# PROBLEM LIST FROM 'CURRENT TRENDS IN HARMONIC ANALYSIS' FIELDS INSTITUTE, JANUARY—JUNE 2008

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Abstract. These are problems that were suggested by participants in the *Thematic Program on New Trends in Harmonic Analysis* , held January—June 2008 at the Fields Institute, at the University of Toronto, Canada.

The presentation of the problems is kept to a minimum, largely sticking to a statement of the problem, a small number of remarks, and and equality small bibliography. Those intriguied by a particular problem should survey the literature, starting with the cited references, for a fuller understanding of the problem and its subtlies.

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## 1. Interpolating Blashke Products

*From the talk of Artur Nicolau* Recall that a sequence $\{z_n\} \in \mathbb{D}$ is *interpolating* iff the trace of $H^\infty(\mathbb{D})$ on the set $\{z_n\}$ is all of $\ell^\infty$. Namely,

$$\bigl\{\{\varphi(z_n) \,:\, n \in \mathbb{N}\} \,:\, \varphi \in H^\infty(\mathbb{D})\bigr\} = \ell^\infty(\mathbb{N})\,.$$

**Theorem** (Carleson's Theorem characterizing interpolating sequences)**.** *$\{z_n\}$ is interpolating iff*

a. *(Zeros are separated in the hyperbolic metric.)*

$$\inf_{m\neq n} \frac{|z_m - z_n|}{|1-\overline{z_m} z_n|} > 0\,.$$

b. *(Zeros determine a Carleson measure.) There is a constant $C$ so that for all balls $B$ with center on the boundary of the disk,*

$$\sum_n 1-|z_n| \le C\,\mathrm{radius}(B)\,.$$

Call a Blaschke product $B$ an *interpolating Blaschke product* iff the zeros of $B$, written $Z(B)$ form an interpolating sequence. This is equivalent to the two conditions below.

$$\inf_{z\in Z(B)} (1-|z|^2)|B'(z)| > 0\,,$$

$$|B(z)| \simeq \inf_{z\in Z(B)} \frac{|z-w|}{|1-\overline{z}w|} \qquad z\in\mathbb{D}\,.$$

Recall that analytic $I$ is *inner* iff almost all radial limits of $I$ at the boundary have complex modulus one.

**Question** (J. Garnett and Peter Jones, 1981)**.** *For all inner $I$ and $\epsilon>0$ does there exist an interpolating Blaschke product $B$ such that $\|B-I\|_\infty<\epsilon$?*

*Remark.* One can construct two Blaschke products $B$ and $B^*$ so that the zeros of $B^*$ are a hyperbolic perturbation of $B$, with the perturbation of arbitrarily small size, and yet $\|B-B^*\|_\infty = 2$.

**Question.** *For all inner $I$ and $\epsilon>0$ does there exist an interpolating Blaschke product $B$ such that $\|\arg(B)-\arg(I)\|_{\mathrm{BMO}(\partial\mathbb{D})}<\epsilon$? Equivalently, does there exist $h\in H^\infty$, $h$ invertible in $H^\infty$, such that $\|I-Bh\|_\infty<\epsilon$?*

**Question** (Nikolskii, 1984)**.** *For all inner $I$ does there exist an interpolating Blaschke product $B$ such that*

$$\mathrm{dist}(I\overline{B}, H^\infty) < 1\,,$$
$$\mathrm{dist}(\overline{I}B, H^\infty) < 1\,.$$

*Remark.* The last condition has an equivalent formulation in terms arising from the Helson-Szego Theorem.

**Question.** *For all $\epsilon > 0$ and any sequence of zeros $\{z_n\}$ arising from a Blaschke product $B$ does there exist an interpolating sequence $\{y_n\}$ so that*

$$\Bigl\| \sum_n \ln|x - z_n|^2 - \ln|x - y_n|^2 \Bigr\|_{\mathrm{BMO}(\mathbb{R})} < \epsilon\,.$$

That this is true with $\epsilon$ replaced by a fixed constant $C$ is a result of Nicolau and Suárez.

Jones, Peter W. 1981. *Ratios of interpolating Blaschke products*, Pacific J. Math. **95**, no. 2, 311–321. MR **632189 (82m:**30032)

Nicolau, Artur and Daniel Suárez. 2006. *Approximation by invertible functions of $H^\infty$*, Math. Scand. **99**, no. 2, 287–319. MR 2289026

Nikol′skiĭ, N. K. 1986. *Treatise on the shift operator*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 273, Springer-Verlag, Berlin. Spectral function theory; With an appendix by S. V. Hruščev [S. V. Khrushchëv] and V. V. Peller; Translated from the Russian by Jaak Peetre. MR **827223 (87i:**47042)

## 2. Corona Theorem with Bounds

*From Sergei Treil* Let $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be decreasing, bounded, and satisfying

$$\int_0^\infty \psi(x)\, dx = \infty\,.$$

Set $\varphi(x) = x^2\psi(x)$. Is the following true? There exist $\tau \in H^\infty(\mathbb{D})$ and integer $n$, $f_1, \dots, f_n \in H^\infty(\mathbb{D})$ so that

$$|\tau(x)| \le \varphi(|(f_1, , \dots, f_n)|)$$

where the norm of the vector is euclidean norm, and $\tau$ is not in the ideal generated by $f_1, \dots, f_n$.

Let $H^\infty$ be any of the spaces $H^\infty(\mathbb{D}^n)$ or $H^\infty(\mathbb{B}^n)$ for $n \ge 2$.

**Problem.** *Give a 'real variable' description of $(H^\infty)^\perp$. In the case of the polydisc this is the space*

$$(H^\infty(\mathbb{D}^n))^\perp = \bigl\{\varphi \in L^1((\partial\mathbb{D})^n) \,:\, \langle \varphi, \psi\rangle = 0 \;\forall\; \psi \in H^\infty(\mathbb{D}^n)\bigr\}\,.$$

By 'real variable' is meant a description which if specialized to the case of $n = 1$ coincides with the description of $H^1(\mathbb{D})$ in terms of maximal function, area integral or some cognate description.

**Problem.** *Let $F$ be a (rectangular) matrix with entries in $H^\infty$ with $H^\infty$ left inverse. Can $F$ be completed to a square $H^\infty$ invertible matrix?*

*Remark.* True for $n = 1$. The proof is short, can be found in standard references, such as Nikol′skiĭ. Beurling's theorem is used in the proof.

Treil, Sergei. 2007. *The problem of ideals of $H^\infty$: beyond the exponent 3/2*, J Funct. Analy. **253**, 220-240, available at arXiv.org/abs/math/0702806.

## 3. Quasiconformal Maps

*From Ignacio Uriarte-Tuero, by way of K. Astala* A function $f : \mathbb{C} \longrightarrow \mathbb{C}$ is $K$-quasiconformal if it is an orientation preserving homeomorphism, $f \in W^{1,2}_{\mathrm{loc}}$ and

$$|Df|^2 \le K\, \mathrm{J}(f)$$

where J is the Jacobian. Equivalently, $f$ is a homeomorphism solving the Beltrami equation $\overline{\partial} f = \mu \partial f$, $\|\mu\|_\infty \le k < 1$ where

$$K = \frac{1+k}{1-k}\,.$$

**Theorem** (Astala)**.** *If $E \subset \mathbb{C}$ is compact,*

$$\frac{1}{K}\Big(\frac{1}{\dim(E)} - \frac{1}{2}\Big) \le \frac{1}{\dim(f(E))} - \frac{1}{2} \le K\Big(\frac{1}{\dim(E)} - \frac{1}{2}\Big)$$

All dimensions are Hausdorff. This means that if $\dim(E) \le d$, then $\dim(f(E)) \le d'$, where $d'$ can be computed from the Theorem above. This next problem asks for an essential strengthening of this observation.

**Question** (Astala)**.** *For $d'$ as above,*

$$\mathcal{H}^d(E) = 0 \quad \text{implies} \quad \mathcal{H}^{d'}(f(E))?$$

Astala, Kari. 1994. *Area distortion of quasiconformal mappings*, Acta Math. **173**, no. 1, 37–60. MR **1294669 (95m:**30028b)

## 4. Hilbert Space valued Holomorphic maps

*From Joseph Cima* Let $B$ be the open unit ball of a separable Hilbert space $H$. For $F : B \longrightarrow H$ the Frechet derivative of $F$ at $x \in B$ is a bounded linear map $L : H \longrightarrow H$ such that

$$\lim_{\|y\| \to 0} \frac{F(x+y) - F(x) - L(y)}{\|y\|} = 0\,.$$

Write $L = DF$. Say that $F$ is holomorphic if $DF(x)$ exists for all $x \in B$ and the map $x \longrightarrow DF(x)$ is continuous in the uniform topology on $B(H)$, the space of bounded linear operators on $H$.

**Question.** *Suppose that $F : B \longrightarrow H$ is holomorphic, injective and $F(B)$ is an open set. Is $F$ biholomorphic, that is is every $DF(x)$ invertible for all $x \in B$.*

*Remark.* The previous quesiton is true for $H = \mathbb{C}^n$.

**Question.** *Assume $F : B \longrightarrow H$ is holomorphic and $K \subset B$ is compact. If $F$ is injective on $B$ and biholomorphic on $B - K$, is $F$ biholomorphic on $B$?*

For this last question, the answer is yes in $\mathbb{C}^n$.

**Question.** *If $F : B \longrightarrow H$ is holomorphic and injective ond $B$ and $DF = 0$ can $F(B)$ be an open set?*

Mujica, Jorge. 2005/06. *Holomorphic functions on Banach spaces*, Note Mat. **25**, no. 2, 113–138. MR **2259962 (2007f:**46040)

## 5. Hankel Forms on Dirichlet Spac

*From Richard Rochberg's Talk* Let $X$ denote a Hilbert space of functions holomorphic functions on $\mathbb{D}$. A *Hankel form on* $X$ is defined by

$$\mathrm{H}_b^X = \langle fg, b\rangle_X \,.$$

A positive measure $\mu$ on $\mathbb{D}$ is a $X$*-Carleson Measure*, written $\mu \in \mathrm{CM}(X)$ iff

$$\int_{\mathbb{D}} |\varphi|^2 \mu(dx, dy) \le C^2 \|\varphi\|_X^2 \,.$$

The constant $C$ is independent of the choice of $\varphi \in X$. The least such constant is denoted by $\|\mu\|_{\mathrm{CM}(X)}$

**Theorem.** *For $X = H^2(\mathbb{D})$, $\mathrm{H}_b^{H^2}$ is bounded iff $|b'|^2 dxdy \in \mathrm{CM}(H^2)$.*

Define the Dirchlet space $D$ to be the space of analytic functions on the disk for which the norm below is finite.

$$\|f\|_D = |f(0)| + \left[ \int_{\mathbb{D}} |f'| \; dxdy \right]^{1/2} .$$

This is a Hilbert space, with the pairing

$$\langle f, g\rangle_D = f(0)\overline{g(0)} + \int_{\mathbb{D}} f'\overline{g'} \; dxdy \,.$$

**Problem.** *Characterize the bounded Hankel forms on $\mathcal{D}$.*

**Question.** *Do we have the equivalence below for $b$ analytic on $\mathcal{D}$?*

$$\sup_{\|f\|_{\mathcal{D}} = \|g\|_{\mathcal{D}} = 1} |H_b(f, g)| \simeq \||b'|\|_{\mathcal{D}\text{-Carleson}}$$

The inequality '$\lesssim$' above is straight forward.

Arcozzi, Nicola and Richard Rochberg. 2004. *Topics in dyadic Dirichlet spaces*, New York J. Math. **10**, 45–67 (electronic). MR **2052364 (2005f:**30069)

Maz′ya, Vladimir G. and Igor E. Verbitsky. 2002. *The Schrödinger operator on the energy space: boundedness and compactness criteria*, Acta Math. **188**, no. 2, 263–302. MR **1947894 (2004b:**35050)
Rochberg, Richard and Zhi Jian Wu. 1993. *A new characterization of Dirichlet type spaces and applications*, Illinois J. Math. **37**, no. 1, 101–122. MR **1193132 (93j:**30039)

## 6. Cauchy Transforms of Measures

*From Alexander Volberg* For measure $\mu$ on the complex plane, set

$$\mathrm{C}^{\mu}(z) = \int \frac{d\mu(\zeta)}{z-\zeta}$$

**Problem.** *Is there a positive measure $\mu$ on $[0,1]^2$ such that $\|\mu\| < \epsilon$, but*

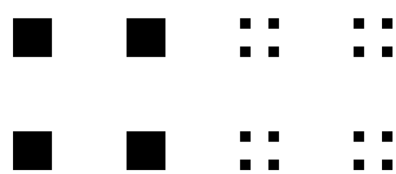
$$\forall\, x \in [0,1]\ \exists\, y \in [0,1]\ \ni\ |\mathrm{C}^{\mu}(x+iy)| > 1$$

*Failing that, is the following true?*

$$|\{x \in [0,1]\ \exists\, y \in [0,1]\ \ni\ |\mathrm{C}^{\mu}(x+iy)| > 1\}| = 1$$

*Remark.* In the last display, examples show that one can get a positive proportion of $x$.

Consider the set $G_1$ which consists of the four closed squares and their interiors of sidelength of $\frac{1}{4}$ at each of the four corners of the unit square. Let $G_2$ be the recursive extension of this consisting of 16 squares of side length $\frac{1}{16}$, and so on. $G_1$ and $G_2$ are pictured below.



**Problem.** *Is there a positive measure $\mu$ on the complex plane such that $\|\mu\| \asymp n^{-1/2}$ and $|\mathrm{C}^{\mu}(z)| > 1$ for all $z \in G_n$.*

*Remark.* The second problem implies the first.

A measure $\mu$ on $\mathbb{C}$ is called *reflectionless* iff $\mu$ is positive, has compact support and $\mathrm{C}^{\mu}(z) = 0$ $\mu$–a.e.

**Problem.** *Show that there is a positive $\delta$ such that all reflectionless measures are orthogonal to Hausdorff measure $H^{2-\delta}$.*

Peherstorfer, F., A. Volberg, and P. Yuditskii. 2007. *Two-weight Hilbert transform and Lipschitz property of Jacobi matrices associated to hyperbolic polynomials*, J. Funct. Anal. **246**, no. 1, 1–30. MR 2316875

## 7. Positive Polynomials on the Tridisk

*From Gregory Knese* Let $n_1, n_2, n_3$ be positive integers and $D$ a contractive $(n_1 + n_2 + n_3) \times (n_1 + n_2 + n_3)$ matrix. By *contractive* we mean that the norm of the matrix as an operator on Euclidean space has norm at most one. Observe that the polynomial

$$p(z_1, z_2, z_3) = \det \left( I - D \begin{bmatrix} z_1 I_{n_1} & 0 & 0 \\ 0 & z_2 I_{n_2} & 0 \\ 0 & 0 & z_3 I_{n_3} \end{bmatrix} \right)$$

has no zeros on the tridisk $\mathbb{D}^3 = \mathbb{D} \times \mathbb{D} \times \mathbb{D}$.

In contrast to the bidisk, not all polynomials with no zeros on $\mathbb{D}^3$ can be written this way, up to constant multiples.

**Question.** *Are there any 'geometric' properties of the polynomials of type p above that distinguish them from the class of all polynomials with no zeros on $\mathbb{D}^3$?*

Agler, Jim, John E. McCarthy, and Mark Stankus. 2006. *Toral algebraic sets and function theory on polydisks*, J. Geom. Anal. **16**, no. 4, 551–562. MR **2271943 (2007j:**32002)

## 8. On Certain $L^p$ Operators

*From Tao Mei* Consider an operator T defined simultaneously on $L^p(\mathbb{R})$ for all $1 \le p \le \infty$. Suppose that T satisfies

(*i*) $\|\mathrm{T}\|_{p \to p} \le 1$, $1 \le p \le \infty$.
(*ii*) T is self adjoint on $L^2(\mathbb{R})$.
(*iii*) $\mathrm{T}(\mathbf{1}_{\mathbb{R}}) = \mathbf{1}_{\mathbb{R}}$.
(*iv*) there is a constant $C$ so that

$$\mathrm{T}^{2j} f \le C \cdot \mathrm{T}^j f\,, \qquad f \ge 0\,,\ j \in \mathbb{N}\,.$$

**Question.** *Does there exist $\alpha > 0$ such that*

$$\mathrm{T}^k f \le \left(\tfrac{k}{j}\right)^\alpha \mathrm{T}^j f \qquad f \ge 0\,,\ k \ge j \in \mathbb{N}\,.$$

*Remark.* The case $k \ge 2j$ is trivial above.

## 9. Bounds for Polynomials on the Bidisk

*From Tavan Trent*

**Question.** *Does there exist a universal constant $C < \infty$ so that for each polynomial $p(z, w)$ with $|p(z, w)| > 0$ for $|z| < 1$ and $|w| < 1$, and*

$$\int_{\mathbb{T}} \int_{\mathbb{T}} |p|^2 d\sigma\, d\sigma = 1$$

*there exists a* $h \in H^2(\mathbb{D}^2)$ *with norm at most* $C$ *and*

$$1 + |p(z,w)|^2 \le |h(z,w)|^2 \,, \qquad |z| \,,\ |w| < 1 \,.$$

This question can be rephrased as the question as to whether or not this quantity is finite.

$$\sup_{\substack{p \text{ poly in } z,\, w \\ |p|>0 \text{ on } \mathbb{D}^2 \\ \|p\|_2=1}} \inf\bigl\{ \|q\|_{H^2(\mathbb{D}^2)} \,:\, 1 + |p(z,w)|^2 \le |q(z,w)|^2 \,,\ \forall\, |z|, |w| < 1 \bigr\}$$

The following is true, in which we change the formula above in two ways.

$$\sup_{\substack{p \text{ poly in } z,\, w \\ \|p\|_2=1}} \inf\bigl\{ \|q\|_{H^2(\mathbb{D}^2)} \,:\, 1+|p(z,w)|^2 \le \operatorname{Re}(q(z,w))^2 \,,\ \forall\, |z|, |w| < 1 \bigr\} = \infty \,.$$

## 10. Trilinear Hilbert Transform

*From Michael Lacey.*

**Question.** *Is there any finite* $1 < p < \infty$ *for which there is a finite constant* $C$ *for which we have the inequality below?*

$$\Bigl\| \int f_1(x-y) f_2(x-2y) f_3(x-3y) \frac{dy}{y} \Bigr\|_{p/3} \le C \prod_{j=1}^{3} \|f_j\|_p$$

The integral is taken in the principal value sense. The arguments of the $f_j$ are the simplest possible choices.

Tao, Terrance. 2007. *Open question: boundedness of the trilinear Hilbert transform*, available at http://terrytao.wordpress.com/2007/05/10/open-question-boundedness-of-the-trilinear-hilbert-transform/.

## 11. Small Ball Inequality

Let $h_R$ be a $L^\infty$ normalized Haar function adapted to dyadic rectangle $R$. In particular, $h_R$ has mean zero in each coordinate seperately. We consider a 'reverse triangle inequality' for such functions.

**Problem.** *In dimensions* $d \ge 3$*, show that there is a constant* $C$ *so that for all integers* $n \ge 1$*, and all choices of real coefficients* $a_R$*,*

$$n^{(d-2)/2} \Bigl\| \sum_{\substack{R \subset [0,1]^d \\ |R| = 2^{-n}}} a_R \cdot h_R(x) \Bigr\|_\infty \ge C 2^{-n} \sum_{\substack{R \subset [0,1]^d \\ |R| = 2^{-n}}} |a_R| \,.$$

*Remark.* (1) With $n^{d/2}$ on the left above replaced by $n^{(d-1)/2}$, the inequality is trivial. (2) The power on $(d-2)/2$ would be optimal. (3) The case of two dimensions, $d = 2$, is a Theorem of Talagrand. (4) In dimensions $d \geq 3$, there is partial information about this conjecture in the article by Dmitriy Bilyk, Michael Lacey, and Armen Vagharshakyan.

Lacey, Michael T, Dmitriy Bilyk, and Armen Vagharshakyan. *On the Small Ball Inequality in All Dimensions*, J Func Analy, to appear, available at arXiv:0705.4619.