

# Kato's $\sqrt{-}$ problem in Banach spaces

Tuomas Hytönen

University of Helsinki, Finland

[tuomas.hytonen@helsinki.fi](mailto:tuomas.hytonen@helsinki.fi)

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Alan McIntosh & Pierre Portal (Canberra)

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Consider

$$L = -\operatorname{div} A(x)\nabla = -\sum_{i,j=1}^n \partial_i A_{ij}(x) \partial_j,$$

where  $A \in L^\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^n))$

- second-order, divergence-form operator with bounded, measurable coefficients.

Let further

$$\operatorname{Re}(w, A(x)w) = \operatorname{Re} \sum_{i,j=1}^n \bar{w}_i A_{ij}(x) w_j \geq \lambda > 0 \quad (\text{EII})$$
$$\forall w \in B_{\mathbf{C}^n}, x \in \mathbf{R}^n$$

- elliptic.

Let  $\mathcal{Y}$  be a function space on  $\mathbf{R}^n$  and suppose:

$$\left\{ (I + t^2 L)^{-1}, t\sqrt{-\Delta}(I + t^2 L)^{-1}, (I + t^2 L)^{-1}t\sqrt{-\Delta}, \right. \\ \left. t\sqrt{-\Delta}(I + t^2 L)^{-1}t\sqrt{-\Delta} : t > 0 \right\} \text{ unif. bounded on } \mathcal{Y}. \quad (\text{Bd})$$

For  $\mathcal{Y} = L^2(\mathbf{R}^n)$ , (EII)  $\Rightarrow$  (Bd).

Then, as an operator in  $\mathcal{Y}$ ,  $\sigma(L) \subseteq S_\theta^+ \cup \{0\}$ , where  $S_\theta^+ := \{\zeta \in \mathbf{C} \setminus \{0\} : |\arg(\zeta)| < \theta\}$ .

Functions of  $L$  by extended Dunford–Riesz Calculus:

$$f(L) = \frac{1}{2\pi i} \int_{\partial S_\nu^+} f(\lambda)(\lambda - A)^{-1} d\lambda,$$

$f$  holomorphic (and decaying) in  $S_\omega^+$ ,  $\theta < \nu < \omega$ .

Let  $\mathcal{Y} = L^2(\mathbf{R}^n)$ .

T. Kato's conjecture:  $D(\sqrt{L}) = D(\nabla)$ , and  $\|\sqrt{L}u\|_{\mathcal{Y}} \asymp \|\nabla u\|_{\mathcal{Y}^n}$ .

P. Auscher, S. Hofmann, M. Lacey, A. McIntosh and Ph. Tchamitchian (Annals of Math, 2002) proved: (EII) $\Rightarrow$

$$\begin{aligned} \|f(L)\|_{\mathcal{L}(\mathcal{Y})} &\leq C \|f\|_\infty \quad \forall f \in H^\infty(S_\omega^+) \quad \text{and} \\ \text{satisfies } &\|\sqrt{L}u\|_{\mathcal{Y}} \asymp \|\nabla u\|_{\mathcal{Y}^n}. \end{aligned} \tag{Cal}$$

What about  $\mathcal{Y} = L^p(\mathbf{R}^n)$  (or even  $\mathcal{Y} = L^p(\mathbf{R}^n; X)$ ,  $X$  Banach)?

For quite general  $\mathcal{Y}$ ,  $(\text{Cal}) \Rightarrow (\text{Bd})$ .

But for  $\mathcal{Y} = L^p(\mathbf{R}^n)$  and  $p \neq 2$ ,  $(\text{EII}) \not\Rightarrow (\text{Bd})$ , hence  $(\text{EII}) \not\Rightarrow (\text{Cal})$ .

We assume:

$$\left\{ (I + t^2 L)^{-1}, t\sqrt{-\Delta}(I + t^2 L)^{-1}, (I + t^2 L)^{-1}t\sqrt{-\Delta}, t\sqrt{-\Delta}(I + t^2 L)^{-1}t\sqrt{-\Delta} : t > 0 \right\} \quad R\text{-bounded on } \mathcal{Y}. \quad (\text{Rbd})$$

For  $\mathcal{Y} = L^2(\mathbf{R}^n)$ ,  $(\text{Bd}) \Leftrightarrow (\text{Rbd})$ .

Quite generally,  $(\text{Cal}) \Rightarrow (\text{Rbd})$ .

## *R*-boundedness

Let  $\varepsilon_k$  be independent random sings on some probability space  $\Omega$  with  $\mathbb{P}(\varepsilon_k = +1) = \mathbb{P}(\varepsilon_k = -1) = 1/2$ .

**Definition** (Berkson, Gillespie 1994):  $\mathcal{T} \subset \mathcal{L}(X)$  is *R*-bounded if  $\exists C < \infty$  such that  $\forall K \in \mathbf{N}$ ,  $\forall x_1, \dots, x_K \in X$ ,  $\forall T_1, \dots, T_K \in \mathcal{T}$ :

$$\left\| \sum_{k=1}^K \varepsilon_k T_k x_k \right\|_{L^2(\Omega; X)} \leq C \left\| \sum_{k=1}^K \varepsilon_k x_k \right\|_{L^2(\Omega; X)}.$$

$$\mathcal{R}(\mathcal{T}) := \inf C.$$

Quite a general rule of thumb in vector-valued harmonic analysis:  
replace classical boundedness assumptions by *R*-boundedness.

## **Theorem (H., McIntosh, Portal):**

Let  $X$  be a UMD space (unconditionality of martingale differences,  $\Leftrightarrow$  Hilbert transform bounded on  $L^2(\mathbf{R}; X)$ ).

Let further  $X$  and  $X^*$  be “RMF” (Rademacher maximal function property – more below).

Let  $(\text{Ell}) \wedge (\text{Rbd})$  in  $\mathcal{Y} = L^p(\mathbf{R}^n; X)$  for all  $p \in (p_-, p_+) \subseteq (1, \infty)$ .

Then  $(\text{Cal})$  holds in the same spaces.

Note: UMD is necessary for  $A = I$ . Then  $L = -\Delta$ , and  $\|-\Delta u\|_{L^p(\mathbf{R}^n; X)} \asymp \|\nabla u\|_{L^p(\mathbf{R}^n; X)^n} \Leftrightarrow X \text{ is UMD}$ .

We use the first-order framework developed by A. Axelsson, S. Keith and A. McIntosh (Inventiones Math, 2006) with two “factorizations”  $\Pi_A$  and  $\tilde{\Pi}_A$  of  $L$  acting in  $\mathcal{X} := \mathcal{Y}^{n+1}$ :

$$\Gamma = \begin{pmatrix} 0 & 0 \\ \nabla & 0 \end{pmatrix}, \quad \Gamma^* = \begin{pmatrix} 0 & -\operatorname{div} \\ 0 & 0 \end{pmatrix}, \quad \Pi = \Gamma + \Gamma^*,$$

$$\Pi_A = \begin{pmatrix} 0 & -\operatorname{div} A \\ \nabla & 0 \end{pmatrix}, \quad \tilde{\Pi}_A = \begin{pmatrix} 0 & -\operatorname{div} \\ A\nabla & 0 \end{pmatrix},$$

$$\Pi_A^2 = \begin{pmatrix} L & 0 \\ 0 & -\nabla \operatorname{div} A \end{pmatrix}, \quad \tilde{\Pi}_A^2 = \begin{pmatrix} L & 0 \\ 0 & -A\nabla \operatorname{div} \end{pmatrix}.$$

**Why first-order?** —  $\|\nabla u\|_{\mathcal{Y}^n} \asymp \|\sqrt{L}u\|_{\mathcal{Y}}$  means  $\|\nabla L^{-1/2}u\|_{\mathcal{Y}} \asymp \|u\|_{\mathcal{Y}}$ , but the Riesz transform  $\nabla L^{-1/2} \neq f(L)$ .

However,

$$\begin{aligned}\Pi_A^2 &= \begin{pmatrix} -\operatorname{div} A \nabla & 0 \\ 0 & -\nabla \operatorname{div} A \end{pmatrix} = \begin{pmatrix} L & 0 \\ 0 & \tilde{L} \end{pmatrix}, \\ \operatorname{sgn}(\Pi_A) &= \Pi_A \frac{1}{\sqrt{\Pi_A^2}} = \begin{pmatrix} 0 & -\operatorname{div} A \tilde{L}^{-1/2} \\ \boxed{\nabla L^{-1/2}} & 0 \end{pmatrix}.\end{aligned}$$

∴ Functional calculus estimates

$$\|f(\Pi_A)\|_{\mathcal{L}(\mathcal{X})} + \|f(\tilde{\Pi}_A)\|_{\mathcal{L}(\mathcal{X})} \lesssim \|f\|_\infty$$

⇒ Functional calculus of  $L$  and the  $\sqrt{\phantom{x}}$  estimate!

By abstract reasons, functional calculus of  $\Pi_A \Leftrightarrow$  “quadratic estimates” (abstract “Littlewood–Paley inequality”)

$$\mathbb{E} \left\| \sum_{k \in \mathbf{Z}} \varepsilon_k Q_{2^k}^A u \right\|_{\mathcal{X}} \lesssim \|u\|_{\mathcal{X}}, \quad Q_t^A = \frac{t\Pi_A}{I + t^2\Pi_A^2}$$

+ “dual” estimate with  $((Q_t^A)^*, \mathcal{X}^*)$  in place of  $(Q_t^A, \mathcal{X})$ ,

and similarly for  $\tilde{\Pi}_A$ .

By algebra, further splitting into

$$\mathbb{E} \left\| \sum_{k \in \mathbf{Z}} \varepsilon_k Q_{2^k}^A u \right\|_{\mathcal{X}} \lesssim \|u\|_{\mathcal{X}}, \quad u \in R(\Gamma) \quad (*)$$

+ a similar estimate on  $R(\Gamma^*)$  + the dual estimates.

Concentrate on (\*).

## Tools: harmonic & dyadic approximate identities

$$P_t := (I - t^2 \Delta)^{-1},$$

$$A_t u(x) := A_{2^k} u(x) := \frac{1}{|Q|} \int_Q u(y) \, dy$$

if  $2^{k-1} < t \leq 2^k$ ,  $x \in Q$  =dyadic cube of side-length  $2^k$ .

Philosophy: “ $P_t \approx A_t$ ”.

**Principal part** of  $Q_t^A$ : “ $\gamma_t := Q_t^A(1)$ ”; precisely,  
 $\gamma_t(x)w := Q_t^A(w \otimes 1)(x)$ ,  $w \in X^{n+1}$ .

**Divide**  $Q_t^A = Q_t^A(I - P_t) + (Q_t^A - \gamma_t A_t)P_t + \gamma_t A_t(P_t - A_t) + \gamma_t A_t$ ,  
**and control:**

$Q_t^A(I - P_t)$  — high-frequency part: use simple algebra.

$(Q_t^A - \gamma_t A_t)P_t$  — off-diagonal part:  $1_F Q_t^A 1_E$  is small if  $d(E, F)/t$  is large; use Poincaré's inequality, Fourier multipliers, etc.

$\gamma_t A_t(P_t - A_t)$  — martingale part: use UMD explicitly to prove

$$\mathbb{E} \left\| \sum_{k \in \mathbf{Z}} \varepsilon_k (P_{2^k} - A_{2^k}) u \right\|_p \lesssim \|u\|_p.$$

$\gamma_t A_t$  — principal part: use a new Carleson's inequality, control by a new maximal function.

**$L^p$ -version of Carleson's inequality:**

$$\left\| \left( \sum_{k \in \mathbf{Z}} |b_{2^k} A_{2^k} u|^2 \right)^{1/2} \right\|_{L^p(\mathbf{R}^n)} \leq \|b\|_{\text{Car}^{p+\epsilon}(\mathbf{R}^n)} \|u\|_{L^p(\mathbf{R}^n)},$$

where  $b = (b_{2^k})_{k \in \mathbf{Z}}$  — a sequence of functions, and

$$\|b\|_{\text{Car}^p(\mathbf{R}^n)} = \sup_Q \left( \frac{1}{|Q|} \int_Q \left[ \sum_{k: 2^k \leq \ell(Q)} |b_{2^k}(x)|^2 \right]^{p/2} dx \right)^{1/p}.$$

Can take  $\epsilon = 0$  if  $p \in (1, 2]$ .

Note:  $\|b\|_{\text{Car}^2(\mathbf{R}^n)}$  = Carleson constant of the measure

$$d\mu(t, x) = \sum_{k \in \mathbf{Z}} |b_{2^k}(x)|^2 dx d\delta_{2^k}(t),$$

i.e.,  $\sup_Q \left( \frac{1}{|Q|} \mu((0, \ell(Q)] \times Q) \right)^{1/2}$ .

Vector-valued version of Carleson:

$$\mathbb{E} \left\| \sum_{k \in \mathbf{Z}} \varepsilon_k b_{2^k} A_{2^k} u \right\|_{L^p(\mathbf{R}^n, X)} \leq \|b\|_{\text{Car}^{p+\epsilon}(\mathbf{R}^n)} \|M_R u\|_{L^p(\mathbf{R}^n)},$$

where  $M_R u(x) := \mathcal{R}(\{A_{2^k} u(x) : k \in \mathbf{Z}\})$  — Rademacher maximal function (RMF).

Note:  $A_{2^k} u(x) \in X \eqsim \mathcal{L}(\mathbf{C}, X)$ .

Can take  $\epsilon = 0$  if  $X$  has type  $p$ .

Say that  $X$  has RMF if  $\|M_R u\|_{L^p(\mathbf{R}^n)} \lesssim \|u\|_{L^p(\mathbf{R}^n, X)}$  for one (all)  $p \in (1, \infty)$ .

$X$  has RMF if

- it has type 2 ( $M_R \lesssim$  usual maximal function — easy), or
- it is a UMD function lattice ( $M_R \lesssim$  lattice maximal function; bounded by J. L. Rubio de Francia), or
- if it is a noncommutative  $L^p$  space,  $1 < p < \infty$  ( $M_R \lesssim$  non-commutative maximal function; bounded by M. Junge).

$\ell^1$  does not have RMF.

Open: Does every UMD space (or even every uniformly convex space) have RMF?

How to check the  $R$ -bisectoriality assumptions of the Theorem?

**Corollary:**

Let  $X$  be a UMD function lattice.

Let  $A : \mathbf{R} \rightarrow \mathbf{C}$  be bounded, measurable, elliptic ( $n = 1!$ )

Then  $L = -d/dx A(x)d/dx$  has bounded  $H^\infty$  calculus and satisfies the  $\sqrt{\cdot}$  estimate in  $L^p(\mathbf{R}, X)$  for all  $p \in (1, \infty)$ .

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