Some remarks on vector partition functions

Corrado De Concini

July 12, 2006

We start with a

$n \times m$ integer matrix A.
We always think of A as a LIST of vectors in $\mathbb{Z}^n \subset \mathbb{R}^n$, its columns:

$$A := (a_1, \dots, a_m)$$

Consider the system of linear equations:

$$\sum_{i=1}^{m} a_i x_i = b, \quad \text{or} \quad Ax = b, \quad A := (a_1, \dots, a_m) \tag{1}$$

b a vector in $\mathbb{Z}^n$.

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The Problem, variable polytopes

This assumption implies that our system has a finite number $P(b)$ of positive integral solutions.

We would like to compute the number

$$P(b)$$

A way to look at $P(b)$ is the following. Consider the

$$\Pi_A(b) := \{x \mid Ax = b, x_i \geq 0, \ \forall i\}$$

which are convex and bounded for every b.

If we identify the spaces $Ax = b$ and $Ax = 0$ then we may think of $\Pi_A(b)$ as a variable polytope in the space $Ax = 0$

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The finite set $I_A(b)$ of points in $\Pi_A(b)$ with integer coordinates. is the set of solutions to our system. So

$$P_A(b) = |I_A(b)|$$

and our problem is the computanion of the number of integral points in $\Pi_A(b)$.

What we are really going to do is to give a qualitative study of the function $P_A(b)$ as $b \in \mathbb{Z}^n$ varies.

Notice that it is natural to think of an expression like: $b = t_1 a_1 + \cdots + t_m a_m$ with t_i not negative integers as a:

partition of b in t_1 parts of "size" a_1 plus $\ldots t_m$ parts of "size" a_m, hence the name partition function for the number $P_A(b)$, thought of as a function of the vector b.

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Obviously, the set of vectors b such that $\Pi_A(b)$ is not empty is equal, by definition, to:

The cone generated by A

$$C_A := \{\sum_{i=1}^{m} x_i a_i \,|\, x_i \geq 0\}$$

C_A is a convex cone in $\mathbb{R}^n$.

By assumption its non zero elements lie entirely in the interior of a half space so C_A does not contain a line. Thus

C_A is a pointed cone and 0 is its vertex.

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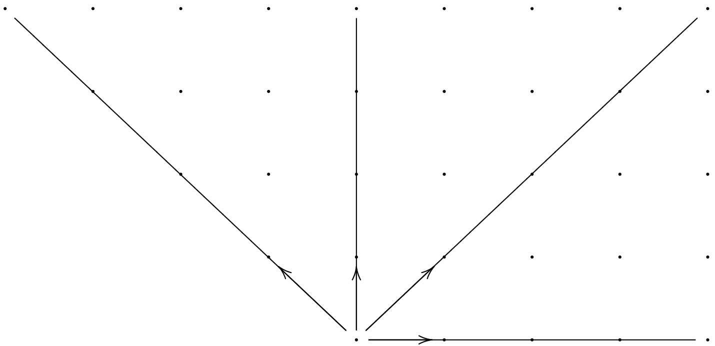
A 2–dimensional example

$$A = \begin{vmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 1 \end{vmatrix}$$

the associated cone $C(A)$ has three big cells (the definition is coming)



Big Cells Small Cells

Let us describe some combinatorial geometry of the cone $C(A)$. This will be needed to state the results.

Set

$$\mathcal{S}(A) = \{B \subset A | B \textit{ does not span } V\}$$

$$Y = \cup_{B \in \mathcal{S}(A)} C(B)$$

$$X = C(A) - Y.$$

X is called the set of regular points of $C(A)$.
Y is called the set of singular points of $C(A)$
A connected component of X is called a **big cell** If instead of the cones $C(B)$ we take the linear span $\langle B \rangle$ we get the notion of a small cell. A connected component of $C(A) - \cup_{B \in \mathcal{S}(A)} \langle B \rangle$ is called a **small cell.**

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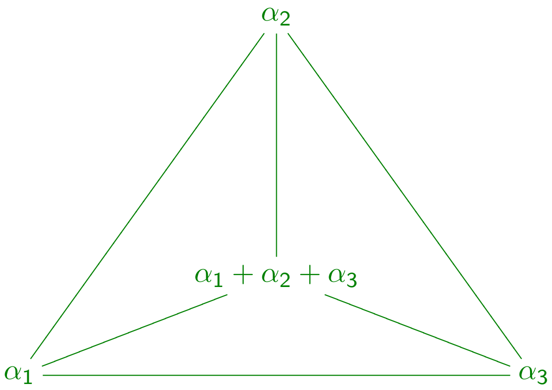
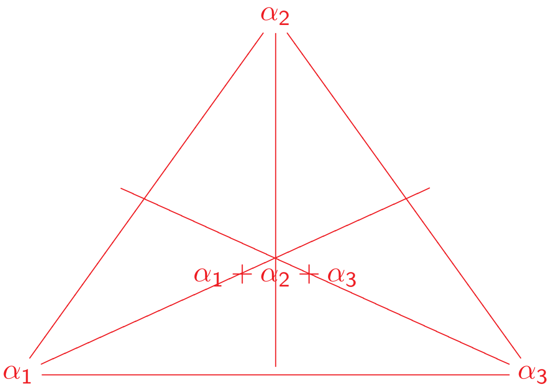
An example (two dimensional section):

There can be more small cells than big cells



We have 3 big cells and

6 small cells.



We go back to the partition function.
Denote by $\Lambda = \mathbb{Z}^n$ the integral lattice.
It is convenient to think of a function f on Λ as the distribution

$$\sum_{\lambda \in \Lambda} f(\lambda)\delta_\lambda.$$

with "Laplace transform"

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The partition function

As a special case let us take the partition function

$$P_A(b) = \#\{t_1, \ldots, t_m \in \mathbb{N} \mid \sum_{i=1}^{m} t_i a_i = b\}$$

An easy computation gives its Laplace transform

of the partition distribution

$$\mathcal{P}_A = \sum_{\lambda \in \Lambda} P_A(b)\delta_b.$$

We get

$$L\mathcal{P}_A = \prod_{a \in A} \frac{1}{(1 - e^{-a})}$$

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Two non commutative algebras

The algebra $\mathcal{D}(T)$

is defined as the Weyl algebras of differential operators on the torus T with coefficients regular functions. i.e. it is generated by the functions e^λ, $\lambda \in \Lambda$ and the derivations ∂_ϕ with $\phi \in \Lambda^*$.
The relations are given by

$$[\partial_\phi, e^\lambda] = \langle \phi, \lambda \rangle$$

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Fourier transform

The relations tell us that there is an algebraic Fourier isomorphism F between $\mathcal{D}(T)$ and $W(\Lambda)$, given by

$$F(e^{\lambda}) = \tau_{\lambda} \qquad F(\partial_{\phi}) = -\phi.$$

So any D_T module M becomes a $W(\Lambda)$ module $\hat{M}$ and viceversa.

Let us see how we can use the Laplace transform as a Fourier isomorphism.

Start by remarking the following property:

$$L(\partial_{\phi} f)(u) = \phi L f(u), \qquad L(\tau_{\lambda} L f)(u) = e^{-\lambda} L f(u).$$

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Theorem

The following two modules are Fourier isomorphic:

1. *The $W(\Lambda)$-module Σ_A generated, in the space of distributions, by the partition distribution $\mathcal{P}_A$ under the action of the algebra $W(\Lambda)$.*

2. *The algebra $S_A := \mathbb{C}[\Lambda][\prod_{a\in A}(1-e^{-a})^{-1}]$ obtained from the coordinate ring $\mathbb{C}[T]$ by inverting $u_A := \prod_{a\in A}(1-e^{-a})$ i.e. the coordinate ring of the open set in T obtained by removing the kernels of the character e^{-a}, $a \in A$. considered as a D_T-module*

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Before we go on here is an "explicit" description of Σ_A.

Σ_A is the space of functions on Λ which are linear combinations of polynomial functions on the cones $C(B) \cap \Lambda$, $B \subset A$ a linearly independent subset and their translates.

What we are going to do now is to obtain informations on Σ_A studying its Fourier transform S_A. In particular we shall obtain some partial fraction expansions of

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$$\mathcal{A} = T / \cup_{a \in A} \mathcal{T}_a$$

We now define

The toric arrangement as the set of connected components of all the intersections of the subgroups $\mathcal{T}_a,\ a \in A$.

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Among the elements of the toric arrangement the one more relevant to partition functions are the points. These are obtained as follows

Given a basis $\underline{b}$ extracted from A, consider the lattice $\Lambda_{\underline{b}} \subset \Lambda$ that it generates in Λ.

We have that $\Lambda/\Lambda_{\underline{b}}$ is a finite group of order $[\Lambda : \Lambda_{\underline{b}}] = |\det(\underline{b})|$.

Its character group is the finite subgroup $T(\underline{b})$ of T which is the intersection of the kernels of the functions e^a as $a \in \underline{b}$.

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points of the arrangement

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We define a filtration on S_A.

$$S_{A,k} = \sum_{\substack{B \subset A \\ dim\ \langle B \rangle \leq k}} \mathbb{C}[T][u_B^{-1}]$$

with $u_B = \prod_{b \in B}(1 - e^{-b})$.

Each $S_{A,k}$ is a D_T submodule, $\quad S_{A,k} \supset S_{A,k-1}$.

Theorem

$S_{A,n}/S_{A,n-1}$ is a semisimple D_T-module whose isotipic components are indexed by the points of the arrangements.

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The filtration

We define a filtration on S_A.

$$S_{A,k} = \sum_{\substack{B \subset A \\ dim\ \langle B \rangle \leq k}} \mathbb{C}[T][u_B^{-1}]$$

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$$A_\phi := \{a \in A \,|\, e^{\langle a \,|\, \phi \rangle} = 1\}.$$

$A_\phi = \{b_1, \dots b_t\}$ is a sublist of A. A_ϕ spans V. Before we go on we need a general definition.

Suppose now we have a list $X = \{x_1, \dots x_s\}$ of vectors spanning V. A basis $\sigma = \{x_{i_1}, \dots x_{i_n}\}$ is called NB (no broken) if for all h there is no $t < i_h$ such that the vectors $\{x_t, x_{i_h}, \dots x_{i_n}\}$ are linearly dependent.

As an example let us take the list

$$\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3.$$

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NB bases have a lot to do with the big cells. Indeed given a regular point p we associate to p the collection $\mathcal{N}_p$ of NB bases such that p lies in the cone they generate. Then

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If p, q are two regular points in $C(X)$, $N_p = N_q$ if and only if p and q lie in the same big cell.

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with γ_ϕ the minimal idempotent in $\mathbb{C}[\Lambda/\Lambda_\sigma]$ associated to $e^\phi \in (\Lambda/\Lambda_\sigma)^*$ (it is easy to see that this makes sense).

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$$S_{A,n}/S_{A,n-1} = \oplus_{\phi,\sigma} D_T n_{\phi,\sigma}$$

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The module $D_T n_{\phi,\sigma}$ is irreducible and it is isomorphic to D_T/J_ϕ where J_ϕ is the left ideal generated by maximal ideal in $\mathbb{C}[T]$ defining the point e^ϕ.
As a M module $S_{A,n}/S_{A,n-1}$ is free with basis the elements $n_{\phi,\sigma}$.

This result tells us that in particular there exist uniquely defined differential operators with constant coefficients $q_{\phi,\sigma}(-\partial)$ with the property that in $S_{A,n}/S_{A,n-1}$

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Quasi Polynomiality

At this point we go back to the partition function applying the inverse Laplace transform. In order to make our translation we need two remarks

1. $L^{-1}(\frac{\gamma_\phi}{\prod_{a\in\sigma}(1-e^{-a})}) = e^\phi \chi_{C(\sigma)}$, $\chi_{C(\sigma)}$ the characteristic function of $C(\sigma)$.
2. Any identity holding in $S_{A,n}/S_{A,n-1}$ is transformed by L^{-1} into and identity on functions on Λ which is valid outside a finite union of translates of cones of the form $C(B)$ with $B \subset A$ $\langle B\rangle \subsetneq V$.

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With these two remarks, we can safely apply L^{-1} and using the fact that it gives a Fourier isomorphism, deduce

Theorem

If $\lambda \in \Lambda$ does not lie in a suitable finite set of translates of cones of the form $C(B)$ with $B \subset A$ $\langle B \rangle \subsetneq V$, we have

$$P_A(\lambda) = \sum_{(\phi,\sigma)} e^{\langle \phi | \lambda \rangle} \mathfrak{q}_{\underline{b},\phi}(\lambda) \chi_{C(\sigma)}(\lambda).$$

Notice that e^{ϕ} is a point of finite order so that the function $e^{\langle \phi | \lambda \rangle}$ is constant on the cosets of a sublattice of finite index.

Now fix a big cell C. Take the set

$$M_C = \{(\phi, \sigma)| \ , e^{\phi} \in \mathcal{Q}(A), \sigma \ NB \ basis \ in \ A_{\phi}, C \subset C(\sigma)\}.$$

Taking into account the fact that a big cell is the intersection of the $C(\sigma)$ containing it we get that (outside a finite union of translates of cones of the form $C(B)$ with $B \subset A \ \langle B \rangle \subsetneq V$) if $\lambda \in C$

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One can remove the not so nice "outside a finite union of translates of cones of the form $C(B)$ with $B \subset A \ \langle B \rangle \subsetneq V$". Indeed a better result holds

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Z_A is called the zonotope generated by A.

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For any big cell C, the formula

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As stated the result is due to Szenes-Vergne (2003) by a refined partial fraction expansion. For small cells instead of big cells had been obtained before by Dahmen-Micchelli (1988). This result plus the above considerations imply the result for big cells.

A key observation is that the quasi polynomials appearing in the formula for $\mathcal{P}_A$ satisfy special difference equations. Define

Difference operators

For $a \in \Lambda$ and f a function on Λ we define the

$$\nabla_a f(x) = f(x) - f(x-a), \ \nabla_a = 1 - \tau_a.$$

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The ideal J_A

Consider the class of u_A^{-1} in $S_{A,n}/S_{A,n-1}$. Its annihilator J_A in $\mathbb{C}[T]$ is the ideal generated by the products $u_Y := \prod_{v \in Y}(1 - e^{-v})$ as Y runs over the subsets of A such that the complent of Y does not span V. These subsets are called cocircuits.

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Associated to J_A we can consider the space $\nabla(A)$ which consists of functions f on Λ which are simultaneous solutions of the equations

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$\nabla(A)$ has various remarkable properties:

1. $\nabla(A)$ is finite dimensional
2. $\nabla(A)$ is invariant under translations
3. $\nabla(A)$ consists of quasi polynomials which are polynomials on the cosets of the lattice

$$\Psi = \cap_{\substack{B \subset A \\ B \ basis}} \Lambda_B$$

of degree at most equal to $|A| - dim V$.

4. $$dim\ \nabla(A) = \delta(A) = \sum_{\substack{B \subset A \\ B \ basis}} |\frac{\Lambda}{\Lambda_B}|$$

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Theorem

For any big cell C the quasi polynomial P_A^C lies in $\nabla(A)$.

In fact much more is true. Consider a point $v \in C$ very near to zero and take the translated zonotope $v - Z_A$. It turns out the the number of lattice points in $v - Z_A$ equals $\delta(A)$, the dimension of $\nabla(A)$ and these points impose independent conditions on $\nabla(A)$. One can show

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In fact this gives a way of computing P_A^C by first imposing the recursion given by the difference equations and the the initial conditions given by the theorem.
Before giving an example let me mention two more results. The first is a remarkable reciprocity formula also due to Dahmen and Micchelli which can be proved as a consequence of the above theorem.

Theorem

For every $\lambda \in \Lambda$ and a big cell C,

$$P_A^C(\lambda) = (-1)^{|A| - dim\ V} P_A^C(-\lambda - \rho_A)$$

with $\rho_A = \sum_{a \in A} a$.

The second is a closed formula which can be proved using the corresponding, analogous but simpler theory which gives the volume of the polytope $\Pi_A(b)$. To give it we need to introduce some auxiliary homogeneous polynomials. Given $a \in A$ denote by D_a the derivative in the direction a and for a subset $Y \subset A$ set $D_Y = \prod_{a \in Y} D_a$. For any $e^{\phi} \in \mathcal{Q}(A)$ and no broken basis $B \in A_\phi$ we define the polynomial $\mathfrak{p}_{\underline{b},A_\phi}(x)$ as the homogeneous polynomial of degree $|A_\phi| - dim\ V$, unique solution of the system

$$\begin{cases} D_Y f = 0 & Y \text{ a cocircuit in } A_\phi \\ D_C f = \delta_{B,C} & C \text{ a NB basis in } A_\phi \end{cases}$$

Theorem

Set

$$Q_\phi = \prod_{a \notin A_\phi} \frac{1}{1 - e^{-D_a - \langle \phi \mid a \rangle}} \prod_{a \in A_\phi} \frac{D_a}{1 - e^{-D_a}}$$

Then

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To finish let us compute a simple example. We want to to write down the function $P(n)$ giving in how many ways a number n can be written as

$$n = h + 2k + 3s$$

Since $|A| - dim\ V = 3 - 1 = 2$ and $\Psi = 6\mathbb{Z}$ we need to determine 6 degree 2 polynomials. $P_0, P_1, \ldots P_5$ such that if $n \equiv j$ modulo 6 $P(n) = P_j(n)$. Write

$$P_j(x) = a_j x^2 + b_j x + b_j.$$

Our recursion is (this relation appears to be due to Euler)

$$P(n) - P(n-1) - P(n-2) + P(n-4) + P(n-5) - P(n-6) = 0$$

The initial conditions are

$$P_0(0) = 1$$

$$P_1(-5) = P_2(-4) = P_3(-3) = P_4(-2) = P_5(-1) = 0$$

The reciprocity law gives

$$P_j(x) = P_j(-x-6).$$

This translates into the relation $b_j = 6a_j$.
Now the the Euler relation gives $P_1(1) = 1$. This together with $P_1(-5) = 0$ gives the system

$$\begin{cases} 7a_1 + c_1 = 1 \\ -5a_1 + c_1 = 0 \end{cases}$$

which implies

$$P_1 = \frac{1}{12}x^2 + \frac{1}{2}x + \frac{5}{12}$$

Completely similar considerations give

$$P_0 = \frac{1}{12}x^2 + \frac{1}{2}x + 1$$

$$P_1 = P_5 = \frac{1}{12}x^2 + \frac{1}{2}x + \frac{5}{12}$$

$$P_2 = P_4 = \frac{1}{12}x^2 + \frac{1}{2}x + \frac{2}{3}$$

$$P_3 = \frac{1}{12}x^2 + \frac{1}{2}x + \frac{3}{4}$$

The algebra of the box spline

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