

Local well-posedness of nonlinear dispersive equations on modulation spaces

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Outline

1 The Modulation Spaces

- Background and Definition

2 Fourier Multipliers on $M^{p,q}$

- Characterization and Sufficient Conditions
- Unimodular Fourier Multipliers

3 Applications: Local well-posedness for nonlinear PDEs

- Nonlinear Schrödinger Equations
- Remarks and References

The Sjöstrand's Algebra.

- (Sjöstrand, 1994). Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$.
 $\sigma \in S(1)$ if and only if

$$\|\sigma\|_{S(1)} = \int_{\mathbb{R}^d} \sup_{k \in \mathbb{Z}^d} |\mathcal{F}[\sigma \cdot \chi(\cdot - k)](\xi)| d\xi < \infty.$$

- $S(1) \rightarrow$ bounded pseudodifferential operators on L^2 .
- $S_{\rangle,0}^0 \subset S(1)$: “improved” Calderon-Vaillancourt Theorem.
- $S(1)$ is a “Wiener” algebra of pseudodifferential operators.
- $S(1)$ is an example of a modulation space.

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Definition.

$$V_g f(x, \omega) = \int_{\mathbb{R}^d} f(t) \overline{g(t-x)} e^{-2\pi i \omega t} dt.$$

Definition (Feichtinger, 1983)

Let $1 \leq p, q \leq \infty$ and $g \in \mathcal{S}$.

The modulation space $M^{p,q}$ is

$$M^{p,q} = \{f \in \mathcal{S}' : \|f\|_{M^{p,q}} = \|V_g f\|_{L^{p,q}} < \infty\}$$

$$\|V_g f\|_{L^{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x, y)|^p dx \right)^{q/p} dy \right)^{1/q}$$

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More Background

- Many work on modulation spaces and pseudodifferential operators, e.g., Tachizawa (1994), Rochberg and Tachizawa (1998), Heil, Ramanathan, and Topiwala (1997); Gröchenig, Heil (1999). More results by Czaja, Labate, Toft, Kobayashi, Sugimoto, Tomita etc...

Theorem

(K. Gröchenig and C. Heil, 1999) If $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$ then the pseudodifferential operator T_σ with symbol σ is bounded on $M^{p,q}(\mathbb{R}^d)$ for $1 \leq p, q \leq \infty$ and $d \geq 1$.

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Some Facts

$$\widehat{H_\sigma f}(\xi) = \sigma(\xi) \hat{f}(\xi) \iff H_\sigma f(x) = \int_{\mathbb{R}^d} \sigma(\xi) \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi.$$

Theorem (Feichtinger, Narimani (2006))

Let $\chi \in \mathcal{C}_0^\infty$. H_σ is bounded on $M^{p,q}(\mathbb{R}^d)$ if and only if
 $\sup_{k \in \mathbb{Z}^d} \|H_{\sigma \cdot \chi(\cdot - k)}\|_{L^p \rightarrow L^p} < \infty$.

Lemma (Bényi, Gröchenig, Okoudjou, Rogers: JFA 2007)

The Fourier multiplier H_σ is bounded on all modulation spaces $M^{p,q}(\mathbb{R}^d)$ for $d \geq 1$ and $1 \leq p, q \leq \infty$ under each of the following conditions:

- (i) $\sigma \in W(\mathcal{F}L^1, \ell^\infty)$.
- (ii) $\sigma \in M^{\infty,1}$.
- (iii) $\sigma \in \mathcal{F}L^1$.

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A class of Fourier Multipliers

- Let $\alpha \geq 0$ and $\xi \in \mathbb{R}^d$. $\sigma_\alpha(\xi) = e^{i|\xi|^\alpha}$.
- Littman (1963): H_{σ_1} on $L^2(\mathbb{R}^d)$ if $d \geq 1$, and on $L^p(\mathbb{R})$, $1 < p < \infty$.
- Hörmader (1963): H_{σ_2} bounded only on $L^2(\mathbb{R}^d)$, $d \geq 1$.

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Time-Frequency of Unimodular Fourier Multipliers

Theorem (Bényi, Gröchenig, Okoudjou, Rogers: JFA 2007)

If $\alpha \in [0, 2]$, then the Fourier multiplier H_{σ_α} with $\sigma_\alpha(\xi) = e^{i|\xi|^\alpha}$ is bounded from $M^{p,q}(\mathbb{R}^d)$ into $M^{p,q}(\mathbb{R}^d)$ for all $1 \leq p, q \leq \infty$ and in any dimension $d \geq 1$.

Proof.

$\alpha = 2$. H_{σ_2} is an example of metaplectic transform that leaves invariant the modulation spaces.

If $\alpha \in [0, 1]$ then $\sigma_\alpha = \chi\sigma_\alpha + (1 - \chi)\sigma_\alpha \in M^{\infty, 1}$.

$\alpha \in (1, 2)$ more technical but $\sigma_\alpha \in W(\mathcal{F}L^1, \ell^\infty)$.



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Time-Frequency of Unimodular Fourier Multipliers

Theorem (Bényi, Okoudjou 2007)

Let $d \geq 1$, $s \geq 0$, and $0 < q \leq \infty$ be given. Define $m_\alpha(\xi) = e^{i|\xi|^\alpha}$. If $1 \leq p \leq \infty$ and $\alpha \in [0, 2]$, then the Fourier multiplier operator H_{m_α} extends to a bounded operator on $M_{0,s}^{p,q}(\mathbb{R}^d)$.

Moreover, If $\alpha \in \{1, 2\}$ and $\frac{d}{d+1} < p \leq \infty$, then the Fourier multiplier operator H_{m_α} extends to a bounded operator on $M_{0,s}^{p,q}(\mathbb{R}^d)$.

Proof.

When $p, q < 1$ we use an equivalent definition of the modulation spaces due to Kobayashi. □

A Local Well Posedness Result

We consider the Schrödinger equation

$$i \frac{\partial u}{\partial t} + \Delta_x u + \lambda |u|^{2k} u = 0, \quad u(x, 0) = u_0(x). \quad (1)$$

Theorem (Bényi, Okoudjou 2007)

Assume that $u_0 \in M_{0,s}^{p,1}(\mathbb{R}^d)$, $\frac{d}{d+1} < p \leq \infty$. Then, there exists $T^* = T^*(\|u_0\|_{M_{0,s}^{p,1}})$ such that (1) has a unique solution

$u \in C([0, T^*], M_{0,s}^{p,1}(\mathbb{R}^d))$. Moreover, if $T^* < \infty$, then

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$$\|S(t)u_0\|_{M_{0,s}^{p,1}} \leq C(t^2 + 4\pi^2)^{d/4} \|u_0\|_{M_{0,s}^{p,1}}$$

$$\|S(t)u_0\|_{M_{0,s}^{p,1}} \leq C_T \|u_0\|_{M_{0,s}^{p,1}},$$

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$$\begin{aligned} \left\| \int_0^t S(t-\tau) (|u|^{2k} u)(\tau) d\tau \right\|_{M_{0,s}^{p,1}} &\leq \int_0^t \|S(t-\tau) (|u|^{2k} u)(\tau)\|_{M_{0,s}^{p,1}} d\tau \\ &\leq T C_T \sup_{t \in [0,T]} \||u|^{2k} u(t)\|_{M_{0,s}^{p,1}}. \end{aligned}$$



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$$\left\| \int_0^t S(t-\tau) (|u|^{2k} u)(\tau) d\tau \right\|_{M_{0,s}^{p,1}} \lesssim C_T T \|u(t)\|_{M_{0,s}^{p,1}}^{2k+1}.$$

Standard contraction arguments complete the proof. □

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Standard contraction arguments complete the proof. □

Remarks and References

- Similar results hold for the NLW and NLKG equations.
- W. Baoxiang, Z. Lifeng, and G. Boling (2006–2007), proved related results.
- E. Cordero and F. Nicola (2006–2007) used similar time-frequency techniques to obtain related results on the Fourier transform of the modulation spaces.
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Thank You!