

Global Hopf Bifurcation of Differential Equations with State-dependent Adaptive Delay

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Outline

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Introduction

Differential Equations with state-dependent delay

$$\begin{cases} \dot{x}(t) = f(x(t), x(t - \tau(t)), \sigma), \\ \dot{\tau}(t) = g(x(t), \tau(t), \sigma), \end{cases} \quad (1)$$

where $x \in \mathbb{R}^N$, $\tau \in \mathbb{R}$ and $\sigma \in \mathbb{R}$.

Application Background

- ① Stage-structured population model (Aiello, Freedman, Wu);
- ② Commodity price model (Mahaffy, Bélair, Mackey);
- ③ Robots arm control model (Walther);
- ④ Other application: electrodynamics, economics, etc.

Current Progress

- ① Existence, uniqueness, differentiability wrt parameters of the solutions (e.g. Driver; Hartung; Wu; Chen; Hu)
- ② Linearization problem (e.g., Brokate; Colonius; Cooke; Huang)
- ③ Existence of periodic solutions (e.g., Stephan; Smith; Arino; Magal; Li; Kuang; Walther)
- ④ Stability of center manifolds; Attractors (e.g., Krisztin; Walther)
- ⑤ Boundary layer phenomena (Mallet-Paret; Nussbaum)
- ⑥ Local and global Hopf bifurcation (Walther; Markus; Wu; Hu)

Major Obstacle in Qualitative Theory of DESDD

The major problem to develop a topological Hopf bifurcation theory for (1) is that for given constants $r > 0$ and $\tau > 0$, the composite operator

$$\begin{aligned} \chi : C^1([-r, T]; \mathbb{R}^N) \times C^1([0, T]; [0, r]) &\rightarrow C([0, T]; \mathbb{R}^N), \\ \chi(x, \tau)(t) &= x(t - \tau(t)), \quad t \in [0, T], \end{aligned} \quad (2)$$

is generally not a C^1 mapping with respect to τ in the super norm even we restrict the domain of χ to be a $|\cdot|_{C^1}$ bounded subset.

Choosing state space

Results of Melvin, Hale, Hartung, Wu, Chen and Hu have shown that the spaces

$$W^{k,\infty}([-r, T]; \mathbb{R}^N), (k \in \mathbb{N}, 0 \leq r < \infty, 0 < T < \infty)$$

endowed with the norm

$$|\cdot|_{W^{k,p}} \quad (1 \leq p < \infty)$$

is most appropriate to obtain a certain type of differentiability. These spaces are not Banach spaces but **locally complete spaces**.

Definition (Locally complete space, l.c.s.)

Let X be a linear space endowed with the norms $|\cdot|$ and $|\cdot|_M$. We say that $(X, |\cdot|)$ is a **locally complete normed linear space** with respect to the norm $|\cdot|_M$ if every closed ball $\overline{B}_{|\cdot|_M}(0; R)$ ($R > 0$) of X is complete with respect to the $|\cdot|$ -norm.

Remark:

- ① Differentiability can be obtained with local completeness only.
- ② Theory of functional analysis in locally complete space is not available until now.
- ③ Our contribution: fundamental theory in locally complete space: [Baire's category theorem; Inverse mapping theorem; Equivalent norms theorem; Closed graph theorem; Uniform boundedness theorem; Uniform contraction principle]

Linearization problem

Notations (Fix $T > 0$)

- ① $\mathcal{W}^{1,p}$: $W^{1,\infty}(\mathbb{R}/T\mathbb{Z}; \mathbb{R}^{N+1})$ endowed with the $|\cdot|_{W^{1,p}}$ norm;
- ② \mathcal{L}^p : $L^\infty(\mathbb{R}/T\mathbb{Z}; \mathbb{R}^{N+1})$ endowed with the $|\cdot|_{L^p}$ norm.
- ③ $\mathcal{W}^{1,p}$ and \mathcal{L}^p are locally complete linear spaces.

Technical assumptions on the system

$$\{\dot{x}(t) = f(x(t), x(t - \tau(t)), \sigma), \dot{\tau}(t) = g(x(t), \tau(t), \sigma)\}$$

- ① $f \in C^1(\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}; \mathbb{R}^N)$ and $g \in C^1(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}; \mathbb{R})$.
- ② f and g satisfy the Lipschitz conditions.
- ③ $\exists L > 0$ s.t. $g(x, \tau, \sigma) < \frac{L}{L+1}$ for all $(x, \tau, \sigma) \in \mathbb{R}^{N+2}$.

Lemma

Let U be a bounded open admissible subset of $\mathcal{W}^{1,p}$ for the system

$$\begin{cases} \dot{x}(t) = f(x(t), x(t - \tau(t)), \sigma), \\ \dot{\tau}(t) = g(x(t), \tau(t), \sigma), \end{cases} \quad (3)$$

Suppose also that the system satisfies the assumptions. Then

$$F(x, \tau, \sigma)(t) = (f(x(t), x(t - \tau(t)), \sigma), g(x(t), \tau(t), \sigma)),$$

is C^1 in the $\|\cdot\|_{L^p}$ norm with respect to $u = (x, \tau) \in U$.

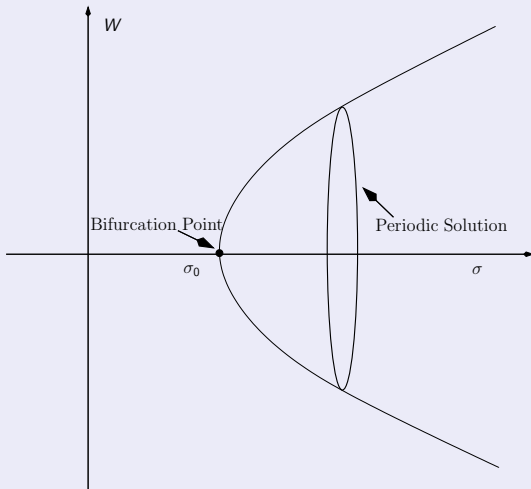
Remark:

- ◆ We can use this lemma to linearize $F(x, \tau, \sigma)(\cdot)$ in $\mathcal{W}^{1,p}$ near the stationary points of the system.

S^1 -Degree and equivariant formulation of Hopf bifurcation

- 1 Degree theory is a topological tool for the detection of solutions of an equation defined over a finite dimensional or infinite dimensional space;
- 2 The standard **normalization, additivity, homotopy invariance** properties are axiomatic properties of degree;
- 3 S^1 -equivariant degree theory is developed for a class of nonlinear mappings preserving the symmetry related to the compact lie group $S^1 = \{z \in \mathbb{C}; |z| = 1\}$;
- 4 S^1 -equivariant degree is one of the main approaches for Hopf bifurcation of DEs;
- 5 Calculation of S^1 -degree can be done near a stationary point if some type of linearization is available.

Hopf Bifurcation



We can rewrite the system equation (1) into $\mathcal{F} : W \times \mathbb{R}^2 \rightarrow W$,

$$\mathcal{F} = u - (L_0 + K)^{-1} \left[\frac{1}{\beta} N_0(u, \sigma, \beta) + Ku \right],$$

where

- ① $u(t) = (x(t/\beta), \tau(t/\beta))$;
- ② The differential operator $L_0 : W \rightarrow V$, W and V are isometric representations of $G = S^1$;
- ③ K is a compact resolvent of L_0 .

Lemma

Let $L_0 : \mathcal{W}^{1,p} \rightarrow \mathcal{L}^p$ be defined by

$$L_0 u(t) = \dot{u}(t)$$

and let $K : \mathcal{W}^{1,p} \rightarrow \mathbb{R}^{N+1} \subset \mathcal{L}^p$ be defined by

$$Ku(t) = \frac{1}{T} \int_0^T u(t) dt.$$

Then $L_0 + K$ has a continuous inverse $(L_0 + K)^{-1} : \mathcal{L}^p \rightarrow \mathcal{W}^{1,p}$.

This lemma is obtained by a version of **Inverse Mapping Theorem** for locally complete spaces.

Crossing number

Let

$$\gamma_{\pm}(u(\sigma_0), \sigma_0, \beta_0) = \deg(\det \Delta_{\sigma_0 \pm \delta}(\cdot), \Omega),$$

then the **crossing number** of the isolated center $(u(\sigma_0), \sigma_0, \beta_0)$ is defined to be

$$\gamma(u(\sigma_0), \sigma_0, \beta_0) := \gamma_- - \gamma_+.$$

where $\Omega := (0, \alpha_0) \times (\beta_0 - \epsilon, \beta_0 + \epsilon)$ is chosen to not contain other zero of the characteristic equation $\det \Delta_{(u(\sigma), \beta)}(\lambda) = 0$.

Crossing numbers are defined to calculate the S^1 -degree of the mapping $\mathcal{F} : W \times \mathbb{R}^2 \rightarrow W$ over each irreducible subrepresentation W_k of W , $(k = 0, 1, 2 \dots)$.

Local Hopf bifurcation of FDE's with Adaptive Delays

Theorem

Suppose the system (1) satisfies the assumptions and the center $(x(\sigma_0), \tau(\sigma_0))$ is isolated. If $\gamma(x(\sigma_0), \tau(\sigma_0), \sigma_0, \beta_0) \neq 0$. Then \exists a bifurcation of nonconstant, periodic solutions near $(x(\sigma_0), \tau(\sigma_0))$. i.e., $\exists \{(x_n, \tau_n, \sigma_n, \beta_n)\}_{n=1}^{\infty}$ s.t. $\sigma_n \rightarrow \sigma_0$, $\beta_n \rightarrow \beta_0$ as $n \rightarrow \infty$, and

$$\lim_{n \rightarrow \infty} |x_n - x(\sigma_0)|_{W^{1,2}} = 0, \quad \lim_{n \rightarrow \infty} |\tau_n - \tau(\sigma_0)|_{W^{1,2}} = 0,$$

where

$$(x_n, \tau_n) \in (W^{1,\infty}([0, 2\pi/\beta_n]; \mathbb{R}^{N+1}); |\cdot|_{W^{1,2}})$$

is a nonconstant $2\pi/\beta_n$ -periodic solution.

Global Hopf bifurcation of FDEs with Adaptive Delays

Let \mathcal{S} denote the closure of the set of all nontrivial periodic solutions of system (1) in the space $\mathcal{W}^{1,2} \times \mathbb{R} \times \mathbb{R}_+$. Assume further that $\exists M > 0$ such that

$$-M \leq g(x, \tau, \sigma) < 1, \text{ for any } (x, \tau, \sigma).$$

We can obtain the following global Hopf bifurcation of DESDD of Robinowitz type.

Theorem

Suppose that system (1) satisfies all the assumptions. Let M be the set of trivial periodic solutions of (1). Assume that all stationary points of (1) are not singular and all the centers are isolated. If $(u_0, \sigma_0, \beta_0) \in M$ is a bifurcation point, then either the connected component $C(u_0, \sigma_0, \beta_0)$ of (u_0, σ_0, β_0) in \mathcal{S} is unbounded, or the number of bifurcation points in $C(u_0, \sigma_0, \beta_0)$ is finite, that is,

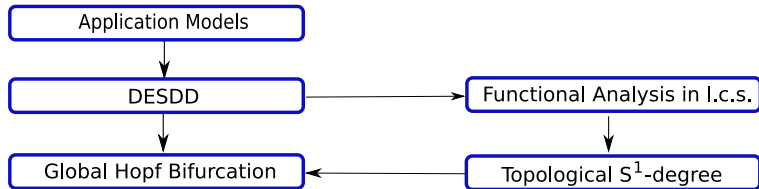
$$C(u_0, \sigma_0, \beta_0) \cap M = \{(u_0, \sigma_0, \beta_0), (u_1, \sigma_1, \beta_1), \dots, (u_q, \sigma_q, \beta_q)\},$$

where $q \in \mathbb{N}$. Moreover, in the latter case, we have

$$\sum_{i=1}^q \gamma(u_i, \sigma_i, \beta_i) = 0.$$

Conclusions

- 1 Linearization of DEs with state-dependent delay can be implemented in locally complete space;
- 2 S^1 -equivariant degree can be applied to the analysis of global Hopf bifurcation of DEs with state-dependent delay with the aid of the established functional analysis in l.c.s.



Selected references

- ① **Survey paper:** Hartung, F., Krisztin, T., Walther, H. -O., and Wu, J., Functional Differential Equations with State-dependent Delays: Theory and Applications, Handbook of Differential Equations: Ordinary Differential Equations, (Volume 3), Elsevier, North Holland (A. Canada eds.), 2006.
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Thank you!