Strongly clean property and stable range one of some rings

by

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Notation and preliminaries

- R: Associative ring with identity $1 \neq 0$.

- $J(R)$: Jacobson radical of R.

- $U(R)$: Set of units of R.

- $C(X)$: The ring of real valued continuous functions.

- $C(X, \mathbb{C})$: The ring of complex valued continuous functions.

Clean and strongly clean rings

- An element $a \in R$ is **clean** if $\exists\ e^2 = e \in R$ and $u \in U(R)$ such that $a = e + u$. R is **clean** if every element of R is clean [Nicholson, 1977].

- An element $a \in R$ is **strongly clean** if $\exists\ e^2 = e \in R$ and $u \in U(R)$ such that $a = e + u$ and $eu = ue$. R is **strongly clean** if every element of R is strongly clean [Nicholson, 1999].

- Matrix rings over strongly clean rings **need not** be strongly clean. (S$\acute{a}$nchez Campos, 2002, Wang and Chen, 2004)

- Strongly clean property of matrix rings over a commutative local ring was **completely solved**. (Borooah,Diesl and Dorsey,2007)

- **Question**: When is $\mathbb{M}_n(R)$ over a commutative strongly clean ring strongly clean?

Regular, strongly regular and strongly π-regular rings

- An element $a \in R$ is **regular** if $a = aba$ for some $b \in R$. R is **regular** if every element of R is regular.

- An element $a \in R$ is **strongly regular** if $a = aba, ab = ba$ for some $b \in R$. R is **strongly regular** if every element of R is strongly regular. R is strongly regular iff R is regular and every idempotent in R is central.

- An element $a \in R$ is **left** π**-regular** if $Ra \supset Ra^2 \supset \cdots \supset Ra^n \supset \cdots$ terminates, a is **right** π**-regular** if $aR \supset a^2R \supset \cdots \supset a^nR \supset \cdots$ terminates, and a is **strongly** π**-regular** if both chains terminate. R is **strongly** π**-regular** if every element of R is strongly π-regular. Strongly π-regular rings are strongly clean [Nicholson, 1999].

P-space

A topological space X is said to be **completely regular** if whenever F is a closed set and x is a point in its complement, there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 1$ and $f[F] = \{0\}$. A completely regular space X is called a **P-space** if every prime ideal in $C(X)$ is maximal.

Finite extension

Let S be a ring and R be a subring of S such that they share the same identity. The ring S is called a **finite extension** of R if S, as an R-module, is generated by a finite set X of generators.

A sufficient condition for $\mathbb{M}_n(C(X))$ to be strongly clean

- **Theorem 1.** Let X be a P-space. Then every finite extension of $C(X)$ is strongly π-regular. In particular, $\mathbb{M}_n(C(X))$ is strongly π-regular, hence strongly clean.

  Proof.

  Let X be a P-space. Then $C(X)$ is regular by [Gillman, 1976]. Since $C(X)$ is commutative, it is strongly regular. By [Hirano, 1990], every finite extension of $C(X)$ is strongly π-regular. $\mathbb{M}_n(C(X))$ is the finite extension of $C(X)$ with generator set $\{E_{ij} : i, j = 1, \cdots, n\}$ where E_{ij} is the matrix with the (i, j)-entry 1 and other entries 0. Hence, $\mathbb{M}_n(C(X))$ is strongly π-regular.

- **Corollary 2.** Let X be a P-space and G a locally finite group. Then $\mathbb{M}_n((C(X)G)[[X]])$ and $\mathbb{M}_n\left(\frac{(C(X)G)[x]}{(x^k)}\right)$ are strongly clean. In particular, $\mathbb{M}_n(C(X))$ is strongly clean.

- **Corollary 3.** If X is a discrete space, then $\mathbb{M}_n(C(X))$ is strongly π-regular, hence is strongly clean.

Rings with stable range one

– A ring R has **stable range one** if $aR + bR = R$ with $a, b \in R$ implies $a + by \in U(R)$ for some $y \in R$.

– **Local rings** have stable range one; **Strongly π-regular rings** have stable range one [Ara, 1996].

– **Question**: Does every strongly clean ring have stable range one [Nicholson, 1999]?

When does $C(X)$ have stable range one?

- A topological space X is called **strongly zero-dimensional** if X is a completely regular Hausdorff space and every finite functionally open cover $\{U_i\}_{i=1}^{k}$ of the space X has a finite open refinement $\{V_i\}_{i=1}^{m}$ such that $V_i \cap V_j = \emptyset$ for any $i \neq j$.

- By the **order** of a family $\mathbb{A}$ of subsets of a set X, we mean the largest integer n such that the family $\mathbb{A}$ contains $n+1$ sets with non-empty intersection, or the "infinite number" ∞ if no such integer exists. The order of a family $\mathbb{A}$ is denoted by ord$\mathbb{A}$. Let X be a space and let n denote an integer ≥ -1, we say:

  1. $dimX \leq n$ if every finite functionally open cover (consisting of non-empty functionally open sets) of the space

X has a finite functionally open refinement of order $\leq n$.

2. $dimX = n$ if $dimX \leq n$ and $dimX \leq n-1$ does not hold.

3. $dimX = \infty$ if $dimX \leq n$ does not hold for any n.

The number $dimX$ is called the **covering dimension** of the space X. By definition, we have $dimX = -1$ iff $X = \emptyset$ and $dimX = 0$ iff X is strongly zero-dimensional.

- **Theorem 4.** Let X be a topological space. Then $C(X)$ is a clean ring iff $C(X)$ has stable range one.

  Proof.

  Following [Vaserstein, 1971], $C(X)$ has stable range one iff $dim X = 0$. By the above definition of covering dimension and its remark, we know $dim X = 0$ iff X is strongly zero-dimensional. So $C(X)$ has stable range one iff $C(X)$ is a strongly clean ring by [Azarpanah, 2002].

- **Corollary 5.** If $C(X)$ has stable range one, then $C(X, \mathbb{C})$ has stable range one.

When does a unital C^*-algebra have stable range one?

- **Theorem 6.** Let R be a unital C^*-algebra in which every element is self-adjoint. Then R is clean iff R has stable range one.

  Proof.

  "$\Rightarrow$". Suppose R is clean. Let $r \in R$. By [Camillo and Yu, 1994], $r = u + v$ where $u \in U(R)$ and $v^2 = 1$. Since $v = v^*$, we have $vv^* = v^*v = 1$, i.e., r is a sum of a unitary (an element $x \in R$ is **unitary** if $xx^* = x^*x = 1$) and a unit. Thus, following [Goodearl and Menal, 1988], R has stable range one.

  "$\Leftarrow$". Suppose R has stable range one. Let $r \in R$. By [Goodearl and Menal, 1988], $r = u + v$ with $u \in U(R)$ and $vv^* = v^*v = 1$.

Since $v = v^*$, we have $v^2 = 1$. Hence, R is clean by [Camillo and Yu, 1994].

- **Question**: Whether the clean property of R implies the clean property of the corner ring eRe [Han and Nicholson, 2001]?

- **Corollary 7.** Let R be a unital C^*-algebra in which every element is self-adjoint. If R is clean, then eRe is clean for any $e^2 = e \in R$.

When is $C(X, \mathbb{C})$ strongly clean?

- **Lemma 8.** $f \in C(X, \mathbb{C})$ is clean iff there exists a clopen set U in X such that $f^{-1}(\{1\}) \subseteq U \subseteq X \backslash z(f)$.

- **Lemma 9.** Let $f \in C^*(X, \mathbb{C})$(the ring of bounded continuous complex valued functions). For each $\alpha \in \mathbb{R}$, set $A_\alpha = \{x \in X : |f(x)| \geq \alpha\}$. If there exist $\alpha, \beta \in \mathbb{R}$ with $0 < \alpha < \beta < 1$ and a clopen set U in X such that $A_\beta \subseteq U \subseteq A_\alpha$, then f is clean in $C^*(X, \mathbb{C})$.

- **Theorem 10.** The following are equivalent:

  1. $C(X, \mathbb{C})$ is a (strongly) clean ring.

  2. $C^*(X, \mathbb{C})$ is a (strongly) clean ring.

3. X is strongly zero-dimensional.

4. $C(X)$ is a (strongly) clean ring.

5. $C^*(X)$ is a (strongly) clean ring.

Proof.

By [Azarpanah, 2002], it suffices to prove the equivalence of $(1), (2)$, and (3).

$(1) \Rightarrow (2)$. Let $f \in C^*(X, \mathbb{C}), A = \{x \in X : |f(x)| \geq \frac{2}{3}\}, B = \{x \in X : |f(x)| \leq \frac{1}{3}\}$. Then A and B are two disjoint zero-sets and they are completely separated. Thus, there exists $g \in C(X, \mathbb{C})$ such that $g(A) = \{1\}$ and $g(B) = \{0\}$. Since g is clean, there exists $e^2 = e \in C(X, \mathbb{C})$ such that $g^{-1}(\{1\}) \subseteq z(e) \subseteq X \backslash z(g)$. By Lemma 9 and $A = A_{\frac{2}{3}} \subseteq g^{-1}(\{1\}) \subseteq z(e) \subseteq X \backslash z(g) \subseteq X \backslash B \subseteq A_{\frac{1}{3}}$, we know f is clean in $C^*(X, \mathbb{C})$.

(2) $\Rightarrow$ (3). Let A and B be two completely separated subsets of X. Then there exists $f \in C^*(X, \mathbb{C})$ such that $|f| \leq \frac{1}{2}, f(A) = \{0\}, f(B) = \{\frac{1}{2}\}$. In this case, $f^{-1}(\{1\}) = \emptyset$ is a clopen set. So f is clean by Lemma 8. By Lemma 8, there exists $e^2 = e \in C(X, \mathbb{C})$ such that $(2f)^{-1}(\{1\}) \subseteq z(e) \subseteq X \backslash z(2f)$. Hence, $B \subseteq (2f)^{-1}(\{1\}) \subseteq z(e) \subseteq X \backslash z(2f) \subseteq X \backslash A$. Since $z(e)$ is clopen, X is strongly zero-dimensional.

(3) $\Rightarrow$ (1). Let $f \in C(X, \mathbb{C})$ and $A = \{x \in X : f(x) = 1\}$. Clearly A is a zero set and $A \bigcap z(f) = \emptyset$. Hence A and $z(f)$ are completely separated. Since X is strongly zero-dimensional, there exists a clopen set $U = z(e)$ for some $e^2 = e \in C(X, \mathbb{C})$ such that $A \subseteq U = z(e) \subseteq X \backslash z(f)$. Thus, $f^{-1}(\{1\}) \subseteq A \subseteq U = z(e) \subseteq X \backslash z(f)$. By Lemma 8, f is clean.