$t^{1/3}$ Diffusivity of Finite-Range Asymmetric Exclusion Processes on $\mathbb{Z}$

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joint work with J. Quastel (U of T)

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Asymmetric Exclusion Process on $\mathbb{Z}$

$p(\cdot)$: finite-range jump rate

$b = \sum_z zp(z) \neq 0$ nonzero drift

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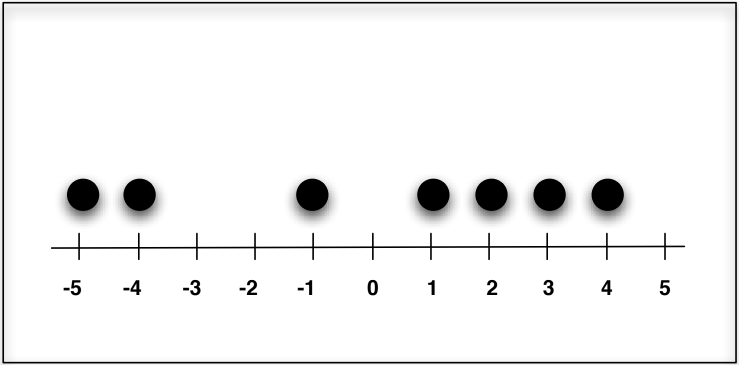

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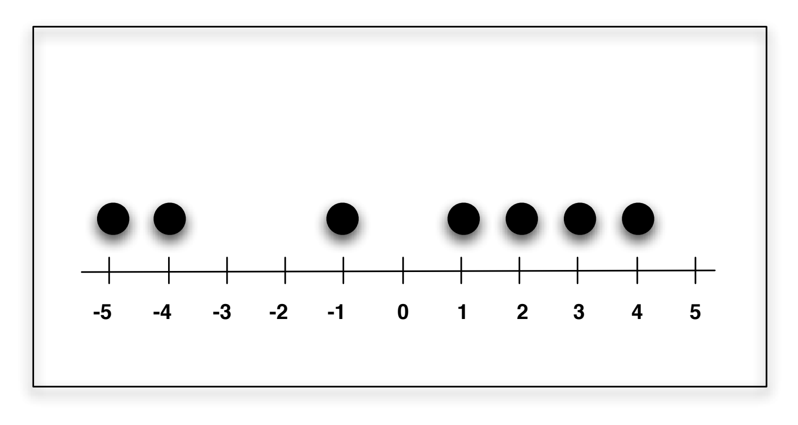

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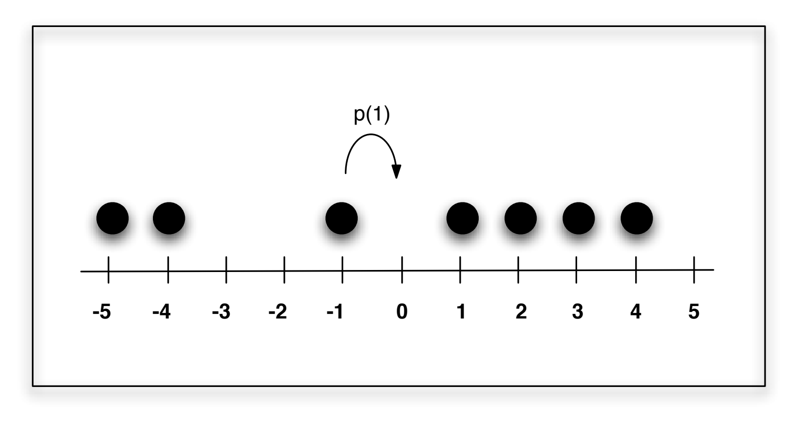

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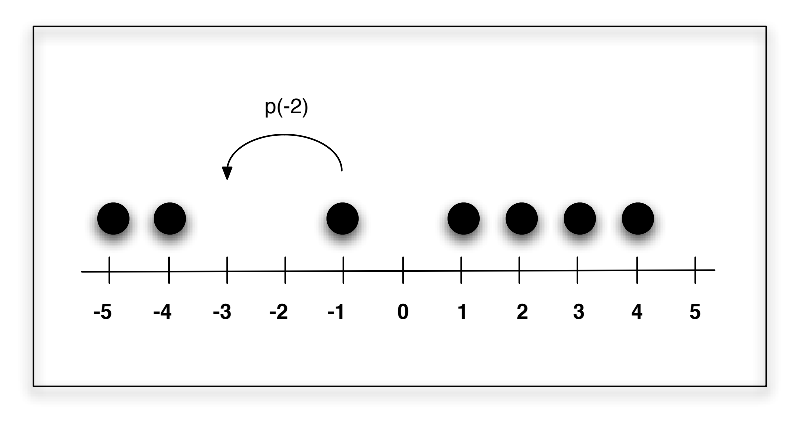

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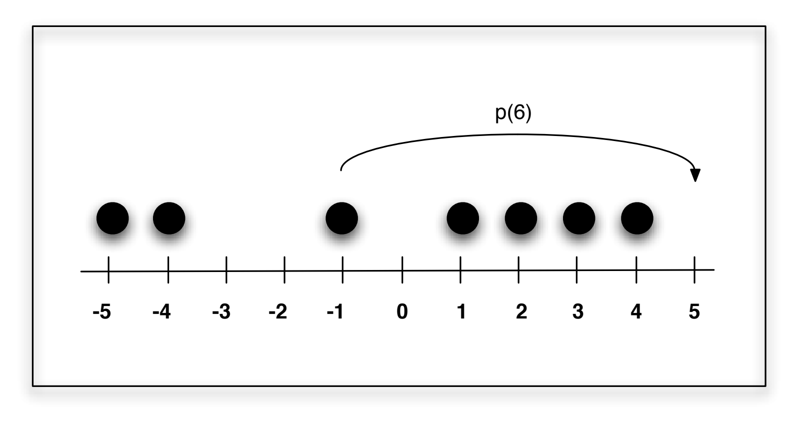

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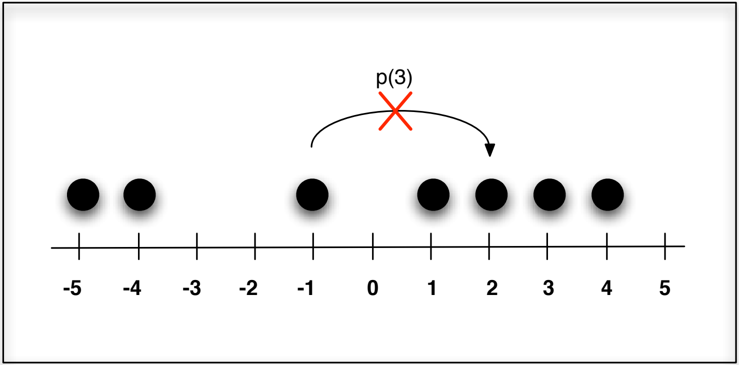

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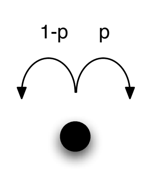
EXCLUSION RULE!

Asymmetric Exclusion Process on $\mathbb{Z}$

Asymmetric Simple Exclusion Process (ASEP)



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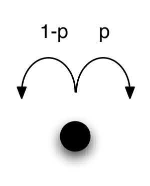
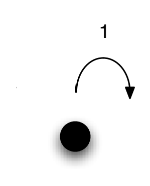


Totally Asymmetric Exclusion Process (TASEP)



The Stationary Process

$\eta_x(t) \in \{0,1\}$: number of particles at site x at time t

$\rho \in (0,1)$: density

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SCALING PROPERTIES?

Space-Time Covariance

$$S(x,t) = \mathsf{E}\big[(\eta_0(0)-\rho)(\eta_x(t)-\rho)\big]$$

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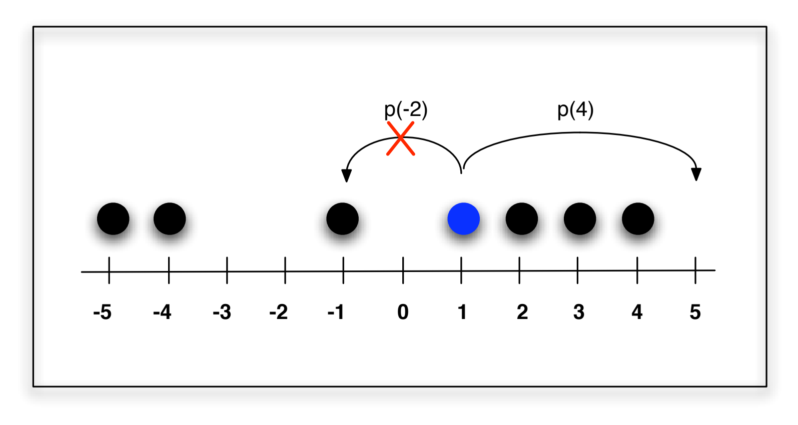
Same jump rate, same exclusion rule



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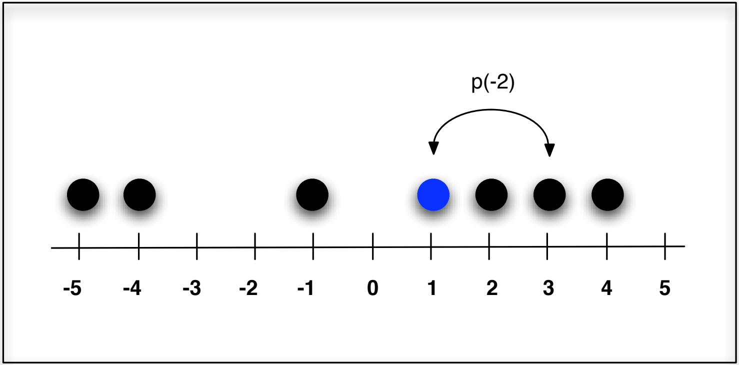
"First Class" particles don't see it



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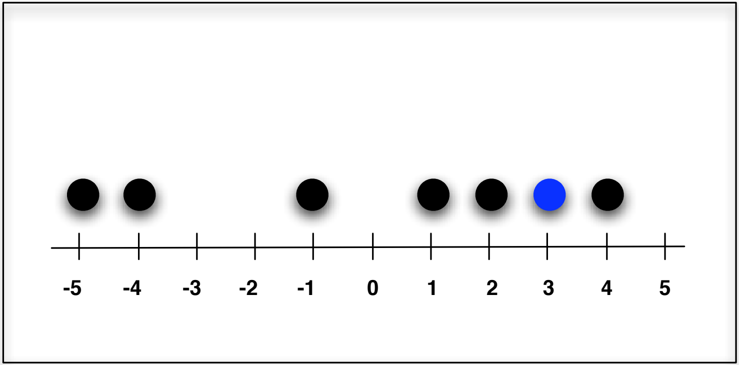

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Moments

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  $\sum_x x^2 S(x,t) - \chi(1-2\rho)^2 t^2 b^2 = ?$

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superdiffusive behavior

Previous Results

Johansson (2000), Baik-Reins (2000):

Limit theorem with $1/3$ exponent for the current fluctuations (TASEP, not the stationary process)

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Landim-Quastel-Salmhofer-Yau (2004)
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'For free':

$$D^{TASEP}(t) \geq C\, t^{1/3}$$

New Results

For any finite-range Asymmetric Exclusion Process on $\mathbb{Z}$:

$$D(t) \geq C\, t^{1/3} \quad \text{in the weak sense}$$

How to get 'strong' bounds?

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For non-negative increasing functions

$$\text{(one-sided) bounds on growth of the Laplace transform as } \lambda \to 0$$
$$\Downarrow$$
$$\text{(one-sided) bounds on the growth of the function as } t \to \infty$$

How to get 'strong' bounds?

Lemma (Q-V):
For ASEP (i.e. nearest neighbor exclusion) $t\,D(t)$ is increasing.

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$$\Downarrow$$

$$D^{ASEP}(t) \geq C\, t^{1/3}(\log t)^{-7/3}$$

Newer Results

Balázs-Seppäläinen (2006):

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$\Downarrow$ (Comparison Theorem)

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$$C_1 t^{1/3} \geq D(t) \geq C\, t^{1/3} \quad \text{in the weak sense}$$

Newer Results

For any finite-range Asymmetric Exclusion Process on $\mathbb{Z}$:

$$Ct^{1/3} \geq D(t) \quad \text{in the usual sense}$$

Main Tools

Green-Kubo formula:

$$D(t) = \sum_z z^2 p(z) + 2\chi t^{-1} \int_0^t \int_0^s \langle\!\langle w, e^{uL} w \rangle\!\rangle du ds$$

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$$D(t) = \sum_z z^2 p(z) + 2\chi t^{-1} \int_0^t \int_0^s \langle\langle w, e^{uL} w \rangle\rangle duds$$

$$\begin{aligned} \langle\langle \phi, \psi \rangle\rangle &= \langle \phi, \sum_x \tau_x \psi \rangle \\ w &= \text{microscopic current} \\ L &= \text{generator} \end{aligned}$$

Main Tools

Taking the Laplace transform

$$\int_0^\infty e^{-\lambda t} t D(t) dt = \lambda^{-2} \left(\sum_z z^2 p(z) + 2\chi \| w^2 \|_{-1,\lambda} \right)$$

where

$$\|\phi\|_{-1,\lambda} = \langle\langle \phi, (\lambda - L)^{-1} \phi \rangle\rangle^{1/2}$$

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Proof of the Comparison Theorem

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- Different norms are comparable
- $\|w^A - kw^B\|_{-1,\lambda}$ is small

Upper Bound from Weak Upper Bound

Landim-Yau (1998)

$$t^{-1}E[\langle\!\langle \int_0^t w(s)ds, \int_0^t w(s)ds \rangle\!\rangle] \leq \|w\|^2_{-1,t^{-1}}$$

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With Green-Kubo the bound follows.

$$D(t) = \sum_z z^2 p(z) + 2\chi t^{-1} \int_0^t \int_0^s \langle\!\langle w, e^{uL} w \rangle\!\rangle duds$$

Open Questions

- Lower bound for $D(t)$
- Limit of $D(t)t^{-1/3}$
- Scaling limit?