Stability of Vortex Solutions of the Two and Three Dimensional Navier-Stokes Equations

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Abstract

I will describe how one can combine ideas from dynamical systems theory and kinetic theory to describe the long-time behavior of solutions of the Navier-Stokes equations. In two dimensions this leads to a very complete description of the behavior of solutions whose initial vorticity is at least slightly localized. In three dimensions it give a better understanding of the existence of the Burgers vortex and its variants.

This is joint work with Th. Gallay of the Univ. of Grenoble.

Work supported in part by the US National Science Foundation.

Introduction

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- In physical flows, these structures are often vortices
- From a mathematical point of view these structures may be invariant manifolds in the phase space of the system.

Examples

Vortices are often prominent in fluid flows in the laboratory.

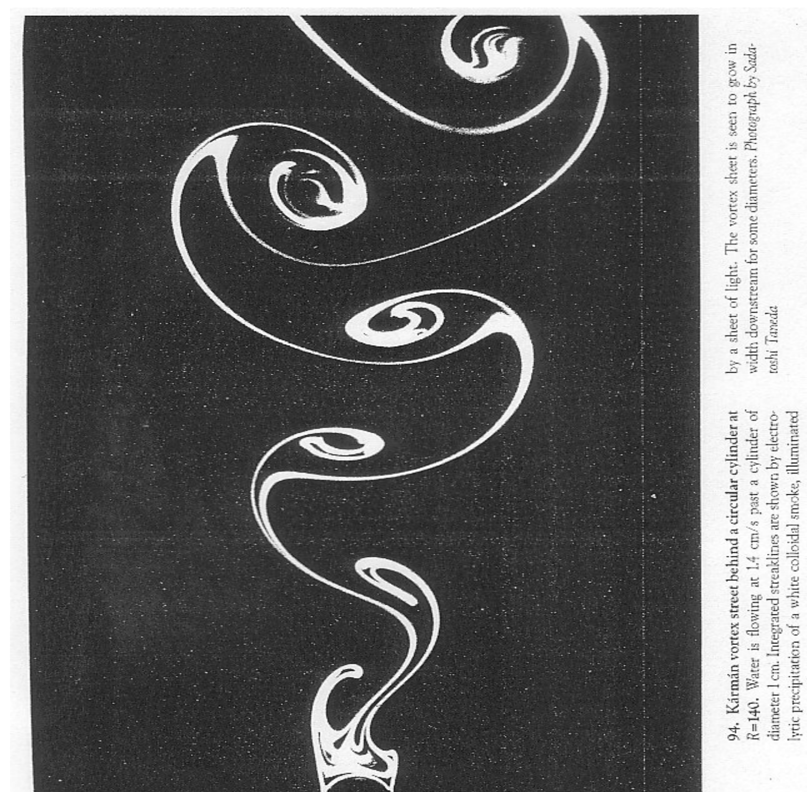


Figure 94 from *An Album of Fluid Motion* (1982) 176 pp, assembled by Milton Van Dyke; Parabolic Press.

Examples (cont.)

More relevant to this talk, vortices are often still obvious structures even after the flow becomes turbulent.

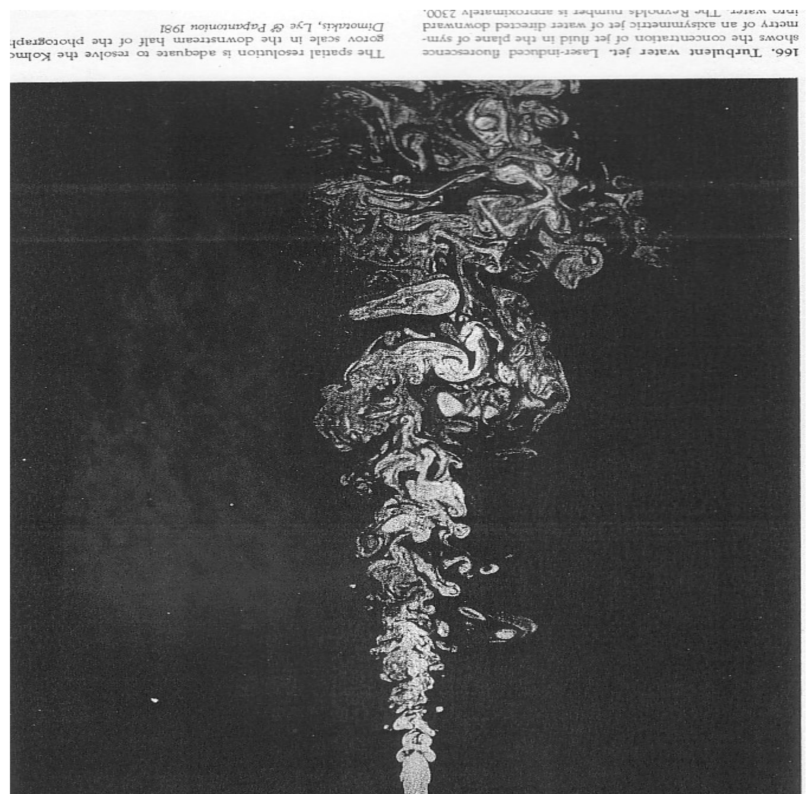
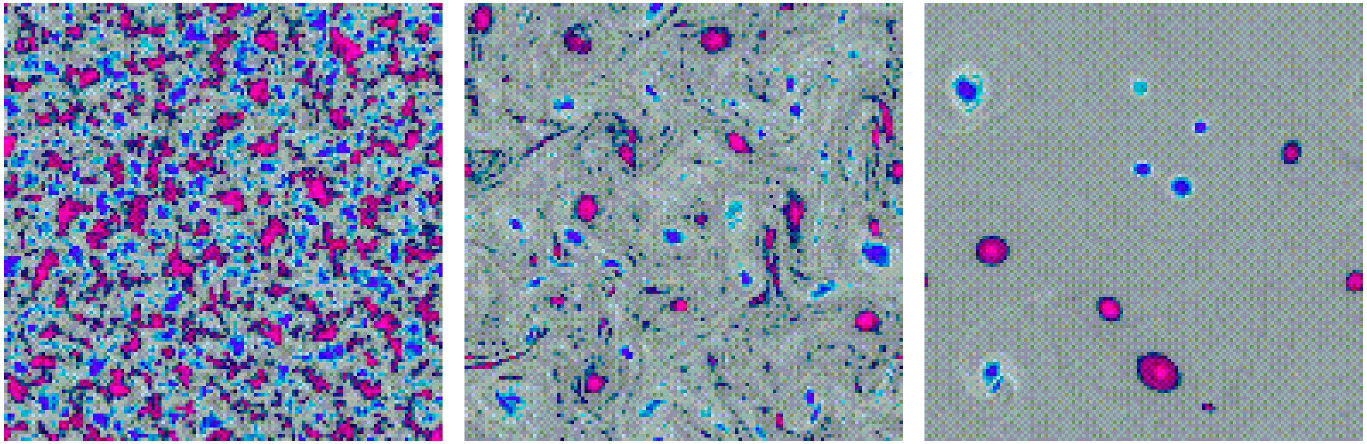


Figure 166 from *An Album of Fluid Motion* (1982) 176 pp. assembled by Milton Van Dyke: Parabolic Press.

Examples (cont.)

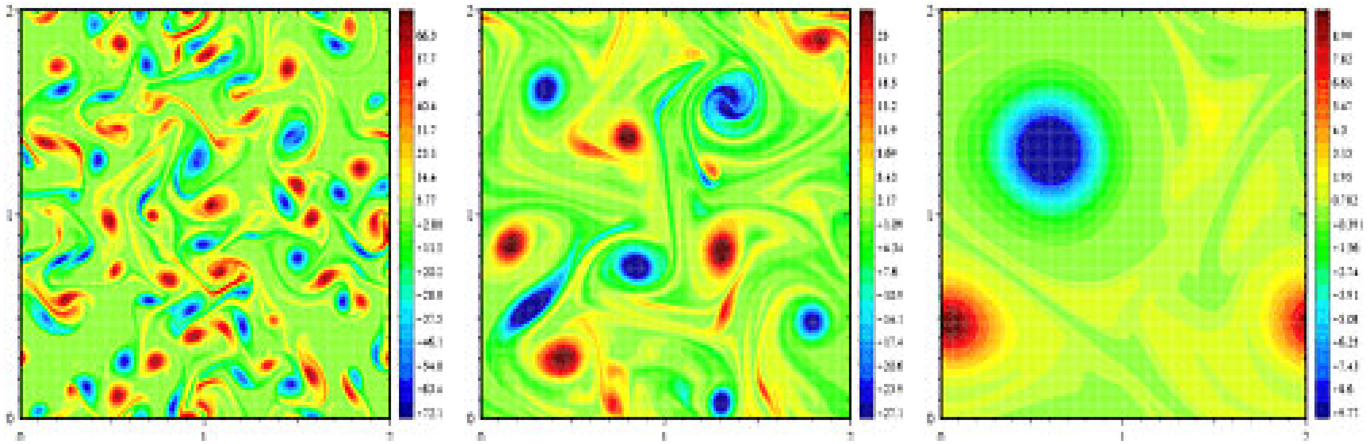
In two dimensional turbulent flows votices undergo a characteristic "coarsening" known as the "inverse cascade".



The images on this slide are from the work of the of the Fluid Dynamics Group, Los Alamos.

`http://gravly.lanl.gov/Turbulence/Dns/2d/2d.html`

Examples (cont.)



The images on this slide are from the work of the of the vortex dynamics group TUE, Netherlands.

`http://www.fluid.tue.nl/WDY/vort/2Dturb/2Dturb.html`

3 Dim. Examples

As we'll discuss later vortices have been used to model three dimensional turbulent flows for many years. There is both experimental and numerical evidence for the importance of vortices in such flows.

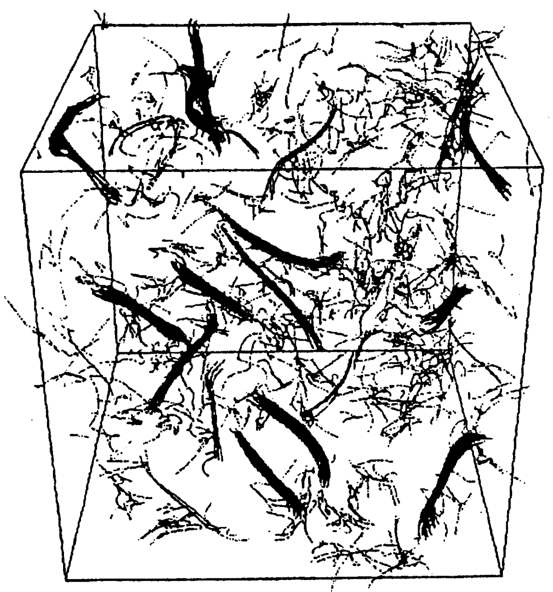


Figure from: She, Jackson, and Orzag; *Proc. R. Soc. London, Ser. A* **434**101-124 (1991).

3 Dim. Examples (cont.)

Another example of the occurrence of such vortex tubes in three-dimensional turbulence is this figure from the work of Jiménez, et. al:

74 *J. Jiménez, A. A. Wray, P. G. Saffman and R. S. Rogallo*

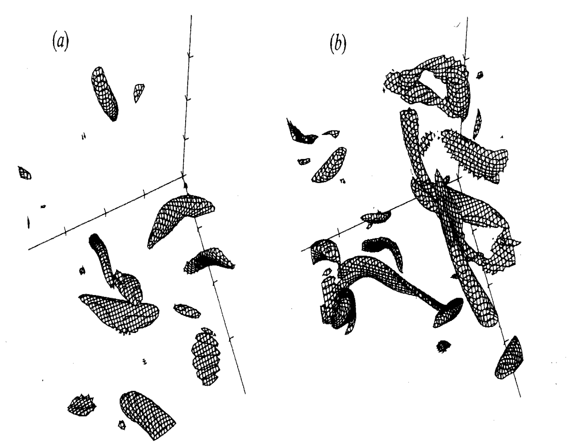


FIGURE 6. Intense-vorticity isosurfaces, $|\omega|/\omega' \geqslant 2.5$, at two different Reynolds numbers. Resolution is similar in both subsets, with the length of each axis $\approx 100\eta$. (*a*) $Re_\lambda = 63$, integral scale $L = 56\eta$; (*b*) $Re_\lambda = 95$, $L = 80\eta$. Thresholds are chosen so that worms contain about 1% of total flow volume.

Figure from: Jiménez, Wray, Saffman and Rogallo; *J. Fluid Mech.* **255** 65-90 (1993).

Two-dim. Navier-Stokes eqn.

A system of nonlinear partial differential equations which describe the motion of a viscous, incompressible fluid.

If $\mathbf{u}(x,t)$ describes the velocity of the fluid at the point x and time t then the evolution of $\mathbf{u}$ is described by:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \nu\Delta\mathbf{u} - \nabla p\ , \quad \nabla\cdot\mathbf{u} = 0\ ,$$

The first of these equations is basically Newton's Law; $F = ma$ while the second just enforces the fact that the fluid is incompressible.

Two-dim. Navier-Stokes (cont.)

We'll begin by focussing on the motion of two dimenional fluids, including possibly turbulent motions.

Although, we live in a three dimensional world, many phenomena are essentially two dimensional – for instance, the behavior of the atmosphere on large scales may for many purposes be treated as two dimensional.

Two dimensional turbulent motions display certain distinctive characteristics which are particularly apparent in this movie from the vortex dynamics group TUE, Netherlands.

`http://www.fluid.tue.nl/WDY/vort/2Dturb/bounded/img/movie_nsbc.gif`

Two-dim. Navier-Stokes (cont.)

We would like to explain the numerical (and experimental) observation that the vorticity of a turbulent two-dimensional flow tends to concentrate itself into isolated vortices. – or more poetically,

> When little whirls meet little whirls,
> they show a strong affection;
> elope, or form a bigger whirl,
> and so on by advection.

This is quoted without attribution on

`http://www.fluid.tue.nl/WDY/vort/2Dturb/2Dturb.html`

2 Dim. Results

In two dimensions we find that the long-time asymptotics of any solution of the Navier-Stokes equation with integrable intial vorticity is governed by a single, explicitly computable votex solution.

Furthermore, the long-time asymptotics of small solutions (or any solution of finite energy) can be computed by restricting the partial differential equation to a finite dimensional, invariant manifold in the infinite dimensional phase space of the equation.

Vorticity

It is convenient to rewrite the Navier-Stokes equations in terms of the vorticity of the fluid rather than the velocity. Roughly speaking, the vorticity describes how much “swirl” there is in the fluid.

$$\omega = \nabla \times \mathbf{u} = (0, 0, \partial_x u_2 - \partial_y u_1) \ .$$

Note that in two dimensions we can treat the vorticity as a scalar!

$$\omega = \partial_x u_2 - \partial_y u_1 \ .$$

Then

$$\omega_t + (\mathbf{u} \cdot \nabla)\omega = \Delta\omega \ ,$$

Biot-Savart

One problem with the vorticity formulation of the Navier-Stokes equation is that the fluid velocity still appears in the nonlinear term. We can recover the velocity given the vorticity via the Biot-Savart law:

$$\mathbf{u}(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(\mathbf{x} - \mathbf{y})^{\perp}}{|x - y|^2} \omega(y) dy \; , \quad x \in \mathbb{R}^2 \; .$$

Here and in the sequel, if $x = (x_1, x_2) \in \mathbb{R}^2$, we denote $\mathbf{x} = (x_1, x_2)^{\mathrm{T}}$ and $\mathbf{x}^{\perp} = (-x_2, x_1)^{\mathrm{T}}$.

Note that this means that the non-linear term is still quadratic (in the vorticity) but now nonlocal.

Oseen vortices

From the simulations we looked at earlier it seems clear that vortex solutions play an important role in two-dimensional fluid motion. There exists a family of explicit vortex solutions of the 2D Navier-Stokes equations known as the Oseen vortices.

$$\Omega^A(x,t) = \frac{A}{4\pi(t+1)} e^{-\frac{x^2}{4(1+t)}} ,$$

with the associated velocity field

$$\mathbf{v}^A(x,t) = \frac{A}{2\pi} \frac{e^{-\frac{x^2}{4(1+t)}} - 1}{|x|^2} \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} .$$

Scaling variables

Note that the formula for the Oseen vortices shows that the size of the vortex increases with time (like $\sqrt{t}$). This is consistent with the simulations we looked at above and suggests that the analysis of these vortices may be more natural in rescaled coordinates. With this in mind we introduce “scaling variables” or “similarity variables”:

$$\xi = \frac{\mathbf{x}}{\sqrt{1+t}} \,, \qquad \tau = \log(1+t) \,.$$

Scaling variables (cont.)

Also rescale the dependent variables. If $\omega(\mathbf{x}, t)$ is a solution of the vorticity equation and if $\mathbf{u}(t)$ is the corresponding velocity field, we introduce new functions $w(\xi, \tau)$, $\mathbf{v}(\xi, \tau)$ by

$$\omega(\mathbf{x}, t) \quad = \quad \frac{1}{1+t}\, w(\frac{\mathbf{x}}{\sqrt{1+t}}, \log(1+t)) \; ,$$

and analogously for $\mathbf{v}$.

Scaling variables (cont.)

In terms of these new variables the vorticity equation becomes

$$\partial_\tau w = \mathcal{L}w - (\mathbf{v} \cdot \nabla_\xi) w \ ,$$

where

$$\mathcal{L}w = \Delta_\xi w + \frac{1}{2}\xi \cdot \nabla_\xi w + w$$

Note that the Oseen vortices take the form

$$W^A(\xi, \tau) = AG(\xi) = \frac{A}{4\pi} e^{-\frac{\xi^2}{4}} \ ,$$

in these new variables. Thus, they are fixed points of the vorticity equation in this formulation.

Dynamical Systems

It is natural to inquire whether or not these fixed points are stable. It turns out (somewhat remarkably) that they are actually globally stable. Any solution of the two-dimensional vorticity equation whose initial velocity is integrable will approach one of these Oseen vortices.

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The proof of this result which draws on ideas from dynamical systems theory combines:

- A local approach, based linearization about the fixed point.
- A global approach based on Lyapunov functionals.

Linearization

We begin with the linearization about the vortex solution. Linearizing about the vortex AG the equation takes the form:

$$\partial_\tau w = \mathcal{L}w + A\Lambda w$$

where

$$\mathcal{L}w = \Delta w + \frac{1}{2}\xi \cdot \nabla w + w$$

and

$$\Lambda w = \mathbf{V}^G \cdot \nabla w + \mathbf{v} \cdot \nabla G$$

In this last expression $\mathbf{V}^G$ is the velocity field associated with the Oseen vortex and $\mathbf{v}$ is the velocity field associated with w via the Biot–Savart law.

The operator $\mathcal{L}$

The analysis of the operator $\mathcal{L}$ is facilitated by the observation that it is can be rewritten as the quantum mechanical harmonic oscillator. In order to compute the spectrum precisely we must specify precisely what function spaces we are working on. For our purposes, square integrable functions with some decay at large distances are appropriate and thus we define:

$$L^2(m) = \left\{ f \in L^2(\mathbb{R}^2) \,|\, \|f\|_m < \infty \right\} ,$$

where

$$\|f\|_m = \left(\int_{\mathbb{R}^2} (1 + |\xi|^2)^m |f(\xi)|^2 d\xi \right)^{1/2} ,$$

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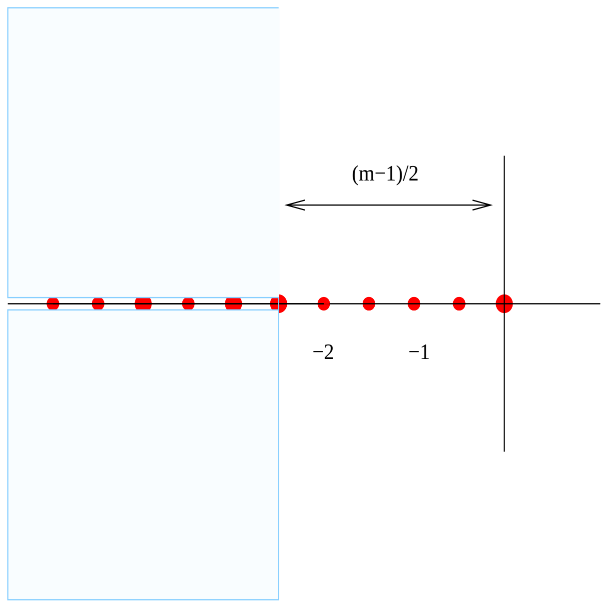

Invariant Manifolds

If we think of this spectral picture in terms of dynamical systems theory we expect that we should be able to construct finite dimensional invariant manifolds tangent at the origin to the eigenspaces of the isolated eigenvalues.

These manifolds reduce the understanding of the long-time asymptotics of these partial differential equations to computing the asymptotics of a finite system of ordinary differential equations.

By this method we are able to compute the long-time asymptotics of solutions of small norm (or in fact, any solution of finite energy) to any inverse power of t.

The operator Λ

In order to analyze the effect of the the second term in the linearized vorticity equation

$$\Lambda w = \mathbf{V}^G \cdot \nabla w + \mathbf{v} \cdot \nabla G$$

we note that:

- Because the effects of Λ are localized it has no effect on the essential spectrum.
- After an appropriate change of coordinates the operator $\mathcal{L}$ is self-adjoint and the operator Λ is anti-self-adjoint.

Roughly speaking this second fact means that the effect of Λ is to move the eigenvalues of $\mathcal{L}$ off the real axis into the complex plane, but not to the right.

Local Stability

With this spectral information the local stability of the Oseen vortices follows easily, namely if one chooses initial conditions for the two-dimensional Navier-Stokes equation which are close to an Oseen vortex the resulting solution converges toward the vortex as $t \to \infty$.

Global Stability

In the two-dimensional case it turns out that these vortices are globally stable. This analysis depends on the existence of a pair of Lyapunov functionals for the two-dimensional vorticity equation.

The first of these is related to the maximum principle and it implies that given any solution whose initial vorticity is integrable the limit of this solution (more precisely, its ω-limit set) must lie in the set of functions which are everywhere non-negative or every where non-positive.

Entropy functional

The other Lyapunov functional is more interesting and is related to similar functionals which arise in kinetic theory – there too one is often looking for convergence to some Gaussian.
In our context this functional takes the form

$$H(\tau) = \int_{\mathbb{R}^2} w(\xi,\tau) \log\left(\frac{w(\xi,\tau)}{G(\xi)}\right) d\xi$$

Proposition $H(\tau)$ *is non-increasing on positive solutions of the rescaled vorticity equation and is strictly decreasing unless* $w(\xi,\tau) = A(\tau)G(\xi)$.

Global Stability

Putting these local and global results together we finally find the following theorem which tells us that any solution of the two-dimensional Navier-Stokes equation whose initial vorticity is integrable eventually converges to one of these Oseen vortices.

Theorem *If* $\omega_0 \in L^1(\mathbf{R}^2)$*, with* $A = \int_{\mathbf{R}^2} \omega_0(x)dx$*, the solution* $\omega(x,t)$ *of the two-dimensional vorticity equation satisfies*

$$\lim_{t\to\infty} t^{1-\frac{1}{p}} |\omega(\cdot,t) - \frac{A}{t} G(\frac{x}{\sqrt{t}})|_p = 0 \, ,$$

for $1 \leq p \leq \infty$*.*

3 Dim. Navier-Stokes

Turning now to the three-dimensional Navier-Stokes we cannot expect such a complete analysis. Indeed, even the existence of smooth, global solutions for general initial data remains an open question.

However, for solutions with small initial data we are still able to describe the long-time asymptotics of solutions in terms of finite-dimensional invariant manifolds as in two-dimensions.

3D Navier-Stokes (cont.)

In 3D, the vorticity equation takes the form:

$$\omega_t + (\mathbf{u} \cdot \nabla)\omega - (\omega \cdot \nabla)\mathbf{u} = \Delta\omega \ ,$$

The extra term in the equation allows for an "amplification" of the vorticity and precludes in general the sort of "relaxation" toward a single vortex that we observed in two-dimensions. This is also evident in the following simulation of a three-dimensional turbulent flow by the research group of Prof. L. Collins.

http://gears.aset.psu.edu/viz/services/projectlist/lance_collins/

3D Navier-Stokes (cont.)

Note that:

- The vortices have a relatively stable shape as they evolve.
- They are not quite circular in cross section.

This second point was also apparent in the figure from Jiménez, et. al that we looked at earlier.

Burgers vortex

These vortices can exist because of a balance between the amplification due to the vortex stretching term and the diffusion due to viscosity.

An explicit example is the Burgers vortex, an exact solution of the Navier-Stokes equation that is is a superposition of a background strain field with a swirling motion in the plane perpendicular to the strain axis.

Burgers vortex (cont.)

The velocity field of the Burgers vortex has the form:

$$\mathbf{U}(x_1, x_2, x_3, t) = \begin{pmatrix} -\frac{\gamma}{2}x_1 \\ -\frac{\gamma}{2}x_2 \\ \gamma x_3 \end{pmatrix} + \begin{pmatrix} u_1(x_1, x_2, t) \\ u_2(x_1, x_2, t) \\ 0 \end{pmatrix} ,$$

where the components u_1 and u_2 of the velocity are exactly the same as those of the Oseen vortex. However, in this case they do not spread with time since the streching due to the background strain field offsets the effects of diffusion.

Burgers vortex (cont.)

Note that the vorticity of the Burgers vortex has only a single non-zero component, and this component is a Gaussian, just as in the case of the Oseen vortex.

$$\mathbf{\Omega}(x_1, x_2, x_3, t) = \begin{pmatrix} 0 \\ 0 \\ \Gamma G(x_1, x_2) \end{pmatrix}, \quad \text{where} \quad \Gamma G = \partial_1 u_2 - \partial_2 u_1 .$$

The explicit formulas for the components u_1 and u_2 of the velocity field are also the same as those of the Oseen vortex (and can be recovered from the vorticity field via the Biot-Savart law.)

Note that there is a two parameter family of Burgers vortices, parameterized by γ and Γ.

Lundgren's transformation

This connection between the Oseen vortex and the Burgers vortex is an example of a remarkable connection between two and three dimensional fbws discovered by Lundgren. Namely, if $\omega(x_1, x_2, t)$ is a solution of the two dimensional Navier-Stokes equations and if $S(t) = exp(\int_0^t \gamma(\tau) d\tau)$, then

$$\mathbf{\Omega}(x_1, x_2, x_3, t) = \begin{pmatrix} 0 \\ 0 \\ S(t)\omega(\sqrt{S(t)}x_1, \sqrt{S(t)}x_2, (\int_0^t S(t')dt')t) \end{pmatrix}$$

is a solution of the three-dimensional Navier-Stokes in a time-dependent background strain field

$$\mathbf{u}^s(x_1, x_2, x_3, t) = \begin{pmatrix} -\frac{\gamma(t)}{2}x_1 \\ -\frac{\gamma(t)}{2}x_2 \\ \gamma(t)x_3 \end{pmatrix}$$

3D/2D connection

In the case where $\gamma(t) = \gamma$ is constant we just recover the Burger's vortex for which the third component of the vorticity ω_3 satisfies

$$\partial_t \omega_3 = \mathcal{L}\omega_3 - (\mathbf{u}^\perp \cdot \nabla^\perp)\omega_3 \ ,$$

where $\nabla^\perp$ is the gradient operator with respect to x_1 and x_2 and $\mathbf{u}^\perp$ is the two dimensional velocity field obtained from ω_3.

This is exactly the same as the two-dimensional vorticity equation we studied earlier rewritten in the rescaled coordinates.

Turbulence modeling

A number of authors have used Burger's vortices to quantitatively model turbulent flow

- Townsend (1951) derived the energy spectrum for a turbulent flow assuming it was a random collection of Burger's vortices and vortex sheets.
- Lundgren (1982, 1993) extended Townsend's work to allow for time dependent strain fields.
- More recently there have been a number of efforts to extend the type of vortex solutions used in these models and to study their stability.

Turbulence modeling (cont.)

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  - They constructed perturbative approximations to these vortices for small Reynolds number and asymmetry parameter.
  - They conducted numerical investigations of their existence up to Reynolds number of about 100.
- Moffatt, Kida and Ohkitani (1994) developed formal asymptotics expansions for the vorticity field for large Reynolds number.
- Prochazka and Pullin (1998) studied numerically the stability of these solutions with respect to two-dimensional perturbations in the plane transverse to the strain axis.

Rigorous results

Using our results from two-dimensions Gallay and I were able to extend some of these results about non-axisymmetric vortices.
We look at solutions of the three-dimensional Navier-Stokes equations in the non-axisymmetric background, strain field:

$$\mathbf{u}^s(x_1, x_2, x_3) = \begin{pmatrix} -\frac{\gamma}{2}(1+\lambda)x_1 \\ -\frac{\gamma}{2}(1-\lambda)x_2 \\ \gamma(t)x_3 \end{pmatrix}$$

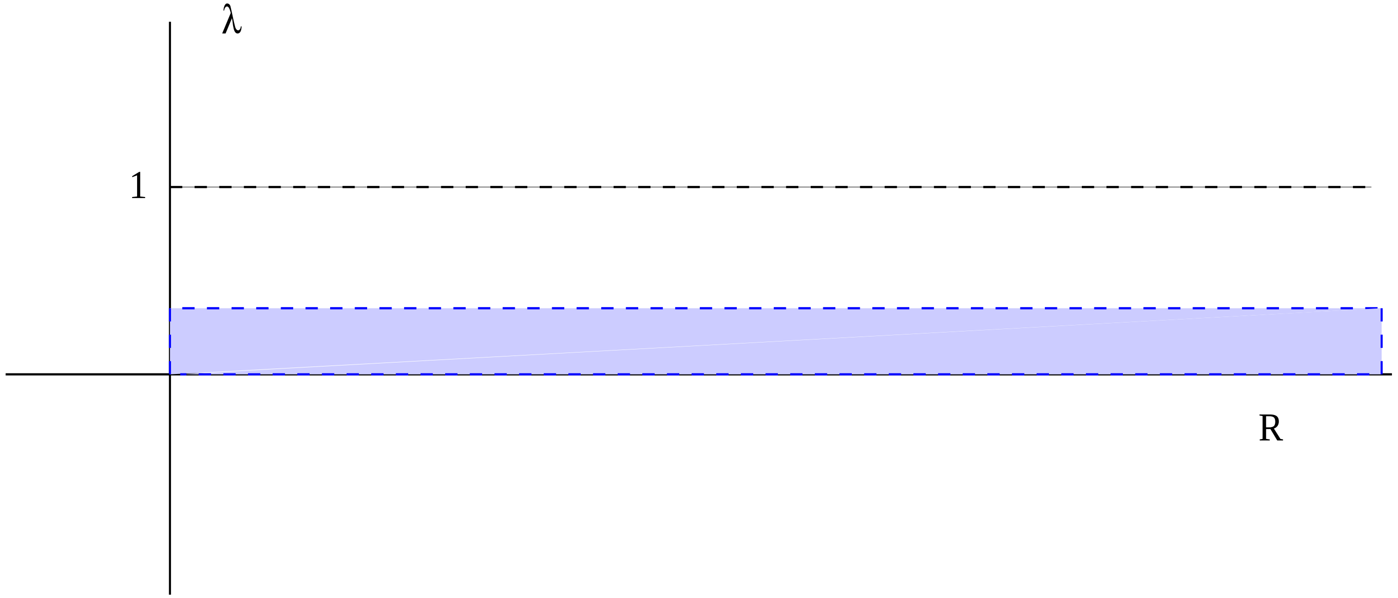
where the asymmetry parameter $\lambda \in [0, 1)$. We begin by looking for stationary solutions of the Navier-Stokes equation of the form

$$\mathbf{\Omega}(x_1, x_2, x_3) = (0, 0, \omega_3(x_1, x_2))^T \ ,$$

in this background field and then we examine their stability.

Rigorous results (Existence)

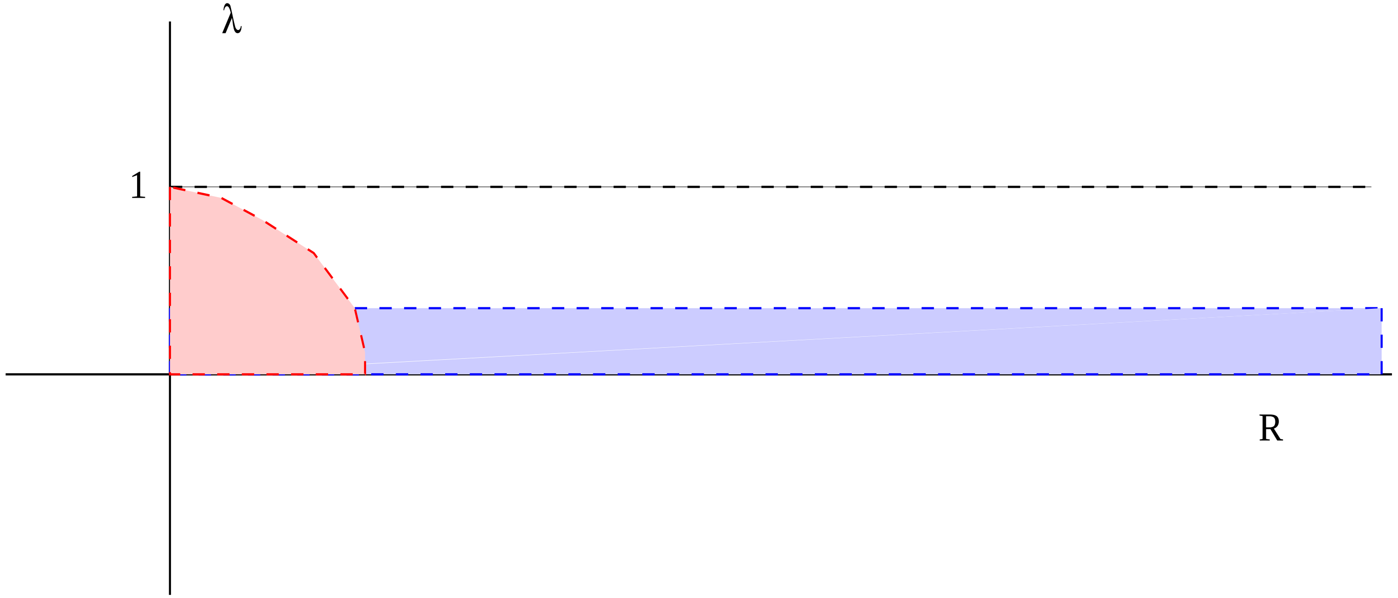
We prove that such axisymmetric vortex solutions exist, if the asymmetry parameter λ is not to large, for all Reynolds numbers.



We recover the formal asymptotics derived by Moffatt, et al in the limit of large Reynolds number.

Rigorous results (Stability)

For any asymmetry parameter $\lambda \in [0, 1)$ we show that for sufficiently small Reynolds number we have an asymmetric vortex solution and that this solution is locally stable with shift with respect to three-dimensional perturbations.



Existence proof

The existence proof is a rigorous perturbation argument, taking as our starting point the known, symmetric Burger's vortex. The difficult part is proving uniformity with respect to the Reynolds number.

- We write the vorticity of the asymmetric vortex as $\omega_3 = \alpha G + w$ (i.e. we regard it as a perturbation of the Burgers vortex. (Here α is proportional to the Reynolds number.)

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- We write the vorticity of the asymmetric vortex as $\omega_3 = \alpha G + w$ (i.e. we regard it as a perturbation of the Burgers vortex. (Here α is proportional to the Reynolds number.)
- w then satisfi es the equation

$$(\mathcal{L} - \alpha)w = \lambda \mathcal{M}(\alpha G + w) + \mathbf{v} \cdot \nabla w$$

where $\mathcal{M}w = (x_1 \partial_{x_1} w - x_2 \partial_{x_2} w)/2$.

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  where $\mathcal{M}w = (x_1 \partial_{x_1} w - x_2 \partial_{x_2} w)/2$.
- Given the information derived earlier about the spectrum of $(\mathcal{L} - \alpha)$ we rewrite this equation as a fixed point problem

$$w = (\mathcal{L} - \alpha\Lambda)^{-1} \Big(\lambda \mathcal{M}(\alpha G + w) + \mathbf{v} \cdot \nabla w \Big)$$

Existence proof (cont.)

One now proves that the fixed point equation for the vorticity has a solution by the contraction mapping theorem.

The uniformity with respect to the Reynolds number comes from analyzing (more or less explicitly) the limit

$$\lim_{\alpha \to \infty} (\mathcal{L} - \alpha\Lambda)^{-1} \mathcal{M}(\alpha G)$$

Stability proof

Once one knows that these asymmetric vortices exist it is natural to ask about their stability.

Their stability with respect to "transverse" perturbations is relatively straightforward to establish using our previous results on stability of two-dimensional vortices and Lundgren's transformation between the two and three dimensional equations.

Stability with respect to perturbations along the axis requires another approach.

For simplicity I'll explain our result in the context of the classical symmetric Burgers vortex.

Stability proof (cont.)

Let $\boldsymbol{\Omega}^{\mathbf{B}}$ be the vorticity of the Burgers vortex and write the vorticity of our perturbed solution as

$$\boldsymbol{\Omega}(\mathbf{x}, t) = \boldsymbol{\Omega}^{\mathbf{B}}(\mathbf{x_1}, \mathbf{x_2}) + \omega(\mathbf{x}, t)$$

Focus on the evolution of the third component of ω

$$\partial_t \omega_3 = (\mathcal{L} + \frac{1}{2} - x_3 \partial_{x_3})\omega_3 + P(\omega) + \mathcal{N}(\omega)$$

Stability proof (cont.)

$$\partial_t \omega_3 = (\mathcal{L} + \frac{1}{2} - x_3 \partial_{x_3})\omega_3 + P(\omega) + \mathcal{N}(\omega)$$

- $\mathcal{L}$ is the linear operator we encountered in analyzing the stability of two-dimensional vortices:

Stability proof (cont.)

$$\partial_t \omega_3 = (\mathcal{L} + \frac{1}{2} - x_3 \partial_{x_3}) \omega_3 + P(\omega) + \mathcal{N}(\omega)$$

- $\mathcal{L}$ is the linear operator we encountered in analyzing the stability of two-dimensional vortices:
- P is a linear term that comes for the interaction of ω_3 with the Burgers vortex around which we perturb:
  - The term $P(\omega)$ will be small for small Reynolds number.

Stability proof (cont.)

$$\partial_t \omega_3 = (\mathcal{L} + \frac{1}{2} - x_3 \partial_{x_3})\omega_3 + P(\omega) + \mathcal{N}(\omega)$$

- $\mathcal{L}$ is the linear operator we encountered in analyzing the stability of two-dimensional vortices:
- P is a linear term that comes for the interaction of ω_3 with the Burgers vortex around which we perturb:
  - The term $P(\omega)$ will be small for small Reynolds number.
- $\mathcal{N}(\omega)$ is the contribution of the nonlinear terms
  - The nonlinear terms will be small if we make a small perturbation of the original vortex

Stability proof (cont.)

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Stability proof (cont.)

Focus on

$$\partial_t \omega_3 = (\mathcal{L} + \frac{1}{2} - x_3 \partial_{x_3}) \omega_3 \ .$$

We can compute an explicit integral representation for the semigroup generated by the equation

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Stability proof (cont.)

We can compute an explicit integral representation for the semigroup generated by the equation

$$\partial_t \omega_3 = (\mathcal{L} + \frac{1}{2} - x_3 \partial_{x_3})\omega_3 \ .$$

Solutions decay exponentially if the initial conditions have zero mean in the transverse direction, i.e if

$$\int_{\mathbb{R}^2} \omega_3(x_1, x_2, x_3, t) dx_1 dx_2 = 0$$

for all x_3.

We force this condition to hold by writing

$$\omega_3(x_1, x_2, x_3, t) = \phi(x_3, t)\mathbf{\Omega}^B(x_1, x_2) + \tilde{\omega}(x_1, x_2, x_3, t)$$

where

$$\phi(x_3, t) = \int_{\mathbb{R}^2} \omega_3(x_1, x_2, x_3, t) dx_1 dx_2 \ .$$

Stability proof (cont.)

Now $\tilde{\omega}$ will decay exponentially in time, but what about ϕ?

Remarkably, the evolution of ϕ decouples completely from the evolution of the other components of the vorticity and one fi nds that

$$\partial_r \phi = \partial_{x_3}^2 \phi - x_3 \partial_{x_3} \phi$$

which has the solution

$$\phi(x_3, t) = (G_t * \phi^0)(x_3 e^{-t}) , \quad x_3 \in \mathbb{R} , \quad t > 0 ,$$

where

$$G_t(y) = \sqrt{\frac{1}{2\pi(1-e^{-2t})}} \exp\left(-\frac{y^2}{2(1-e^{-2t})}\right) , \quad y \in \mathbb{R} , \quad t > 0 .$$

The effect of this formula is to make ϕ approach the constant value

$$\delta\Gamma = (G * \phi^0)(0) .$$

Stability proof (completed)

More physically, we find see that the transverse components of the perturbation decay to zero exponentially rapidly while the vertical component is "smeared out" along the axis of the vortex resulting in a "renormalization" of the Reynolds number of circulation of the vortex as time tends to infinity.

Theorem *Let* $\mathbf{\Omega}(\mathbf{x}, t)$ *be a solution of the three dimensional vorticity equation with initial conditions a sufficiently small perturbation of the Burgers vortex with circulation number* Γ*. As* t *tends to infinity,* $\mathbf{\Omega}(\mathbf{x}, t)$ *tends toward the Burgers vortex with circulation number* $\Gamma' = \Gamma + \delta\Gamma$*.*

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Summary

- Coherent structures, either vortices or invariant manifolds, serve as important organizing features of viscous fluid flows.
- In two dimensions these allow a very complete description of the long-time asymptotics.
- In three-dimensions we can also describe the long-time asymptotics of "small" solutions in terms of finite-dimensional invariant manifolds.
- We also obtain the existence (for all Reynolds numbers) and stability (for low Reynolds numbers) of asymmetric versions of Burgers vortices.

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