Essential dimension and algebraic stacks

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It is easy to see that if F is represented by a scheme X of finite type over k, then $\operatorname{ed} F = \dim X$.

The essential dimension $\operatorname{ed}\xi$ is finite, under weak hypothesis on F. But $\operatorname{ed} F$ could still be $+\infty$.

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In general finding lower bounds is much harder than finding upper bounds.

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It is also known that $\operatorname{ed}\mathrm{PGL}_3 = 2$;

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It is also known that $\operatorname{ed}\mathrm{PGL}_3 = 2$; this follows from the result of Albert on the cyclicity of central division algebras of degree 3.

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Computing $\operatorname{ed} \mathrm{PGL}_n$ when n is a prime is an extremely interesting question, linked with the problem of cyclicity of division algebras of prime degree.

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Theorem. *Let* $\mathcal{X}$ *be an algebraic stack of finite type over a field. Assume that for each object* ξ *of* $\mathcal{X}(K)$*, where* K *is an algebraically closed field, the group scheme* $\operatorname{Aut}_K(\xi)$ *is affine.*

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For example, $\mathcal{X} = \mathcal{M}_g$. What can we say about $\operatorname{ed}\mathcal{M}_g$? The condition of the theorem is satisfied for $g \neq 1$, hence $\operatorname{ed}\mathcal{M}_g < +\infty$ if $g \neq 1$.

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Let $C \to S$ be a family in $\mathcal{M}_g$ where S is integral of finite type over k. A *compression* of $C \to S$ consists of a non-empty open subset $U \subseteq S$ and a cartesian diagram

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Theorem. *Let $\mathcal{X}$ be a smooth connected separated Deligne–Mumford stack of finite type over a field k, $\mathcal{U}$ a non-empty open substack. Then* $\operatorname{ed} \mathcal{X} = \operatorname{ed} \mathcal{U}$.

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In particular, if the stabilizer of a generic point of $\mathcal{X}$ is trivial, then $\operatorname{ed} \mathcal{X} = \dim \mathcal{X}$.

This takes care of the case $g \geq 3$. For more general cases we need a more precise form of the theorem.

Let $\mathcal{X}$ be a smooth connected separated Deligne–Mumford stack of finite type over a field k with moduli space $\mathcal{X} \to \mathbf{X}$. Let K be the function field of $\mathbf{X}$. Let $\mathcal{X}_K \stackrel{\text{def}}{=} \mathcal{X} \times_{\mathbf{X}} \operatorname{Spec} K$ be the *generic gerbe* of $\mathcal{X}$.

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The automorphism group of a generic hyperelliptic curve is $\boldsymbol{\mu}_2 = \{\pm 1\}$. So $(\mathcal{H}_g)_K$ is banded by $\boldsymbol{\mu}_2$.

Let n be a positive integer. By a fundamental result of Grothendieck and Giraud, gerbes over a field K that are banded by $\boldsymbol{\mu}_n$ are classified by $\mathrm{H}^2(K, \boldsymbol{\mu}_n)$.

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A gerbe $\mathcal{X}$ banded by $\boldsymbol{\mu}_n$ has a class $[\mathcal{X}]$ in $\operatorname{Br} K$.

Each element of $\operatorname{Br} K$ comes from a PGL_m-torsor for some m, via the non-commutative boundary operator $\mathrm{H}^1(K, \mathrm{PGL}_m) \to \mathrm{H}^2(K, \mathbb{G}_\mathrm{m})$ coming from the sequence

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The theorem has important applications even in the "classical" case of the essential dimension of an algebraic group.

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How about spin groups?

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All this seemed to suggest that $\operatorname{ed}\operatorname{Spin}_n$ should be a slowly increasing function of n.

Theorem. *If n is not divisible by 4, then*

$$2^{\lfloor (n-1)/2 \rfloor} - \frac{n(n-1)}{2} \leq \operatorname{ed} \operatorname{Spin}_n \leq 2^{\lfloor (n-1)/2 \rfloor}\,.$$

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From this point on the exponential term takes over, the growth becames fast and the gap between the upper and the lower bound relatively smaller.

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In the theorem above, if n is a power of a prime p, then $\operatorname{ind} \partial P$ is also a power of the p.

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Theorem. *Suppose that n is a prime-power, and that P is a Q-torsor. Then*

$$\operatorname{ed} G \geq \operatorname{ind} \partial P - \dim G.$$

In the theorem above, if n is a power of a prime p, then $\operatorname{ind} \partial P$ is also a power of the p. This can be used to show that in many situations the essential dimension is much larger than expected.

The theorem can be applied to the sequence

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Let us sketch a proof of the theorem.

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We may assume that K is finitely generated over k. There exists a variety V over k with quotient field K and a Q-torsor $\widetilde{P} \to V$ whose generic fiber coincides with P. The torsor $\widetilde{P}$ corresponds to a morphism $V \to \mathcal{B}_k Q$.

We have a cartesian diagram

$$\begin{tikzcd}
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$$\operatorname{ed} \delta\widetilde{P} \leq \operatorname{ed} \mathcal{B}_k G + \dim V + \dim G.$$

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Conjecturally, equality holds. This is equivalent to a conjecture of Merkurjev and Colliot-Thélène on the canonical dimension of Brauer–Severi schemes. They proved it for $\operatorname{ind} \partial P = 6$.