ESSENTIAL DIMENSION AND ALGEBRAIC STACKS

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In general finding lower bounds is much harder than finding upper bounds.

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It is also known that $\operatorname{ed} \operatorname{PGL}_3 = 2$; this follows from the result of Albert on the cyclicity of central division algebras of degree 3.

When n is a prime larger than 3, it is only known (due to Reichstein) that

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Computing $\operatorname{ed} \operatorname{PGL}_n$ when n is a prime is an extremely interesting question, linked with the problem of cyclicity of division algebras of prime degree.

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Let \mathcal{X} be an algebraic stack over k. The essential dimension of \mathcal{X} over k, denoted by $\operatorname{ed} \mathcal{X}$ or $\operatorname{ed}(\mathcal{X}/k)$, is the essential dimension of the functor of isomorphism classes of objects of \mathcal{X} defined over extensions of k.

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For example, $\mathcal{X} = \mathcal{M}_g$. What can we say about $\operatorname{ed} \mathcal{M}_g$? The condition of the theorem is satisfied for $g \neq 1$, hence $\operatorname{ed} \mathcal{M}_g < +\infty$ if $g \neq 1$.

If C is a smooth projective curve over an extension K of k, the essential dimension of C is the least transcendence degree of a field of definition of C.

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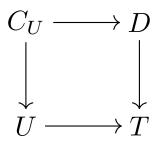
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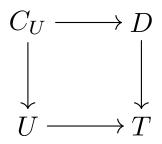
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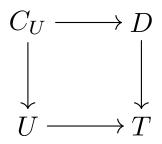
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If K is the function field of S, the essential dimension of the generic fiber $C_K \to \operatorname{Spec} K$ is the minimal dimension of T, taken over all compressions of $C \to S$.

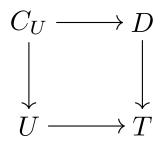
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where $D \to T$ is in \mathcal{M}_q .

If K is the function field of S, the essential dimension of the generic fiber $C_K \to \operatorname{Spec} K$ is the minimal dimension of T, taken over all compressions of $C \to S$. The essential dimension of \mathcal{M}_g is the supremum over all essential dimension of all families $C \to S$. Hence $\operatorname{ed} \mathcal{M}_g \geq 3g - 3$ if $g \geq 2$.

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Theorem. Let \mathcal{X} be a smooth connected separated Deligne–Mumford stack of finite type over a field k, \mathcal{U} a non-empty open substack. Then $\operatorname{ed} \mathcal{X} = \operatorname{ed} \mathcal{U}$.

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This takes care of the case $g \geq 3$. For more general cases we need a more precise form of the theorem.

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The automorphism group of a generic hyperelliptic curve is $\mu_2 = \{\pm 1\}$. So $(\mathcal{H}_g)_K$ is banded by μ_2 .

Let n be a positive integer. By a fundamental result of Grothendieck and Giraud, gerbes over a field K that are banded by μ_n are classified by $\mathrm{H}^2(K,\mu_n)$.

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Let n be a positive integer. By a fundamental result of Grothendieck and Giraud, gerbes over a field K that are banded by μ_n are classified by $H^2(K, \mu_n)$. From the Kummer sequence

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we get an exact sequence

$$0 = \mathrm{H}^1(K, \mathbb{G}_{\mathrm{m}}) \longrightarrow \mathrm{H}^2(K, \boldsymbol{\mu}_n) \longrightarrow \mathrm{H}^2(K, \mathbb{G}_{\mathrm{m}}) \xrightarrow{\times n} \mathrm{H}^2(K, \mathbb{G}_{\mathrm{m}}).$$

The group $H^2(K, \mathbb{G}_m)$ is called the *Brauer group* of K, and is denoted by $\operatorname{Br} K$. If \overline{K} is the algebraic closure of K and \mathcal{G} is the Galois group of \overline{K} over K, then $\operatorname{Br} K = H^2(\mathcal{G}, \overline{K}^*)$. Thus $H^2(K, \mu_n)$ is the n-torsion part of $\operatorname{Br} K$.

A gerbe \mathcal{X} banded by $\boldsymbol{\mu}_n$ has a class $[\mathcal{X}]$ in Br K.

 $\mathrm{H}^1(K,\mathrm{PGL}_m) \to \mathrm{H}^2(K,\mathbb{G}_m)$ coming from the sequence

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The index of class of the generic gerbe $(\mathcal{H}_g)_K$ is 1 if g is odd, 2 if g is even.

Theorem.

$$\operatorname{ed} \mathcal{H}_g = \dim \mathcal{H}_g + \operatorname{ed}((\mathcal{H}_g)_K/K) = \begin{cases} 2g & \text{if } g \text{ is odd} \\ 2g+1 & \text{if } g \text{ is even.} \end{cases}$$

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The theorem has important applications even in the "classical" case of the essential dimension of an algebraic group.

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How about spin groups?

$$\operatorname{ed}\operatorname{Spin}_n \geq \begin{cases} \lfloor n/2 \rfloor + 1 & \text{if } n \geq 7 \text{ and } n \equiv 1, 0 \text{ or } -1 \pmod{8} \\ \lfloor n/2 \rfloor & \text{for } n \geq 11. \end{cases}$$

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$$\begin{array}{lll} {\rm ed}\,{\rm Spin}_3 = 0 & {\rm ed}\,{\rm Spin}_4 = 0 & {\rm ed}\,{\rm Spin}_5 = 0 & {\rm ed}\,{\rm Spin}_6 = 0 \\ {\rm ed}\,{\rm Spin}_7 = 4 & {\rm ed}\,{\rm Spin}_8 = 5 & {\rm ed}\,{\rm Spin}_9 = 5 & {\rm ed}\,{\rm Spin}_{10} = 4 \\ {\rm ed}\,{\rm Spin}_{11} = 5 & {\rm ed}\,{\rm Spin}_{12} = 6 & {\rm ed}\,{\rm Spin}_{13} = 6 & {\rm ed}\,{\rm Spin}_{14} = 7. \end{array}$$

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All this seemed to suggest that $\operatorname{ed} \operatorname{Spin}_n$ should be a slowly increasing function of n.

$$2^{\lfloor (n-1)/2\rfloor} - \frac{n(n-1)}{2} \le \operatorname{ed} \operatorname{Spin}_n \le 2^{\lfloor (n-1)/2\rfloor}.$$

If n is divisible by 4 then

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So for example we get

$$23 \le \operatorname{ed} \operatorname{Spin}_{15} \le 128$$

 $9 \le \operatorname{ed} \operatorname{Spin}_{16} \le 129$
 $120 \le \operatorname{ed} \operatorname{Spin}_{17} \le 256$

From this point on the exponential term takes over, the growth becames fast and the gap between the upper and the lower bound relatively smaller.

The proof of this result is based on the following fact.

$$1 \longrightarrow \boldsymbol{\mu}_n \longrightarrow G \longrightarrow Q \longrightarrow 1.$$

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We get a boundary map

$$\partial \colon \operatorname{H}^{1}(K, Q) \longrightarrow \operatorname{H}^{2}(K, \boldsymbol{\mu}_{n}) \subseteq \operatorname{Br} K.$$

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Theorem. Suppose that n is a prime-power, and that P is a Q-torsor. Then

$$\operatorname{ed} G \ge \operatorname{ind} \partial P - \operatorname{dim} G.$$

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In the theorem above, if n is a power of a prime p, then ind ∂P is also a power of the p.

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In the theorem above, if n is a power of a prime p, then ind ∂P is also a power of the p. This can be used to show that in many situations the essential dimension is much larger than expected.

The theorem can be applied to the sequence

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If P is an SO_n -torsor, $\partial P \in H^2(K, \mu_2) \subseteq Br K$ it the Hasse invariant of P.

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$$= 2^{\left\lfloor \frac{n-1}{2} \right\rfloor} - \frac{n(n-1)}{2}.$$

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Let us sketch a proof of the theorem.

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If P is a Q torsor over an extension K of k, the gerbe δP of liftings of P to a G-torsor is banded by μ_n .

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If P is a Q torsor over an extension K of k, the gerbe δP of liftings of P to a G-torsor is banded by μ_n . Its class in $\mathrm{H}^2(K,\mu_n)$ is the image ∂P in $\mathrm{H}^2(K,\mu_n)$ of the class of P in $\mathrm{H}^1(K,Q)$.

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$$\operatorname{ed} G \ge \operatorname{ed}(\delta P/K) - \dim G.$$

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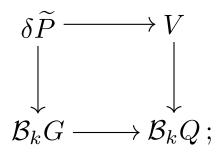
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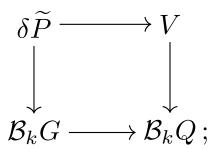
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We may assume that K is finitely generated over k. There exists a variety V over k with quotient field K and a Q-torsor $\widetilde{P} \to V$ whose generic fiber coincides with P. The torsor \widetilde{P} corresponds to a morphism $V \to \mathcal{B}_k Q$.



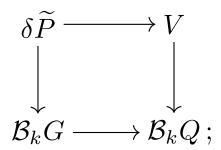
$$\delta \widetilde{P} \xrightarrow{V} V$$
 $\downarrow \qquad \qquad \downarrow$
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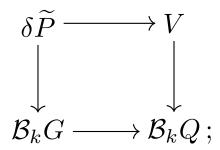
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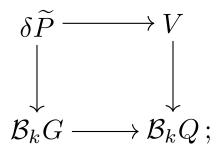
On the other hand $V \to \mathcal{B}_k Q$ is representable, with fibers of dimension $\dim V + \dim Q = \dim V + \dim G$;



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$$\operatorname{ed} \delta \widetilde{P} \leq \operatorname{ed} \mathcal{B}_k G + \dim V + \dim G.$$

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Let P be a Q-torsor. Write the prime factor decomposition

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Conjecturally, equality holds. This is equivalent to a conjecture of Merkurjev and Colliot-Thélène on the canonical dimension of Brauer-Severi schemes. They proved it for ind $\partial P = 6$.