

based on a joint work with Hsian-Hua Tseng
(Comp. Math.) and Zhu DG/0609420,
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Integration of Lie algebroids via stacks

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Lie groupoids

Groupoid = small category where all arrows (morphisms) are invertible.

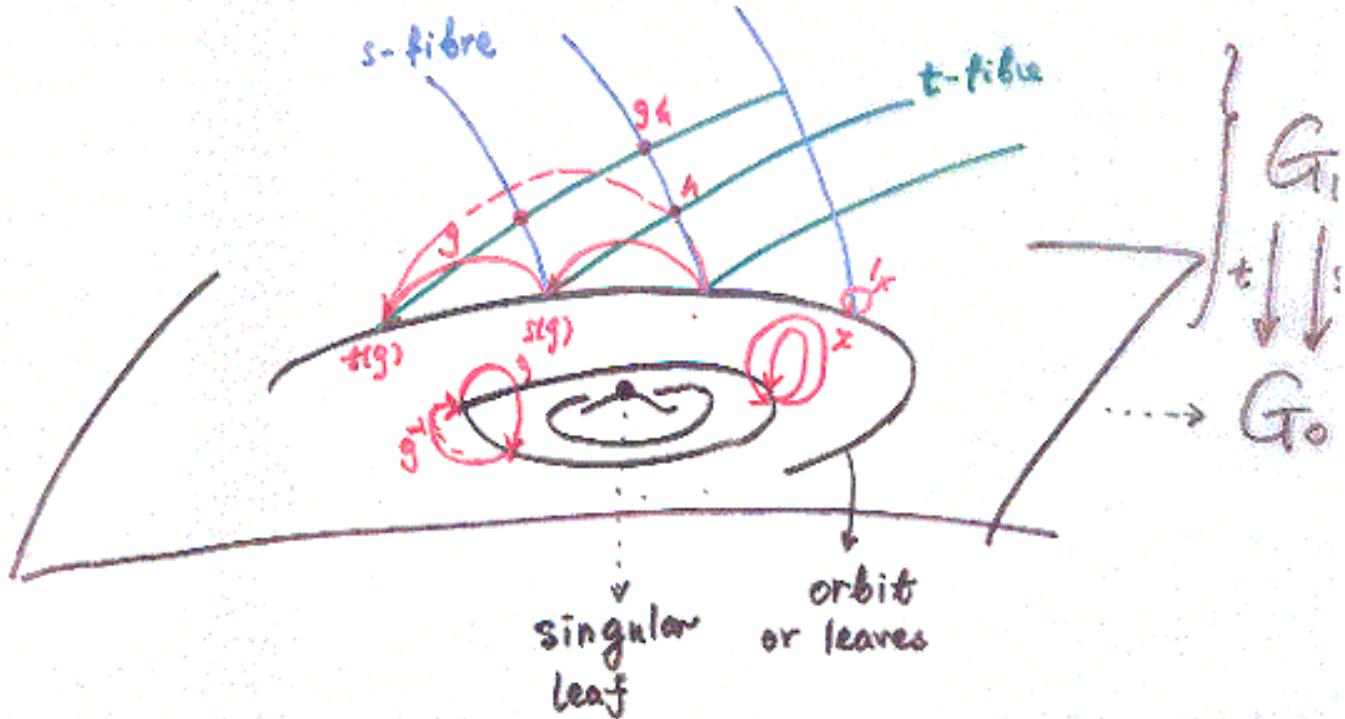
- G_0 = set of objects
- G_1 = set of arrows
- $G_1 \xrightarrow{s, t} G_0$ source & target maps ($s(g)$, $t(g)$)
- $G_1 \times_{G_0} G_1 \xrightarrow{m} G_0$ multiplication, $(g, h) \mapsto gh$
- $G_0 \xrightarrow{e} G_1$, $x \mapsto 1_x$ identity
- $G_1 \xrightarrow{i} G_1$, $g \mapsto g^{-1}$ inverse

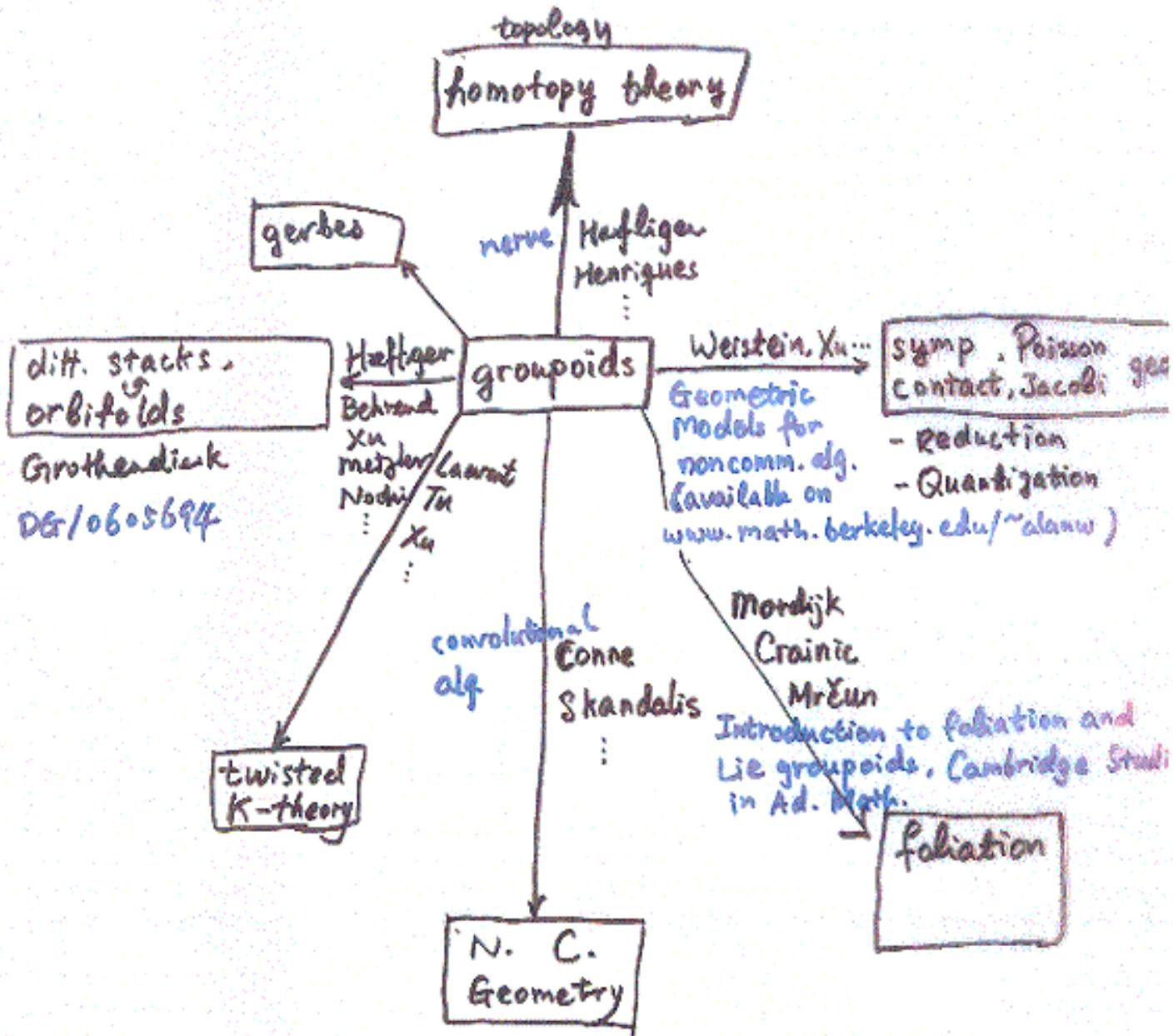
satisfying the usual identities expected

A Lie groupoid $\begin{array}{c} G_1 \\ \downarrow \downarrow \\ G_0 \end{array}$ is a groupoid

where - G_1 , G_0 are smooth mfd's

- s, t, surjective submersions
- e, i, m, smooth maps





kerTslGeo

Ex1 Lie group G

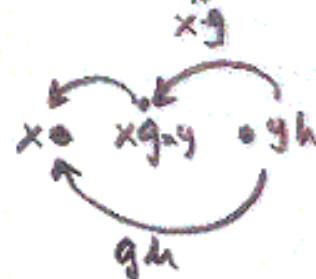


Ex2 Action groupoids $M \times G$
when $M \curvearrowright G$

$$\downarrow \text{b}$$

$$M$$

$$(x, g)(y, h) = (x, gh)$$



Ex3 homotopy qpd.

{paths, $[0, 1] \rightarrow M^q$ } / homotopy

$$\begin{matrix} t & \downarrow & s \\ \downarrow & & \downarrow \\ M & & \end{matrix}$$



"loops" at a pt x . in $\pi_1(M, x)$, the 1st homotopy gp.

Ex4 Monodromy qpd of a foliation \mathcal{F} on M .

(Holonomy)
requires
the same
holonomy.

{leafwise paths} / {leafwise homotopies}

$$\begin{matrix} t & \downarrow & s \\ \downarrow & & \downarrow \\ M & & \end{matrix}$$

(or same holonomy)

Ex5 Čech qpd. of: a covering family $\{U_i\}$ of M .

$\sqcup U_i \sqcap U_j$

$$\begin{matrix} \downarrow & \downarrow \\ \sqcup & \sqcap \end{matrix}$$

$$\begin{matrix} u_i & x_i & u_j \\ \sqcup & \sqcap & \sqcup \end{matrix}$$

$$\begin{matrix} x_{ij} & \cdot & x_{jk} & = & x_{ik} \\ \swarrow & & \downarrow & & \searrow \\ x_i & & x_j & & x_k \end{matrix}$$

The infinitesimal data of a Lie groupoid $G_1 \rightrightarrows G_0$ is a **Lie algebroid**. More precisely, $TG_1|_{e(G_0)} = \ker Ts|_{G_0} \oplus TG_0$. Then the vector bundle

$$\begin{array}{ccc} \ker Ts|_{G_0} & \xrightarrow{Tt} & TG_0 \\ \downarrow & & \downarrow \\ G_0 & & \end{array}$$

satisfies

- Tt is a morphism of vector bundles;
- \exists a Lie bracket $[,]$ on $\Gamma(\ker Ts|_{G_0})$ (comes from $[,]$ of $\Gamma(TG_1)$);
- $[,]$ and Tt go well together: $[X, fY] = f[X, Y] + Tt(X)(f) \cdot Y$.

In general a vector bundle

$$\begin{array}{ccc} A & \xrightarrow{\rho} & TG_0 \\ \downarrow & & \downarrow \\ G_0 & & \end{array}$$

Kosmann-Schwarzbach

with the above condition is a Lie algebroid.

degree 1 super-mfd with a degree 6¹ vector field Q , s.t. $Q^2=0$ \Rightarrow Lie n -algebroid

Problem

Lie algebras differentiation
Lie groups integration

Lie algebroids differentiation
Integration ??

- Cattaneo and Felder—Poisson case, using Poisson-sigma model (a Poisson manifold P give rise to a natural Lie algebroid $T^*P \rightarrow P$ with $[df, dg] = d\{f, g\}$).

- Crainic and Fernandes—classify the criterir of integration and construct a universal topological groupoid.

- Loo-algebra E. Getzler (nilpotent 2004) A. Henrique (2006)

We consider the problem from a little bit

(Severa) → different viewpoint: Sullivan (1977) constructed a spacial realization for every d. a. (differential algebra) D as $\text{Hom}_{d.a.}(D, \Omega^*(\Delta^n))$. When $D = \wedge^* \mathfrak{g}^*$ with

$$d\xi(a_0, a_1, \dots, a_n) = \sum \pm \xi([a_i, a_j], a_0, \dots, \hat{a}_i, \dots, \hat{a}_j, \dots)$$

?

$\sqrt{a_n}$

then

$$\begin{aligned}\text{Hom}_{d.a.}(D, \Omega^*(\Delta^n)) &= \text{Hom}_{algd}(T\Delta^n, \mathfrak{g}) \\ &= \{\alpha \in \Omega^1(\Delta^n, \mathfrak{g}) : d\alpha = \frac{1}{2}[\alpha, \alpha]\} \\ &= \{\text{flat connections of } G \times \Delta^n \rightarrow \Delta^n : \\ &\quad G \text{ a Lie group of } \mathfrak{g}\}\end{aligned}$$

Thus

$$\begin{aligned}\text{Hom}_{algd}(T\Delta^2, \mathfrak{g}) &= \{\alpha \in \Omega^1(\Delta^2, \mathfrak{g}), d\alpha = \frac{1}{2}[\alpha, \alpha]\} \\ &\quad \downarrow \downarrow \downarrow \downarrow \\ \text{Hom}_{algd}(T\Delta^1, \mathfrak{g}) &= \Omega^1(\Delta^1, \mathfrak{g}) \\ &\quad \downarrow \uparrow \\ \text{Hom}_{algd}(T\Delta^0, \mathfrak{g}) &= pt\end{aligned}$$

Thus we end up with a **simplicial manifold**, which is roughly, as you've seen, a tower of manifolds with face and degeneracy maps satisfying compatible conditions.

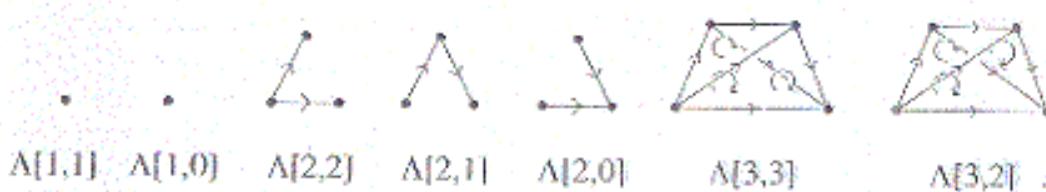
Other examples of simplicial manifolds come

from Lie groupoids.

$$\begin{array}{c} X_3 = G_1 \times_{G_0} G_1 \times_{G_0} G_1 \\ \downarrow \downarrow \downarrow \quad \uparrow \uparrow \uparrow \\ X_2 = G_1 \times_{G_0} G_1 \\ \downarrow \downarrow \quad \uparrow \uparrow \\ X_1 = G_1 \\ \downarrow \downarrow \\ X_0 = G_0 \end{array}$$

But it is not true for the other direction.
For this we have to introduce **Kan conditions**. Consider simplicial manifold:

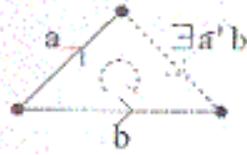
$$(\Delta[m,j])_n = \{f \in (\Delta[m])_n \mid \{0, \dots, j-1, j+1, \dots, m\} \not\subseteq \{f(0), \dots, f(n)\}\}.$$



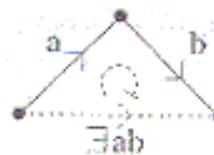
$Kan(m, j)$: Any map $\Lambda[m, j] \rightarrow X$ extends to a map $\Delta[m] \rightarrow X$.

$Kan!(m, j)$: Any map $\Lambda[m, j] \rightarrow X$ extends to a unique map $\Delta[m] \rightarrow X$.

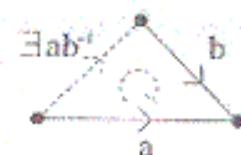
Then the Kan condition corresponds to the possibility of composing various morphisms.



$Kan(2,2)$

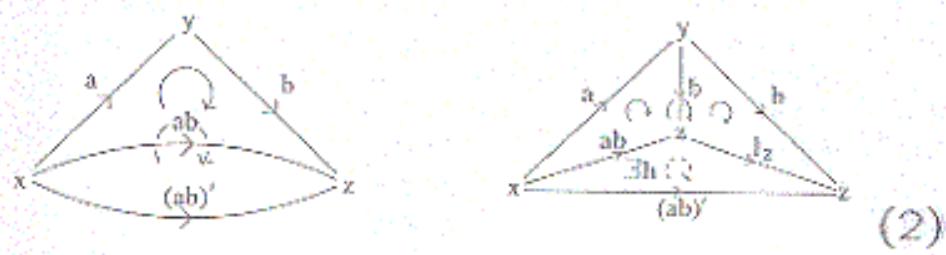


$Kan(2,1)$



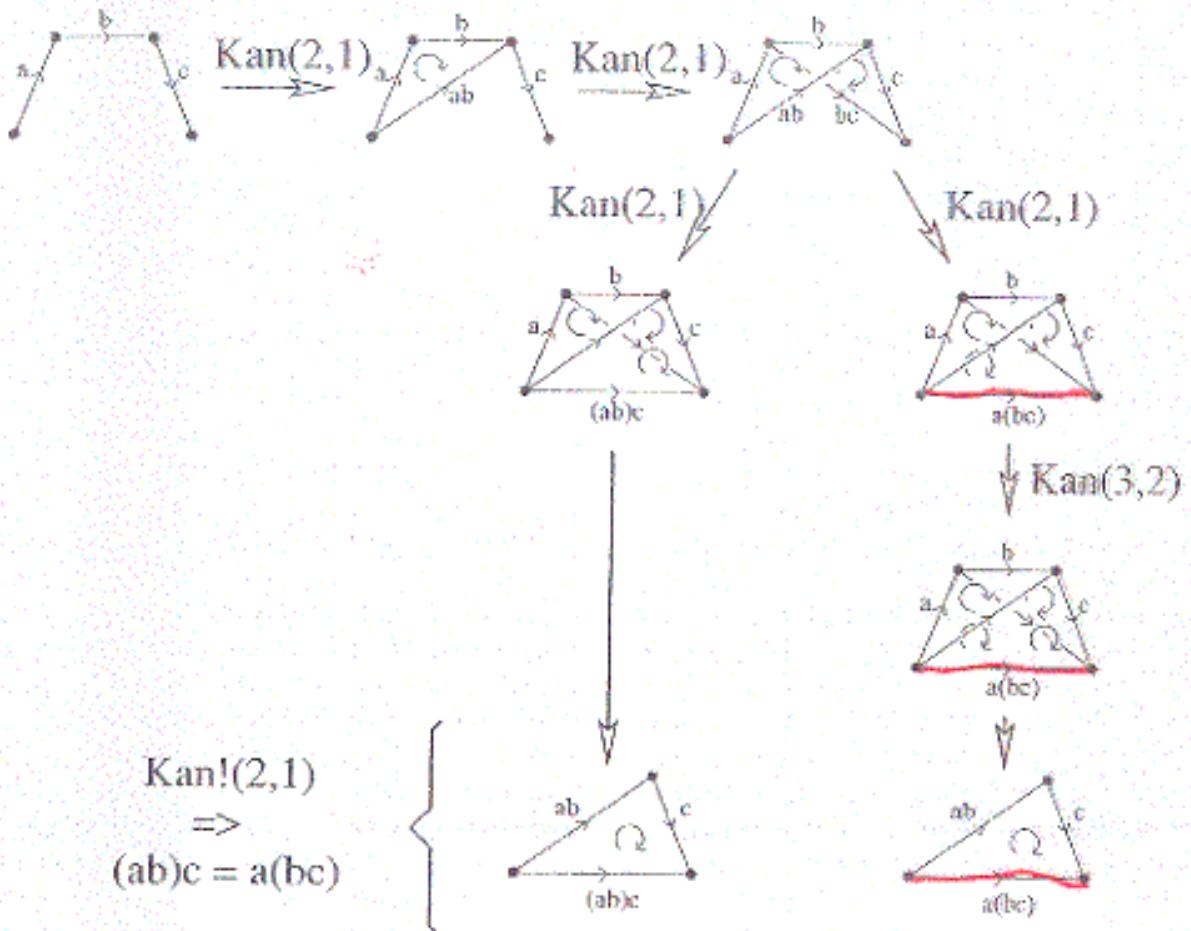
$Kan(2,0)$ (1)

The composition of two arrows is in general not unique,



but any two of them can be joined by a 2-morphism h given by $Kan(3, 1)$.

The associativity is given by $Kan(3, 1)$ and $Kan!(2, 1)$.



Proof of associativity.

This says a simplicial set satisfying $Kan!(\geq 2, j)$ and $Kan(\geq 0, j)$ gives arise to a **groupoid**. This motivate us to define in the **differentiable category**

(André Henriques)

Definition 1. A Lie n -groupoid X ($n \in \mathbb{N} \cup \infty$) is a simplicial manifold that satisfies $Kan(m, j)$ for all $0 \leq j \leq m \geq 1$ and $Kan!(m, j)$ $0 \leq j \leq m > n$, where

Kan(m, j): The restriction map $\text{hom}(\Delta[m], X) \rightarrow \text{hom}(\Delta[m, j], X)$ is a **surjective submersion**.

Kan!(m, j): The restriction map $\text{hom}(\Delta[m], X) \rightarrow \text{hom}(\Delta[m, j], X)$ is a **diffeomorphism**.

Given a Lie algebroid,

$$S_n(A) = \text{Hom}_{algd}(T\Delta^n, A),$$

$$S_0(A) = M$$

$$S_1(A) = \{ a(t) \in A : \rho(a(t)) = \dot{\gamma}(t) \} = \text{mfd}$$

$$S_2(A) = \{a(\epsilon, t) : \exists b(\epsilon, t), \partial_\epsilon a - \partial_t b = T_{\nabla}(\alpha, \beta)\}$$

$b(0, t) = b(1, t) = 0\}$?? Banach manifold

$$S_3(A) = \dots$$

So not sure it's a simplicial manifold or even Kan simplicial manifold. It's true in the case of \mathfrak{g} .

Two natural simplicial manifold for \mathfrak{g} ,

- $S_\bullet(\mathfrak{g})$,
- the nerve, $\dots G \times G \Rightarrow G \rightrightarrows pt$.

$$S_1(\mathfrak{g})/S_2(\mathfrak{g}) \xrightarrow{\text{holonomy}} G$$
$$\alpha \text{ conn} \mapsto hol(\alpha)$$

is an isomorphism when G is the simply connected Lie group of \mathfrak{g} .

So to obtain the second from the first, some sort of "truncation" take place:

$$S_\bullet(\mathfrak{g}) \rightsquigarrow \dots S_2(\mathfrak{g})/S_3(\mathfrak{g}) \Rightarrow S_1(\mathfrak{g})/S_2(\mathfrak{g}) \rightrightarrows pt$$

$\{U_i, \varphi_i\}$ local charts $\hookrightarrow M$ manifolds

Lie groupoids

6
G

10

differentiable stacks

6

G_i

Hilsum-Skandalis

bimodules (generalized morph)

free-transitive

1

11

卷之三

1

$$x \xrightarrow{\text{morphism}} y$$

၅၁၆

from green. free ^{green}.

二十一

1

$$x \xrightarrow{*} y$$

Morita equivalence

$$\begin{array}{ccc} \text{G} & \xrightarrow{+} & \text{H} \\ \downarrow & & \downarrow \\ \text{G}' & \xrightarrow{+'} & \text{H}' \end{array}$$

$$\text{m} \rightarrow \frac{G_1}{G_0} \xrightarrow{\text{S}_\text{p}^2\text{H}_\text{e}^+} \frac{H_1}{H_0}$$

morphism \rightarrow H.S. bisimulate

Do the same thing for A , $S_1(A)/S_2(A)$ is
only a topological space!

$\dots S_2(A)/S_3(A) \Rightarrow S_1(A)/S_2(A) \rightrightarrows M = S_0(A)$

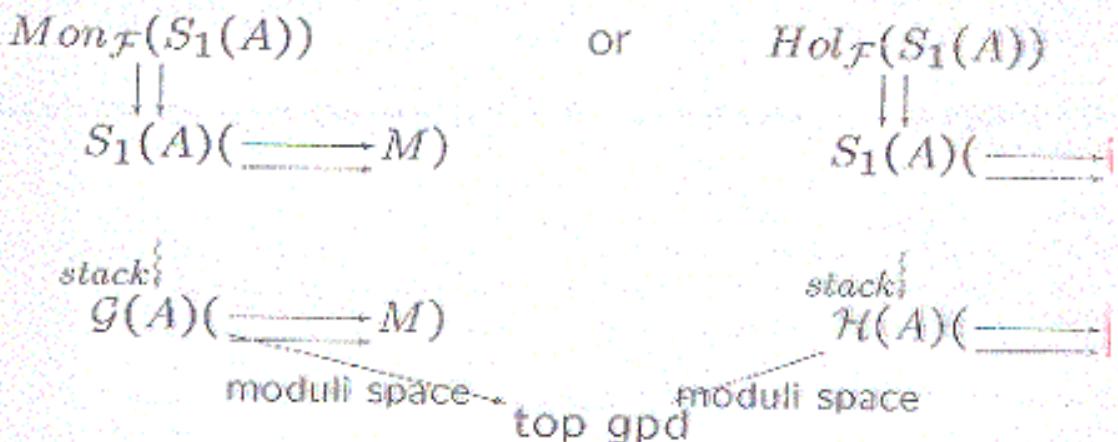
is the nerve of the universal topological groupoid constructed in
Cantaneo-Felder & Crainic-Fanandes.

Theorem 1. A higher truncation:

$\dots S_3(A)/S_4(A) \Rightarrow S_2(A)/S_3(A) \Rightarrow S_1(A) \rightrightarrows M$.

is a Lie 2-groupoid.

Idea: $S_2(A) \rightsquigarrow \mathcal{F}$ on $S_1(A)$ take



A Weinstein groupoid over a manifold M consists:

- the space of objects : M a smooth manifold

- the space of arrows : G an étale differentiable stack

- $G \xrightarrow{\begin{matrix} s \\ t \end{matrix}} M$, source & target maps

surjective submersions

- $m: G_{\times_M^s G} \longrightarrow G$ multiplication : $\bullet t \circ m = t \circ \text{pr}_1, s \circ m = s \circ \text{pr}_1$

- 2-associative, i.e. a 2-morph s.t. the associative diagram

$$\begin{array}{ccc} G \times_G \times_G & \xrightarrow{\text{m id}} & G \times_G \\ \text{id} \times m \downarrow & \nearrow \text{id} \downarrow m & \\ G \times_G & \xrightarrow{m} & G \end{array}$$

is 2-commutat

identity section
↓
immersion

- $e: M \xrightarrow{\text{immersion}} G$, s.t. the following identities

$$m \circ ((e \circ t) \times \text{id}) = \text{id}, m \circ (\text{id} \times (e \circ s)) = \text{id}$$

hold up to 2-morphisms.

- (inverse) isomorphism: $i: G \rightarrow G$ s.t. up to 2-morp, the following identities hold :

$$m \circ (i \circ \text{id} \circ \Delta) = e \circ s, m \circ (\text{id} \times i \circ \Delta) = e \circ t$$

where $\Delta: G \times G \rightarrow G$ is the diagonal map.

Moreover, according to the identity section, the above 2-morphisms for 2-morphisms satisfy higher coherence conditions.
(like pentagon conditions, etc.)

$$\#(g \cdot h) = \#(g)$$

$$s(g \cdot h) = s(h)$$

$$(g \cdot h) \cdot k = g \cdot (h \cdot k)$$

$$1_{\text{Alg}} \cdot g = g, \quad g \cdot 1_{\text{Alg}} = g$$

$$g^{-1} \cdot g = 1_{\text{Alg}}$$

$$g \cdot g^{-1} = 1_{\text{Alg}}$$

Theorem 2. $\mathcal{G}(A) \rightrightarrows M$ and $\mathcal{H}(A) \rightrightarrows M$ are $W\text{-gpd}$.

$\mathcal{G}(A)$ is source 2-connected.
 $\mathcal{H}(A)$ is source 1-connected.

[$\mathcal{H}(A) = \text{the Lie gpd if } A \text{ is integrable.}$]

Hence $\mathcal{G}(A)$ is more universal.

Theorem 3 (Lie III).

$$\boxed{\begin{array}{c} \text{Lie} \\ \text{algebroids} \end{array}} \xrightleftharpoons[\text{integration}]{1-1} \boxed{\begin{array}{c} W\text{-gpd} \\ s\text{-2-conn} \end{array}}$$

Theorem 4 (Lie II).

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & & \downarrow \\ M & \longrightarrow & N \end{array} \Rightarrow \mathcal{G}(A) \xrightarrow{\Phi} \mathcal{H}$$

$\forall \mathcal{H} \quad \text{W-gpd of } B \quad \downarrow$

$$\begin{array}{ccc} N & & N \end{array}$$

Theorem 5. Upto homotopy and some étale conditions,

$$\boxed{\begin{array}{c} \text{Lie 2-gpd} \end{array}} \xrightleftharpoons[1-1]{\quad} \boxed{\begin{array}{c} W\text{-gpd} \end{array}}$$

Under this correspondence the 2-truncation

$$\dots S_2(A)/S_3(A) \rightrightarrows S_1(A) \rightrightarrows M. \quad \mathcal{G}(A) \rightrightarrows M$$

Application: Morita equivalence of Poisson mfd's (via
 Morita equivalence of their sympl ~~W-gpd~~.)