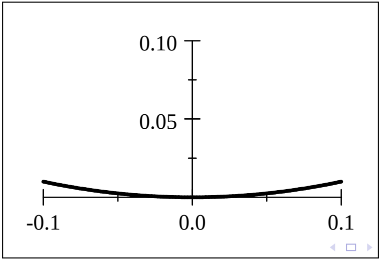


In Terms of Euclidean Geometry

Think of the upper halfplane and a contact region that touches the boundary at the origin. The coordinates of an index point n steps from o are $\sim (0, 2^{-n})$. The rows being majorized are kn steps further so their y coordinates are $\sim 2^{-n-kn}$. At that height the horizontal step size is 2^{-n-kn}. These rows are kn steps below the index point so the width of the row is

$$\text{number of steps } \times \text{ step size } = 2^{kn} 2^{-n-kn} = 2^{-n}$$

Hence the coordinates of the edge of the row are $\left(2^{-n}, 2^{-n-kn}\right)$; we are on the curve $y = |x|^{k+1}$.



Semi-exponential Contacts–Constructing a Majorant

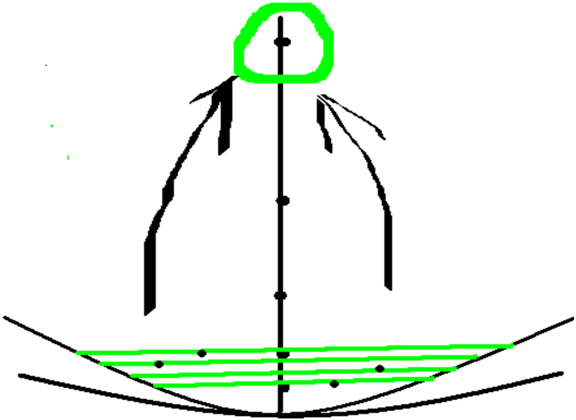


Figure: Now there are more rows to deal with.

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- Hence the previous proof goes through but now it shows that the set on which G is infinite–and hence F has boundary limits–will have null capacity but for the capacity associated to $l^2(T, 2^{\alpha d(o,\cdot)})$.

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- Use the fact that such a G has limits on a set of full measure.
- And hence on a set of full measure $\mathrm{Osc}(G) \to 0$
- Use this to get the *a.e.* $d\theta$ convergence for F.

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$$\text{Carleson measure} \iff \text{capacity}$$

in world of nonlinear potential theory.

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- $p \neq 2$, need to establish

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  in world of nonlinear potential theory.
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- In general when a tree model captures the geometry and there is summability information for the local oscillation these ideas can be used; for instance $p-$harmonic functions.

Example with a Different Function Space on the Tree

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  - F has boundary limits through sets making doubly exponential contact except of a set of measure zero.