Small random perturbation of critical dynamical systems

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OBJECTIVE

Our goal is to study the asymptotic behavior of 1–dimensional dynamical systems perturbed by a weak random noise.

We consider systems of the form

$$x_{n+1} = f(x_n) + \sigma \epsilon_{n+1}, \tag{1}$$

where

1. $f : M \longrightarrow M$ and $M = [-1, 1]$, $\mathbb{R}^1$, or $\mathbb{T}^1$,
2. $(\epsilon_n : n \in \mathbb{N})$ is a sequence of independent random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$
3. $\sigma > 0$ is small.

A BRIEF INTRODUCTION

1. Systems of these type have been consider heuristically. In particular Crutchfield, et. al and Shraiman, et. al (1981,1982) looked at Gaussian perturbations of the fixed point of the doubling period and relations for the standard deviation of the propagation of error.
2. Rigorous results for the scaling relations found in the papers above are done in Khanin, et. al (1984) based on the Thermodynamic formalism.
3. Collet and Lesne (1989) presented a renormalization group analysis for the study of accumulation of period doubling with noise.
4. Most recently, Isaeva, et. al. (2004) studied the effects of small noise of a complex map at the period–tripling accumulation point.

A GENERAL CLT FOR 1-D MAPS

Let us denote by $x_n(x_0, \sigma)$, the value at time n of the process x starting at x_0, for which the scale σ has been fixed.

We are interested in the behavior of $x_n(x_0, \sigma)$ for small noise level σ and large time n.

We define the Lindeberg–Lyapunov functionals

$$\begin{aligned}
\Lambda_p(x,n) &= \sum_{j=1}^{n} \left| \left(f^{n-j}\right)' \circ f^j(x) \right|^p \\
\widehat{\Lambda}(x,n) &= \max_{0 \le i \le n} \sum_{j=1}^{i} \left| \left(f^{i-j}\right)' \circ f^j(x) \right|
\end{aligned}$$

Under the general assumption that

$$0 < c \le \mathbb{E}^{1/2}[\varepsilon_j^2] \le \mathbb{E}^{1/p}[|\varepsilon_j|^p] \le C < \infty$$

we have the following result

Theorem 1. *(Diaz-de la Llave) Assume that for some $x \in M$ and $p > 2$*

$$\lim_{k\to\infty} \frac{\Lambda_p(x, n_k)}{(\Lambda_2(x, n_k))^{p/2}} = 0 \tag{2}$$

Then, for any sequence σ_k decreasing to 0 such that

$$\lim_{k\to\infty} \frac{\hat{\Lambda}^3_{n_k}(x)}{\sqrt{\Lambda_2(x, n_k)}} \mathbb{E}\left[\max_{1\le j\le n_k} \epsilon_j^2\right] \sigma_k = 0,$$

there exists events $B_k \in \mathcal{F}$ with $\lim_k \mathbb{P}[B_k] = 1$ such that

$$w_{n_k} = \frac{x_{n_k}(x, \sigma_k) - f^{n_k}(x)}{\sqrt{\mathrm{var}[x_{n_k}(x, \sigma_k) - f^{n_k}(x)]}} \mathbf{1}_{B_k} \Longrightarrow N(0, 1)$$

REMARKS

(i) An important property of the Lindeberg–Lyapunov functionals that works nice for renormalization is

$$\Lambda_p(x, n_1 + n_2) = \left\{ (f^{n_2})' \left(f^{n_1}(x)\right)\right\}^p \Lambda_p(x, n_1) + \Lambda_p(f^{n_1}(x), n_2)$$

(ii) The standard CLT corresponds to taking $f(x) = x$. In this case, the each outliers is empty. For more complicated f's these outliers may have positive, though small, probability.

(ii) When the point x_0 is hyperbolic, then hypothesis 2 is not satisfied. Systems with enough hyperbolicity satisfy other types of limit theorems even in the absence of noise Liverani (1995), or for weak noise Kiffer (1996)

Example: Consider the map on $\mathbb{R}$ (or $\mathbb{T}^1$) given by $f(x) = 2x$. There is a limit for the scaled noise which depends on the distribution of the sequence ϵ_n.

If (ϵ_n) is an i.i.d $U[-1, 1]$ sequence, then

$$\frac{x_n(x_0, \sigma_n) - 2^n x_0}{3^{-1}\sqrt{4^n - 1}} \Longrightarrow \xi$$

where ξ has characteristic function

$$\phi(z) = \prod_{n=1}^{\infty} \frac{\sin(2^{-n}3z)}{2^{-n}3z}$$

If (ε_n) is an i.d.d. standard normal sequence, we have a Gaussian behavior.

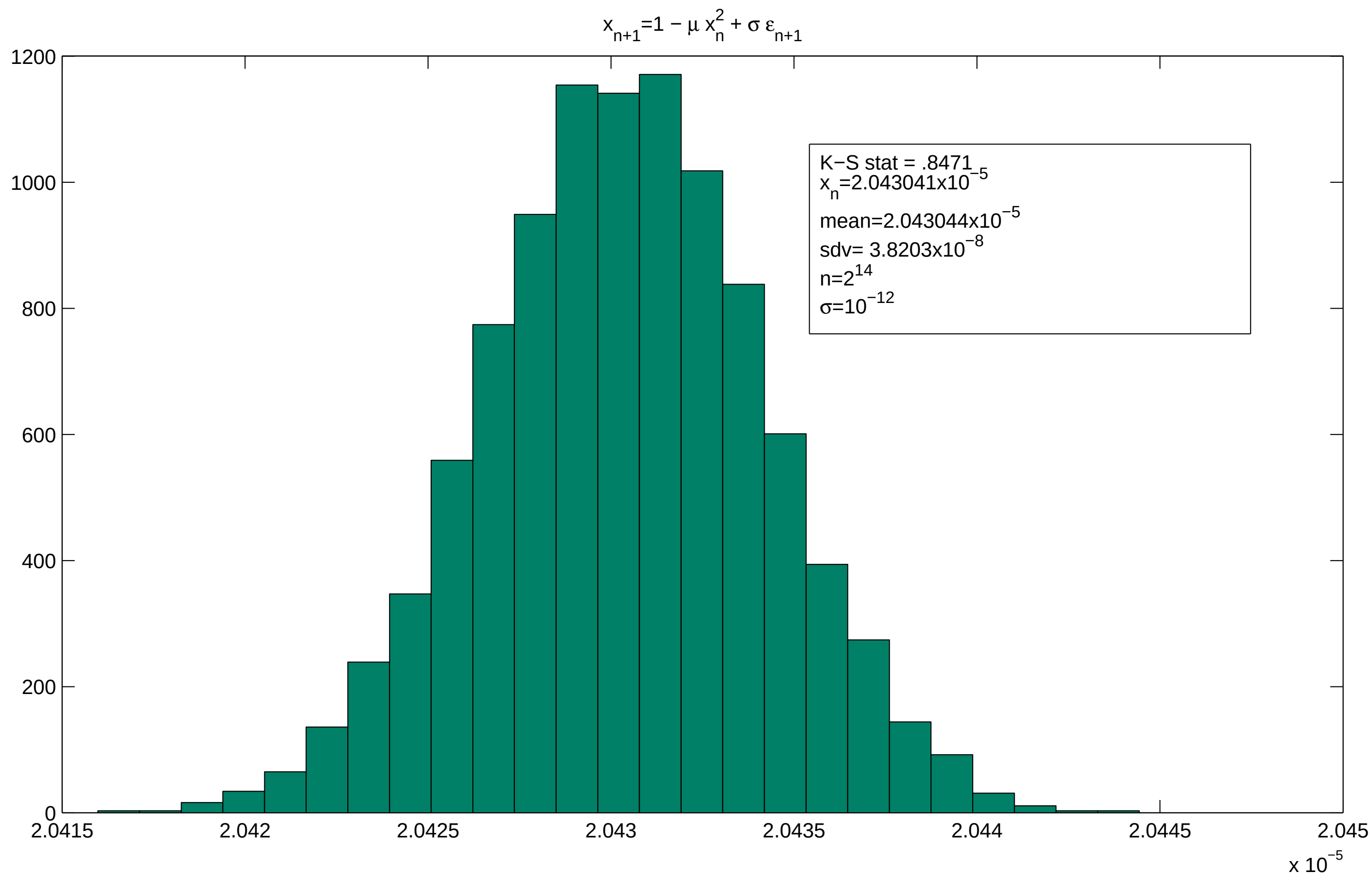
Example: Smooth diffeomorphisms of the circle with whose rotation numbers satisfy a Diophantine condition are smoothly conjugate to linear rotations (Herman, Yoccoz, Khanin). Hence, for this systems, the CLT holds.

Example: One important class of systems satisfying condition (2) are those given by maps in the stable manifold of the the fixed point of period doubling RG operator, as well as analytic maps of the circle with one singularity, whose rotation number is the golden mean.

For instance, for the critical map

$$f(x) = 1 - \mu x^2$$

with $\mu = 1.40115519...$, numerical experiments suggest the existence of a Gaussian limiting behavior for $x_{2^n}(x, \sigma_n)$ with σ_n small.


$x_{n+1}=1-\mu x_n^2+\sigma\,\varepsilon_{n+1}$
K−S stat = .8471
$x_n=2.043041\times10^{-5}$
$\text{mean}=2.043044\times10^{-5}$
$\text{sdv}=3.8203\times10^{-8}$
$n=2^{14}$
$\sigma=10^{-12}$
1200
1000
800
600
400
200
0
2.0415
2.042
2.0425
2.043
2.0435
2.044
2.0445
2.045
$\times 10^{-5}$

SKETCH OF THE PROOF OF THE GENERAL CLT

We divide the proof in tow parts.

LINEAR THEORY

First we use a linear approximation of the map f using Taylor expansion. Notice that system (1) can be expressed as

$$x_n = f^n(x) + \sigma L_n(x,\varepsilon) + \sigma^2 Q_n(x,\varepsilon,\sigma)$$

The term L_n is a sum of independent random variables with weights

$$L_n(x,\varepsilon) = \sum_{k=1}^{n} \left(f^{n-k}\right)' \circ f^k(x)\varepsilon_k$$

The term Q_n contains the nonlinear propagation of noise.

The condition (2) implies that the linear propagation

$$y_n(x, \sigma) = f^n(x) + \sigma L_n(x, \epsilon)$$

satisfies the following CLT:

$$\frac{y_{n_k}(\sigma) - f^{n_k}(x)}{\sqrt{\text{var}[y_{n_k}(\sigma) - f^{n_k}(x)]}} \Longrightarrow N(0, 1)$$

Observe that this result does not depend on the scale σ.

NON–LINEAR THEORY

(i) To complete the proof we show that the linear approximation y_n and the nonlinear process x_n are closed to each other when σ is properly chosen. This is done in a similar way as in the analysis of variational equations for ordinary differential equations.

(ii) The control of nonlinear terms imply the existence of outliers, sets of small probability where fluctuations of noise are large.

FEIGENBAUM FIXED POINT

The doubling period RG transformation T defined by

$$Tf(x) = \frac{f^{(2)}(\lambda_f x)}{\lambda_f} \quad \text{with} \quad \lambda_f = f(f(0))$$

acts on the space $\mathcal{P}$ of unimodal symmetric maps on the unit interval that satisfy

(a) $f(0) = 1$

(b) $f \in C^1$, $xf'(x) < 0$ for $x \neq 0$

(c) $0 < a_f < b_f$ and $f(b_f) < a_f$, with $a_f = -f(f(0))$ and $b_f = f(a_f)$

By induction, it can be seen that

$$T^n f(x) = \frac{f^{2^n}(\Gamma_n^f x)}{\Gamma_n^f}$$

where $\Gamma_n^f = f^{2^n}(0)$. Also,

$$\lambda_{T^{n-1}f} = \frac{\Gamma_n^f}{\Gamma_{n-1}^f} = \frac{f^{2^n}(0)}{f^{2^{n-1}}(0)}$$

It is known (Lanford, 1984) that

1. There is a function g, analytic and even on $V = \{z \in \mathbb{C} : \|z\| < \sqrt{8}\}$ whose restriction to I is a fixed point of T. Restricted to I, g is concave.

2. There is a neighborhood $\mathcal{V}$ of g on the space of even analytic functions having value 1 at 0 where T is differentiable.

3. For $f \in \mathcal{V}$, the derivative $DT(f)$, acting on the space of even functions that vanish at 0, is a compact operator.

4. DT is hyperbolic with one–dimensional expanding subspace. The eigenvalue δ associated to this expanding subspace, δ is larger that one.

CLT FOR THE FEIGENBAUM FIXED POINT

Theorem 2. *Assume ϵ_n is a sequence of independent random variables with uniformly bounded moments of order $p > 2$. There is $\gamma > 1$ such that if $x \in C_g$ and σ_n is such that*

$$\mathbb{E}[\max_{1\leq j\leq n} \varepsilon_j^2]\sigma_n n^{3(\gamma+1)} = 0(1)$$

then, there is a sequence of sets $B_n \in \mathcal{F}$ with large probabilities such that

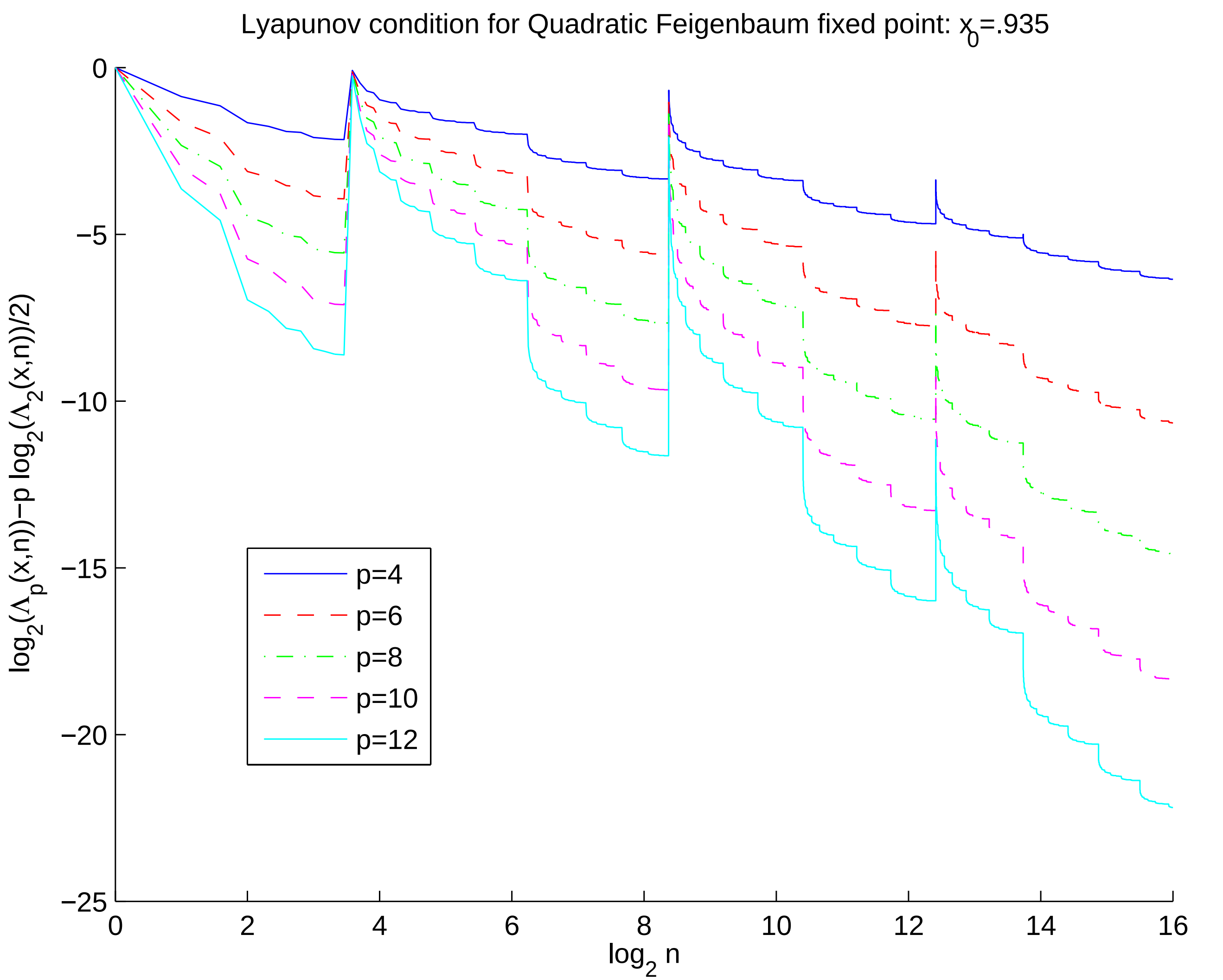
$$\frac{x_n(x,\sigma_n) - g^n(x)}{\sigma_n\sqrt{\Lambda_2(x,n)}}\mathbf{1}_{B_n} \Longrightarrow N(0,1)$$

The key argument for a CLT for maps close to the fixed point g on the stable manifold of T is

Theorem 3. *For $x \in \mathcal{C}_g$ we have*

$$\lim_{n\to\infty} \frac{\Lambda_p(x,n)}{\{\Lambda_2(x,n)\}^{p/2}} = 0$$

1. First we show this is true for $x = 0$ through the sparse subsequence 2^n by analyzing the spectrum of the cumulant operators K_p.
2. Extend the limit to all steps n by renormalization techniques
3. Extend limit to all points in $\{g^k(0)\}$ again by renormalization techniques.


Lyapunov condition for Quadratic Feigenbaum fixed point: x_0=.935
$\log_2(\Lambda_p(x,n))-p \log_2(\Lambda_2(x,n))/2)$
$\log_2 n$
0
−5
−10
−15
−20
−25
0
2
4
6
8
10
12
14
16
p=4
p=6
p=8
p=10
p=12

CUMULANTS

Definition 1. The *cumulant generating function* (c.g.f.), ϕ_X of a random variable X is defined as

$$\psi_X(t) := \log \mathbb{E}\left[e^{itX}\right]$$

The *m–th order cumulant* of X, $\kappa_n(X)$, is defined as the n–th coefficient in formal Taylor of $\psi_X(t) = \sum_{k=1}^{\infty} \frac{\kappa_n}{n!}(it)^n$.

Clearly

$$\psi_{aX+Y}(t) = \psi_X(at) + \psi_Y(t)$$
$$\kappa_n(aX+Y) = a^n \kappa_n(X) + \kappa_n(Y)$$

ANALYSIS OF CUMULANTS AND RENORMALIZATION

Recall the variational equation

$$y_{2^n} = f^{2^n}(x) + \sigma \sum_{j=1}^{n} \left(f^{n-j}\right)' \circ f^j(x)\varepsilon_j$$

Let $W_p^n(x)$ be the $p-th$ order cumulant of y_{2^n}. By independence of ε, one can show that

$$W_p^n(x) \approx \left\{ \left(f^{2^{n-1}}\right)' \circ f^{2^{n-1}}(x) \right\}^p W_p^{n-1}(x) + W_p^{n-1} \circ f^{2^{n-1}}(x)$$

Consider the renormalized cumulants

$$\widetilde{W}_p^n(z) = |\Gamma_n^f|^{-p} W_p^n(\Gamma_n^f z)$$

For each $f \in \mathcal{W}_s(g)$, define the family of *positive cumulant operators* $K_{f,p}$ by

$$K_{f,p}h(y) = \frac{1}{a_f^p} \{\{-f' \circ f(a_f y)\}^p h(a_f y) + h \circ f(a_f y)\}$$

Then

$$\widetilde{W}_p^n(z) = K_{T^{n-1}f,p} \circ \cdots \circ K_{f,p} 1(z)$$

Lyapunov's condition (2) for initial condition near zero can be expressed in terms of the positive cumulant operators by

$$\lim_{n\to\infty} \frac{K_{T^{n-1}f,p} \circ \cdots \circ K_{f,p} 1(z)}{\left\{K_{T^{n-1}f,2} \circ \cdots \circ K_{f,2} 1(z)\right\}^{p/2}} = 0 \tag{3}$$

If f is close enough to g then the operators $K_{f,p}$ are well defined as operators on space of real analytic functions $H^r = H^r(U)$ onto itself, where for $U \subset V$ a strip around I.

SPECTRAL ANALYSIS OF THE CUMULANT OPERATORS

By Montel's theorem these operators are compact. Furthermore, if C be the cone of nonnegative functions on H_r then each operator $K_p = K_{f,p}$ satisfies

(i) $K_m(C \setminus \{0\}) \subset C \setminus \{0\}$

(ii) $K_m(\mathrm{int}(C)) \subset \mathrm{int}(C)$, and

(iii) for each $f \in C \setminus \{0\}$, there is an integer $n = n(m, f)$ such that $K_m^n f \in \mathrm{int}(C)$.

By slight variation on the Krein–Rutman, we have

Theorem 4. *For each K_p*

(a) the spectral radius ρ_p of K_p is a positive simple eigenvalue of K_p;

(b) The eigenvector $f_p \in X \setminus \{0\}$ associated with ρ_p can be taken in int(C);

(c) if μ is in the spectrum of K_p, $0 \neq \mu \neq \rho_p$, then μ is an eigenvalue of K_p and $|\mu| < \rho_p$;

(d) if $h \in C \setminus \{0\}$ is an eigenvector of K_p, then the corresponding eigenvalue is ρ_p.

A consequence of the theorem is that for any $h \in C \setminus \{0\}$, there is a constant $c_p = c_p(h) \neq 0$ such that

$$\lim_{n \to \infty} \frac{K_p^n h}{\rho_p^n f_p} = c_p$$

uniformly. Here f_p is a positive eigenfunction of K_p.

The following result gives a comparison of the sizes among the spectral radii of the operators K_p

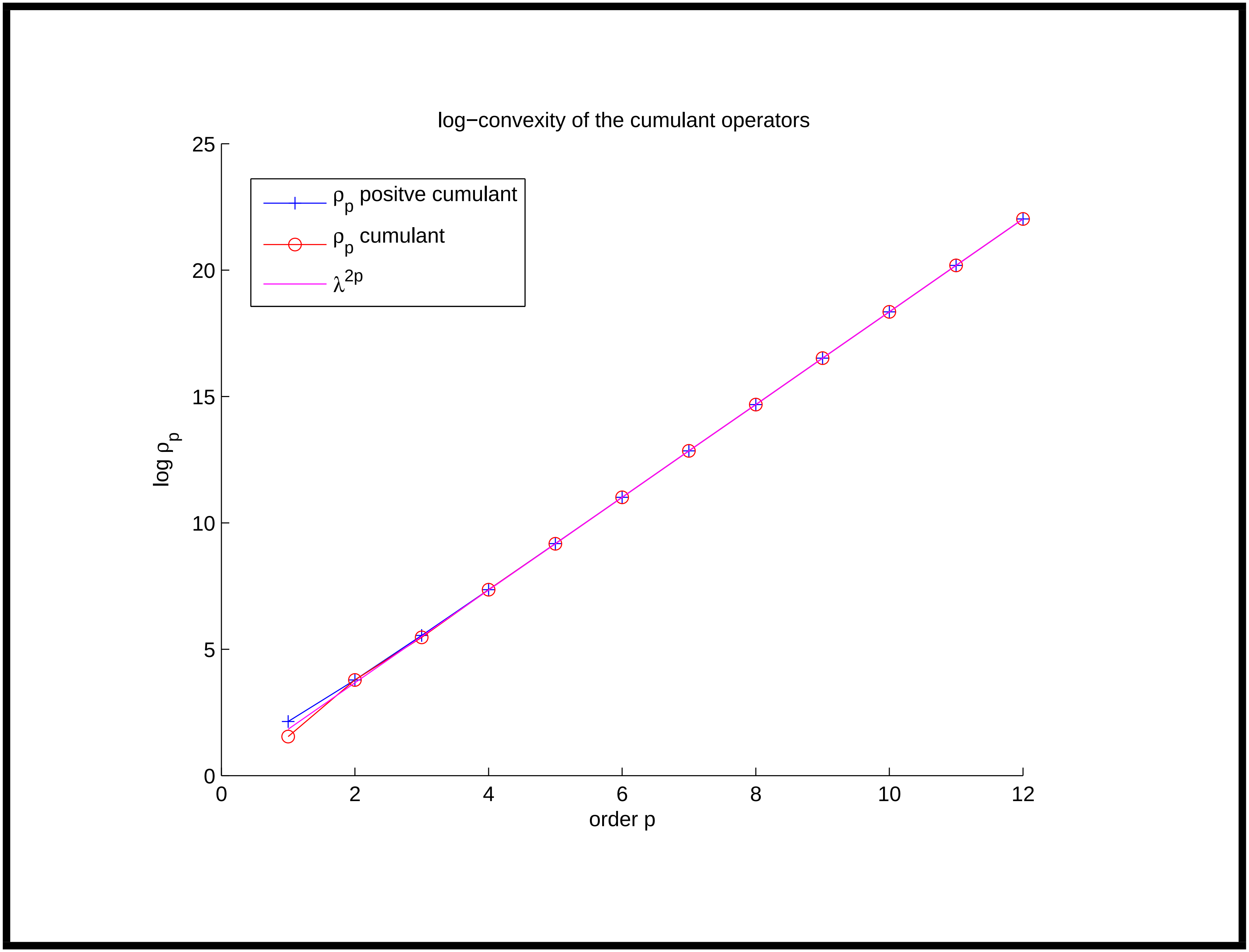
Lemma 5. *Let f be a map in $\mathcal{W}_s(g)$ close enough to g. let $\rho_{f,p} = \rho_p$ be the spectral radius of $K_{f,p} = K_p$. Then,*

$$1 < (\lambda_f^{-1} f'(1))^p < \rho_p < (\lambda_f^{-1} f'(1))^p + (-\lambda)_f^{-p} \tag{4}$$

for all p. In particular, if $f = g$, then $1 < \lambda^{2p}\rho_p < 1 + |\lambda|^p$.
For each $m \in \mathbb{N}$ we have that

$$\rho_{mp} < \rho_p^m$$

The map $p \mapsto \rho_p$ is increasing and $\log$–convex. Moreover, function $p \mapsto \log \rho_p / p$ is decreasing.


log−convexity of the cumulant operators
ρ_p positve cumulant
ρ_p cumulant
λ^{2p}
log ρ_p
order p
0
5
10
15
20
25
0
2
4
6
8
10
12

In particular, if $f = g$,

$$\rho_p < \rho_2^{p/2}$$

for all $p > 2$, and

$$\begin{aligned}\lim_{n\to\infty} \frac{\Lambda_p(\lambda^n z, 2^n)}{\{\Lambda_2(\lambda^n z, 2^n)\}^{p/2}} &= \lim_{n\to\infty} \frac{K_p^n 1(z)}{\{K_2^n 1(z)\}^{p/2}} \\ &= \lim_{n\to\infty} \left(\frac{\rho_p}{\rho_2^{p/2}}\right)^n = 0\end{aligned}$$

uniformly for $z \in [-1, 1]$.

CENTRAL LIMIT THEOREM IN THE DOMAIN OF UNIVERSALITY

We will denote by $\langle x \rangle$ the integer part of x. For each $n \in \mathbb{N}$ we write its binary expansion as

$$n = 2^{m_0(n)} + \cdots + 2^{m_{r_n}(n)} \tag{5}$$

so that $m_0(n) = \langle \log_2(n) \rangle > m_1(n) > \ldots > m_{r_n}(n) \geq 0$.
We denote by $q(n) = r_n + 1$ the number of terms in (5).
Observe that

$$1 \leq q(n) \leq m_0(n) + 1 < \log_2(n) + 1$$

RENORMALIZATION TECHNIQUE

We start by considering $x = |\lambda|^{m_0(n)+1} z$ with $|z| < 1$.
The properties of $q(n)$ times we obtain

$$\Lambda_p(x_n(z), n) = \sum_{j=0}^{r_n} \{\Phi_{j,n}\}^p \Lambda_p(\xi_{j-1}, 2^{m_j(n)})$$

where

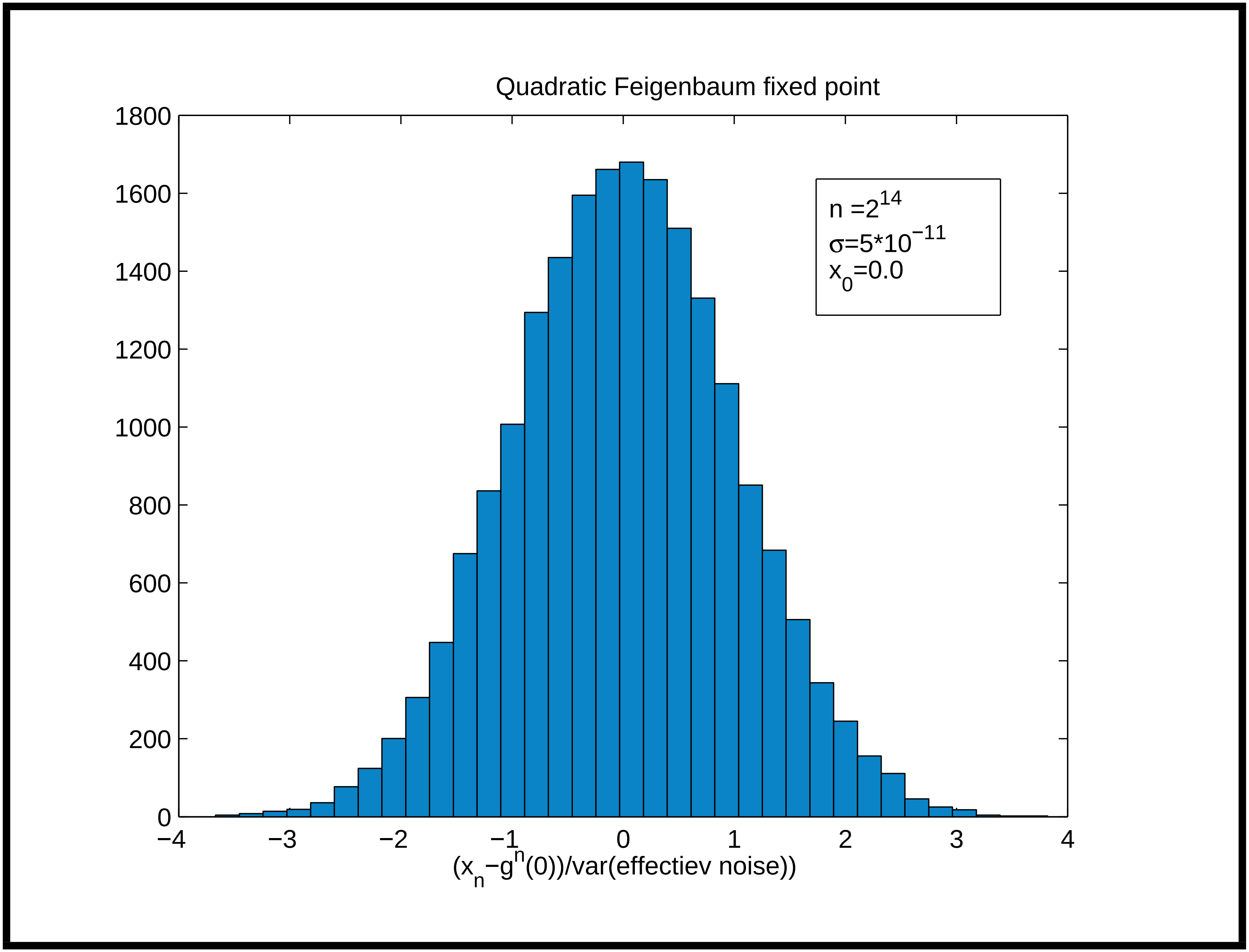
$$\Phi_{j,n} = \prod_{i=j+1}^{r_n} \left(g^{2^{m_i(n)}}\right)' (\xi_{i-1})$$

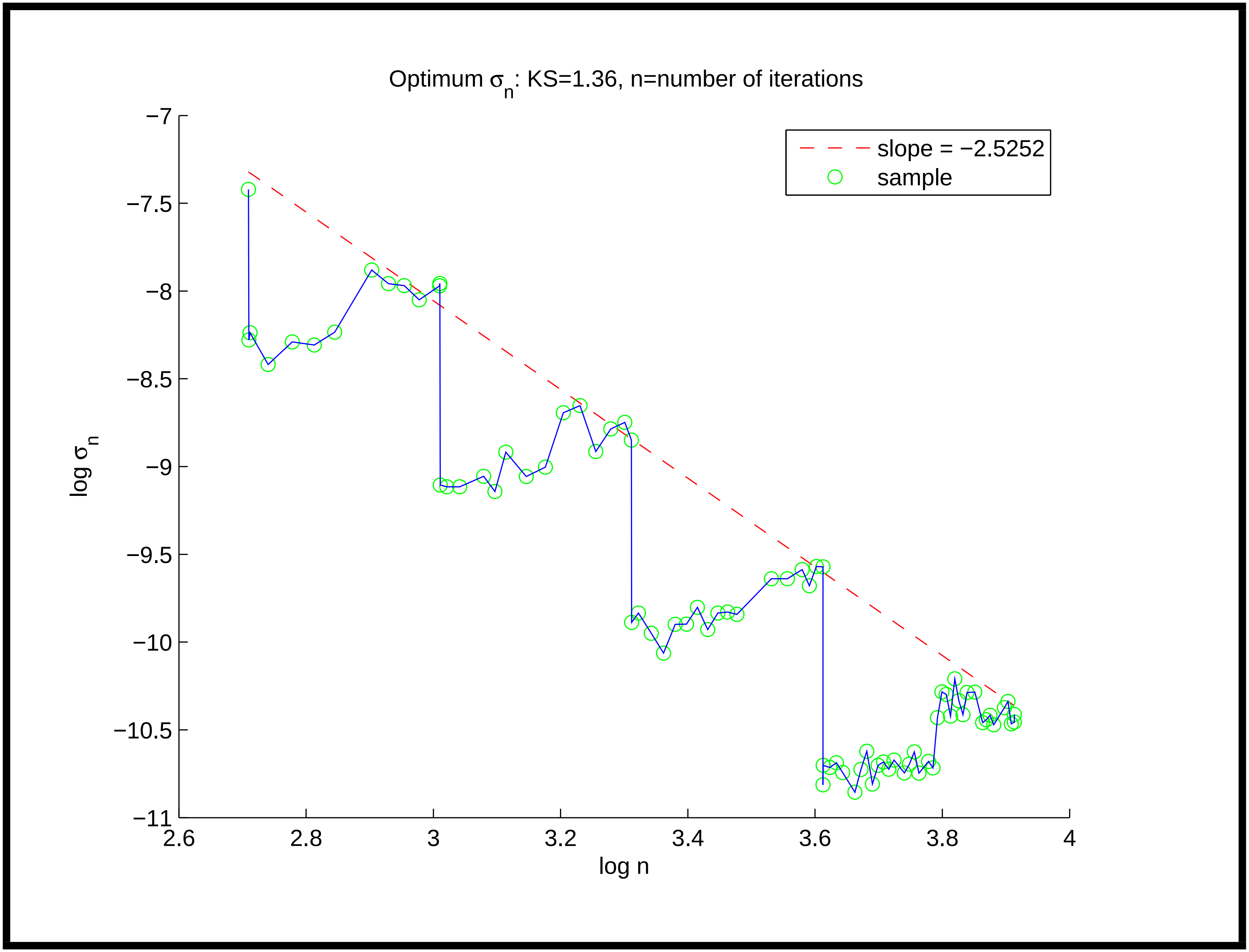
for $j = 0, \ldots, r_n - 1$, and $\Phi_{r_n,n} \equiv 1$.

The Lyapunov condition for orbits starting near zero follows by comparing blocks of the same size in the ratio

$$\frac{\sum_{j=0}^{r_n}\{\Phi_{j,n}\}^p \Lambda_p(\xi_{j-1}, 2^{m_j(n)})}{\sum_{j=0}^{r_n}\{\Phi_{j,n}\}^p \{\Lambda_2(\xi_{j-1}, 2^{m_j(n)})\}^2}$$

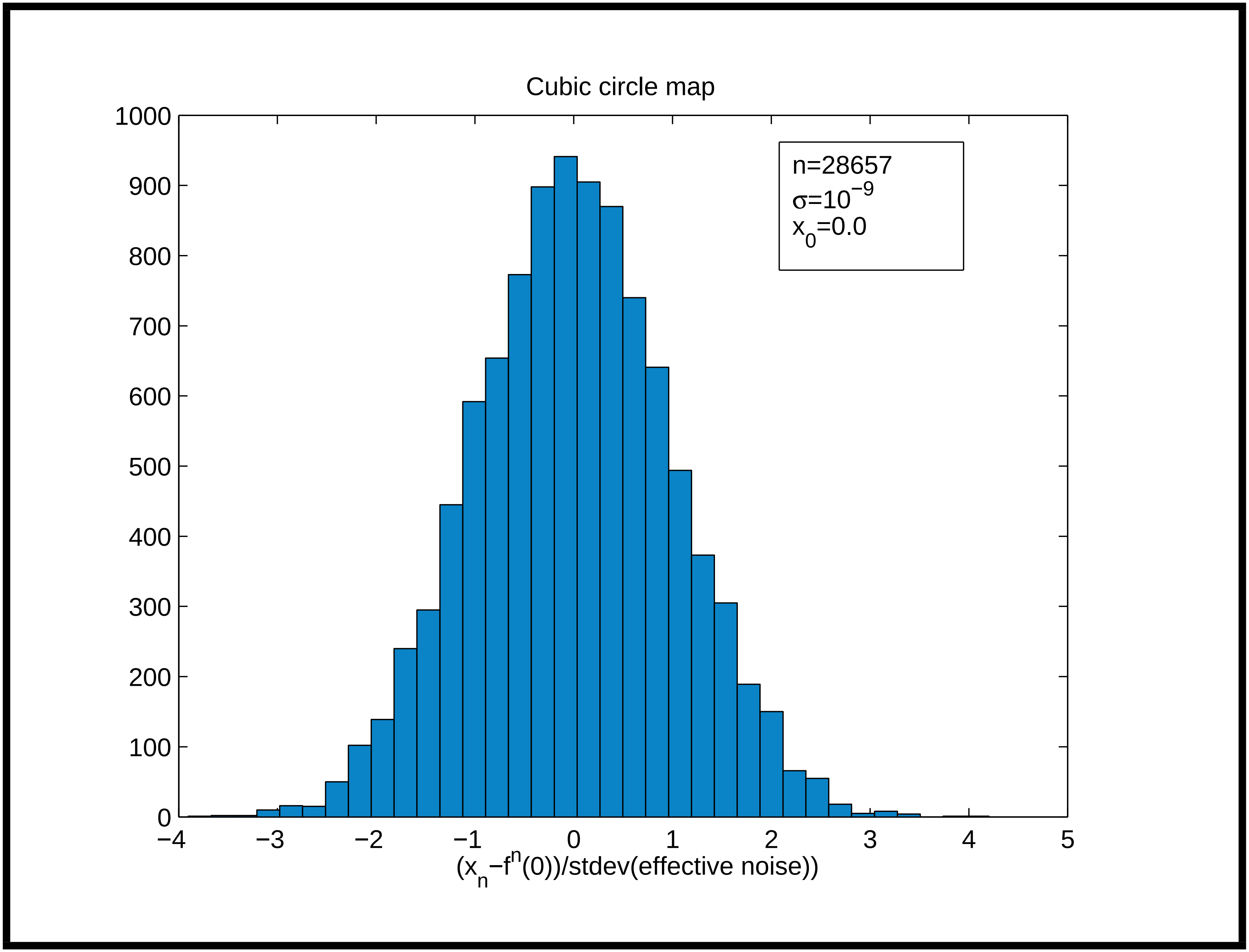
where $p > 2$. The important part of this argument is the control of the weights $\Phi_{j,n}$.

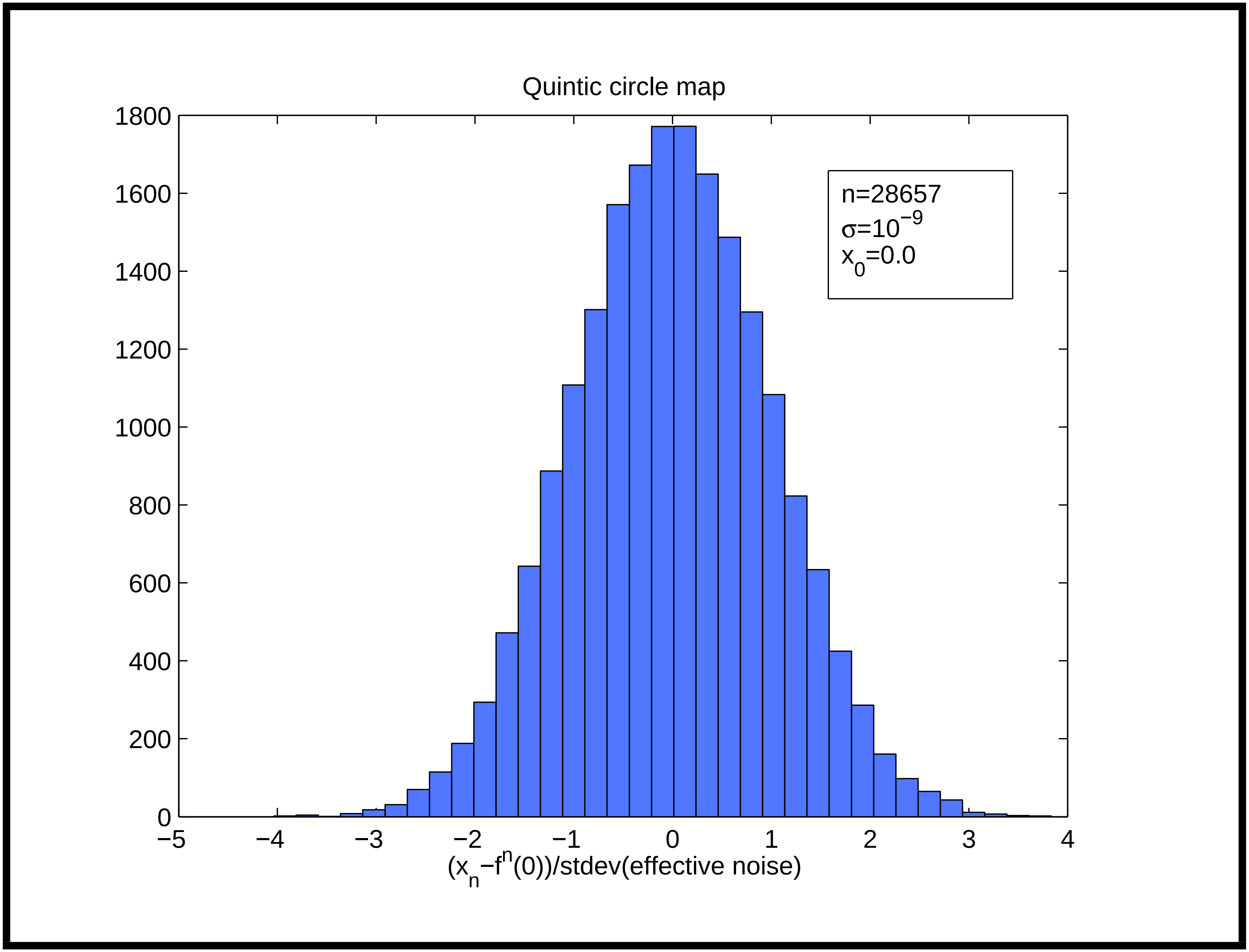

Quadratic Feigenbaum fixed point
n =2^14
σ=5*10^−11
x_0=0.0
1800
1600
1400
1200
1000
800
600
400
200
0
−4
−3
−2
−1
0
1
2
3
4
(x_n−g^n(0))/var(effectiev noise))


Optimum σn: KS=1.36, n=number of iterations
slope = −2.5252
sample
log σn
log n
−7
−7.5
−8
−8.5
−9
−9.5
−10
−10.5
−11
2.6
2.8
3
3.2
3.4
3.6
3.8
4

OTHER CRITICAL MAPS

1. Similar techniques are applied to analytic circle maps with golden mean rotation number with one critical point at the origin in Diaz–de la Llave, 2006.
2. A central limit theorem is obtain for Fibonacci times as well as for all times.
3. Currently work for complex maps that have Siegel discs is underway. Numerical evidence suggests that there is a rotation invariance Gaussian limit.


Cubic circle map
n=28657
σ=10^−9
x0=0.0
1000
900
800
700
600
500
400
300
200
100
0
−4
−3
−2
−1
0
1
2
3
4
5
(xn−f^n(0))/stdev(effective noise))



Quintic circle map
n=28657
σ=10^−9
x0=0.0
1800
1600
1400
1200
1000
800
600
400
200
0
−5
−4
−3
−2
−1
0
1
2
3
4
(xn−f^n(0))/stdev(effective noise)