Statistical Mechanics and Gaussian Integrals

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Abstract

A very long random walk, seen from so far away that individual steps cannot be resolved, is the continuous random path called Brownian motion. This is a rough statement of Donsker's theorem and it is an example of how models in statistical mechanics fall into equivalence classes classified by their scaling limits. One quite general way to understand scaling limits is to exploit combinatorial connections with Gaussian Integration and then to use the renormalisation group to study the resulting almost Gaussian integrals.

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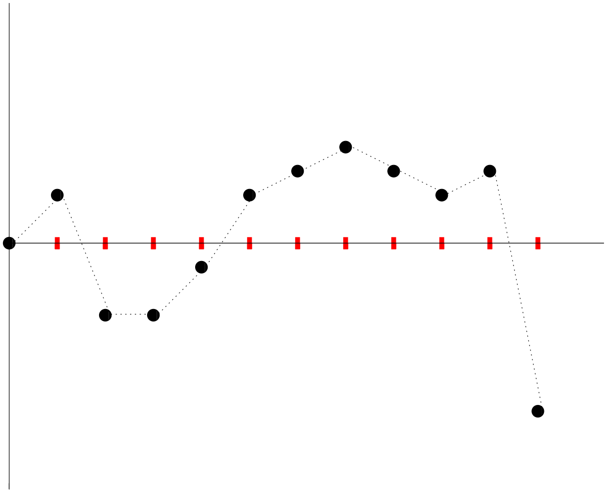

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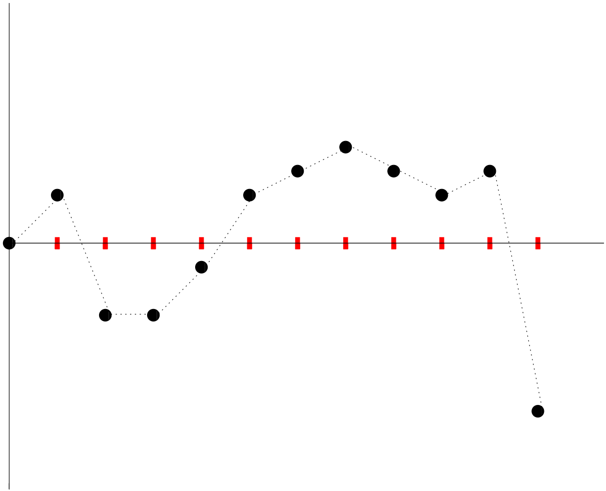

$\mathbb{N}$ is (discrete) time

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Scaling Limit Law for $L^{-\frac{1}{2}}\Phi_{\lfloor Lt \rfloor}$ in limit $L \to \infty$.

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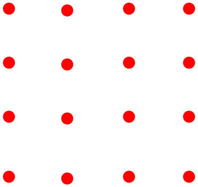

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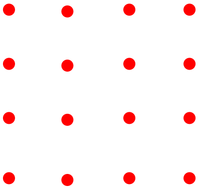

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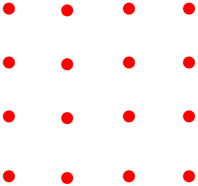

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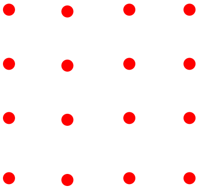

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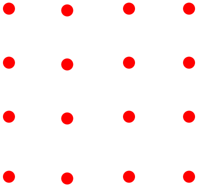

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Then Φ is a sound wave (phonon) in the crystal.

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Intuition

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Intuition $I^\Lambda \approx e^{-\frac{1}{2}\int_\Lambda (\nabla\phi)^2}$

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Brownian motion is case $d = 1$, for which $[\phi] = \frac{1-2}{2} = -\frac{1}{2}$

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5. RG cancels the $\exp(O(\Lambda))$ in numerator and denominator of

$$\frac{1}{\text{Normalisation}} \int \phi_x \phi_y \, I^\Lambda \, d\mu$$

so accurately that we see the correct decay in $x - y$ after taking $\lim_{\Lambda\uparrow}$.

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Proof.

Compare two ways of solving $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u$,

$$(a) \quad u(t,\phi) = \int e^{-\frac{(\phi-\phi')^2}{2t}} P(\phi') \, d\phi',$$

$$(b) \quad u(t,\phi) = e^{\frac{t}{2}\Delta} P(\phi)$$

$\square$

Many variables

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$$\propto \qquad \Delta^2 \phi_a^2 \phi_b^2,$$

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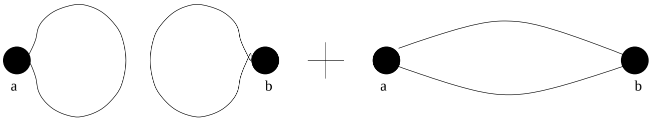
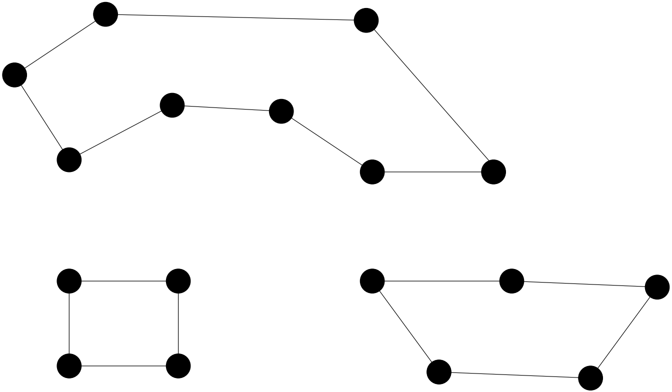
$$=$$

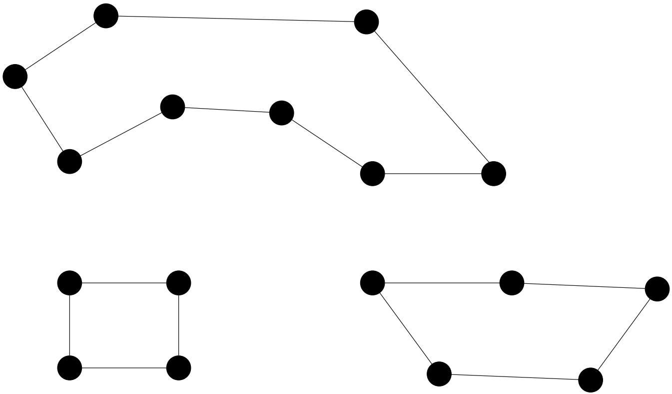


Self-Avoiding Loops

Self-Avoiding Loops



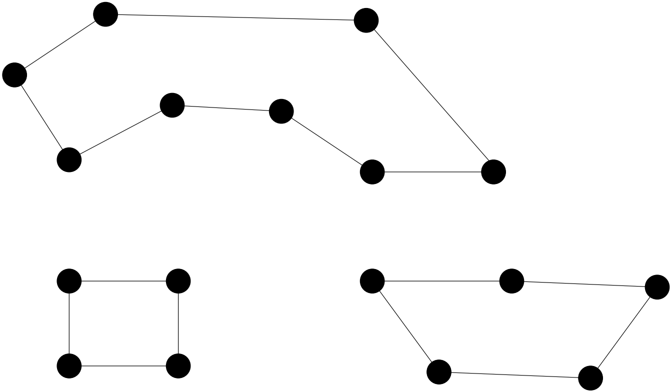
Self-Avoiding Loops



Let

$$I_x = \frac{1}{2}\phi_x^2,$$

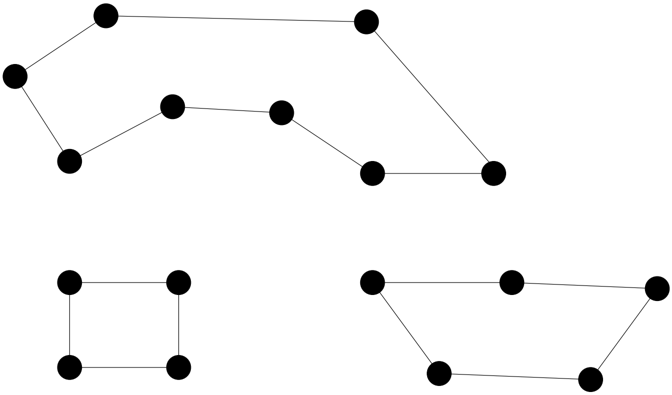
Self-Avoiding Loops



Let

$$I_x = \frac{1}{2}\phi_x^2, \quad \text{respectively} \quad 1 + \frac{1}{2}\phi_x^2.$$

Self-Avoiding Loops



Let

$$I_x = \frac{1}{2}\phi_x^2, \quad \text{respectively} \quad 1 + \frac{1}{2}\phi_x^2.$$

(Thanks to John Imbrie) Evaluate

$$\int e^{-\frac{1}{2}(\phi, A\phi)}\, I^{\Lambda}\ d^{\Lambda}\phi$$

Oriented Loops

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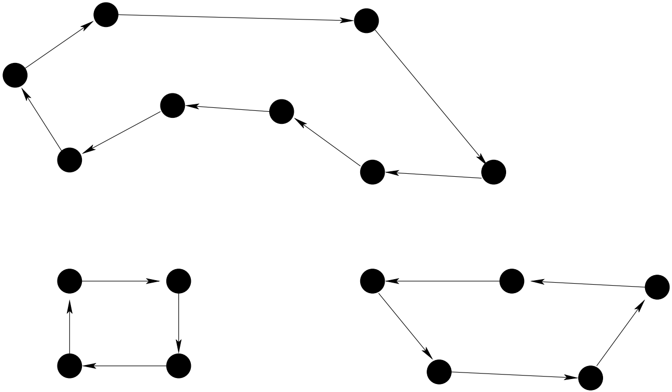
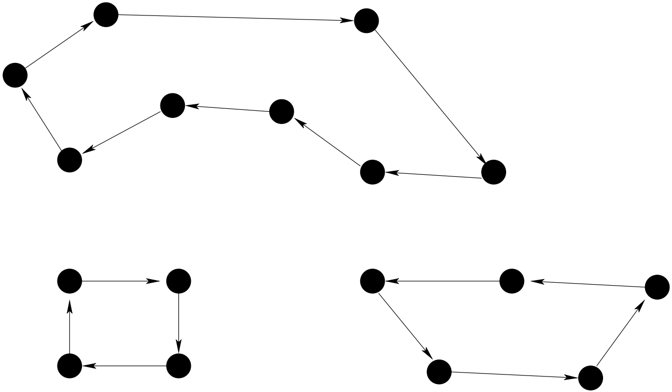


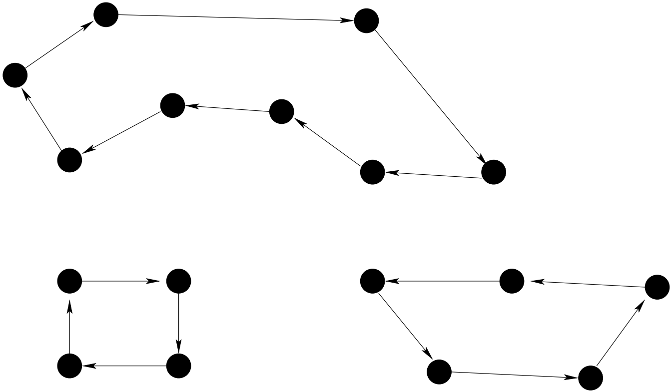
Oriented Loops



Let

$$I_x = c + \phi_x \bar{\phi}_x$$

Oriented Loops



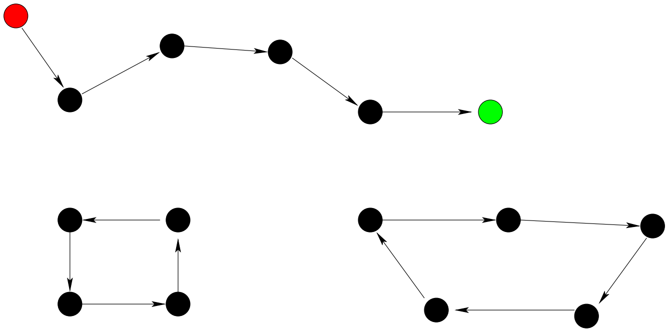
Let

$$l_x = c + \phi_x \bar{\phi}_x$$

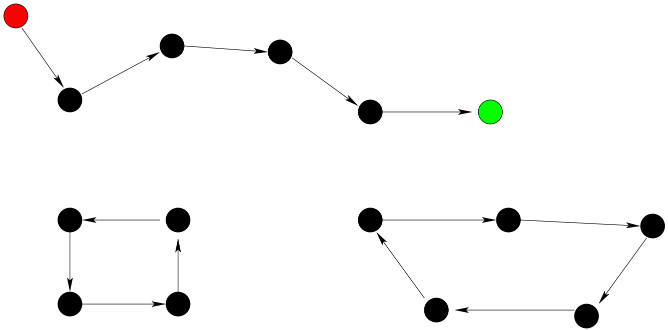
Evaluate

$$\int e^{-(\phi, A\bar{\phi})} \, l^\Lambda \; d^{2\Lambda}\phi$$

Particle finding its way through a sea of loops



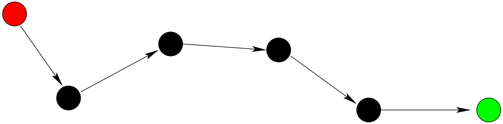
Particle finding its way through a sea of loops



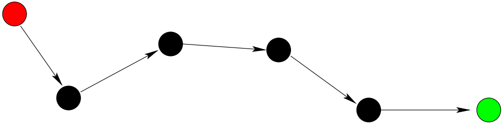
$$\int e^{-(\phi, A\bar{\phi})} \, I^{\Lambda \setminus \{a,b\}} \, \bar{\phi}_a \phi_b \, d^{\Lambda}\phi$$

Self-Avoiding Walk, No Loops, Cohomology and Cosmology

Self-Avoiding Walk, No Loops, Cohomology and Cosmology



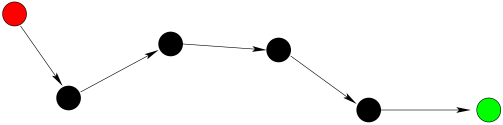
Self-Avoiding Walk, No Loops, Cohomology and Cosmology



Let

$$I_x = 1 + \phi_x \bar{\phi}_x + d\phi_x \wedge d\bar{\phi}_x$$

Self-Avoiding Walk, No Loops, Cohomology and Cosmology



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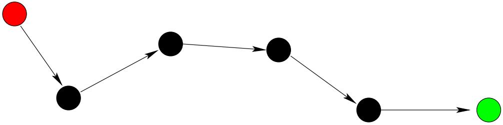
$$e^{-\left(d\phi, A d\bar{\phi}\right)} \stackrel{\text{def}}{=}$$

Self-Avoiding Walk, No Loops, Cohomology and Cosmology



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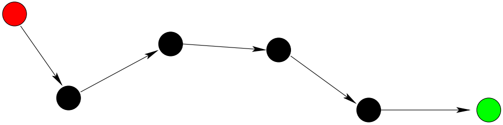
$$e^{-\left(d\phi, A d\bar{\phi}\right)} \overset{\text{def}}{=} \sum \frac{1}{n!} \left(\sum_{x,y} A_{x,y} d\phi_x \wedge d\bar{\phi}_y \right)^{\wedge n}$$

Self-Avoiding Walk, No Loops, Cohomology and Cosmology



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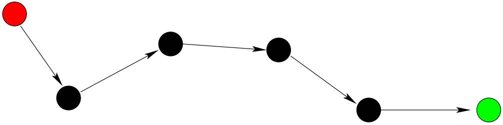
Evaluate

$$\int e^{-(\phi, A\bar{\phi}) - (d\phi, A d\bar{\phi})} \, I^{\Lambda \setminus \{a,b\}} \, \bar{\phi}_a \phi_b$$

Self-Avoiding Walk, No Loops, Cohomology and Cosmology



Let

$$I_x = 1 + \phi_x \bar{\phi}_x + d\phi_x \wedge d\bar{\phi}_x$$

$$e^{-\left(d\phi, Ad\bar{\phi}\right)} \overset{\text{def}}{=} \sum \frac{1}{n!} \left(\sum_{x,y} A_{x,y} d\phi_x \wedge d\bar{\phi}_y \right)^{\wedge n}$$

Evaluate

$$\int e^{-(\phi, A\bar{\phi}) - (d\phi, Ad\bar{\phi})} \, I^{\Lambda \setminus \{a,b\}} \, \bar{\phi}_a \phi_b$$

Supersymmetry $Q = \iota + d$ implies zero vacuum energy.