# Conditioning super-Brownian motion on its exit measure $X_D$

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**Dynkin's definition of a super-Brownian motion:**

Let $E$ be an open domain of $\mathbb{R}^d$.

A super-Brownian motion is a family of measures $X = (X_D)$ indexed by subdomains of $E$ and a family of probability law's for $X$, $P_\mu$, where $\mu$ are finite measures on $E$.

There are four basic properties:

a) Exit property: $(X_D)$ puts mass only on the boundary of $D$ $P_\mu$ a.s. for all $\mu$ with $\mu(D) > 0$.

b) Markov property: If $D' \subset D \subset D''$, then $X_{D'}$ is conditionally independent of $X_{D''}$ given $X_D$.

c) Branching property: $P_\mu$-law of $X_D$ is the superposition of its laws under $P_{\mu_1}$ and $P_{\mu_2}$ if $\mu_1 + \mu_2 = \mu$.

d) PDE of the log-Laplace functional: If $u(x) = -log P_x(e^{-<f,X_D>})$ then $u$ solves the pde

$$Lu = 2u^2$$

where $L = \frac{1}{2}\Delta$ and $u(x_n) \to f(x)$ as $x_n$ converge to $x$ on the boundary of $D$.

We fix $D \subset E$.

For any $P_\mu$, we would like to find and describe the conditional law $P_\mu^{X_D}$ on

$$\mathcal{F}_D = \sigma(X_{D'}, D'\text{a subdomain of } D).$$

given $X_D$.

**Remark:** We will be interested only on non-zero realizations of $X_D$.

A simpler problem:

Consider the same conditioning under a different measure $N_x$, the "excursion law" or Lévy law of $X$.

$X$ can be obtained by adding up $X_1, \dots X_N$, where $X_i$ are sampled from a Poisson random measure with intensity $\int N_x \mu(dx)$.

We first find and describe $N_x^{X_D}$, then describe $P_\mu^{X_D}$ based on $N_x^{X_D}$.

**How to find $N_x^{X_D}$?**

We obtain $N_x^{X_D}$ as an $H$-transform of $N_x$.

$H$-**transform of** $N_x$**:** $N_x^H$ is obtained from $N_x$ and an $X$-harmonic function $H$ defined on the space of finite measures supported in $D$:

Let $D' \subset D$.

$$N_x^H(Y) = N_x(H(X_{D'})Y)$$

for all $\mathcal{F}_{D'}$-measurable non-negative $Y$.

To extend it to all of $\mathcal{F}_D$ requires $H$ to be $X$-harmonic that is

$$P_\mu(H(X_{D'})) = H(\mu)$$

for all $\mu$ and $D' \subset D$.

In our case, for almost every realization $\nu$ of $X_D$, $H^\nu(\mu)$ is a version of

$$\frac{dP_\mu(X_D \in (.))}{dN_x(X_D \in (.))}.$$

**Branching backbone construction of $N_x^\nu$:**

We consider the following system:

$u$: the maximal solution of the non-linear pde in the domain $D$

$$Lu = 2u^2$$

where $L = \frac{1}{2}\Delta$.

$L_{4u}$-diffusion: Brownian motion killed at rate $4u$.

A function $\gamma^\nu(x)$ which is an $L_{4u}$ potential in $x$ for almost all $\nu$.

A fragmentation kernel $K_x(\nu, d(\nu_1, \nu_2))$, which is for fixed $\nu$ a probability measure on the pairs of measures $(\nu_1, \nu_2)$ such that $\nu_1 + \nu_2 = \nu$.

Here is how we construct the branching backbone:

A particle starts from $x$, following a $\gamma^{\nu}$-transform of an $L_{4u}$-diffusion and dies somewhere inside $D$, say at point $y$.

When it dies two new particles start from $y$, following $\gamma^{\nu_1}$ and $\gamma^{\nu_2}$ transforms of $L_{4u}$-diffusion where $\nu_1$ and $\nu_2$ are chosen according to the kernel $K_y(\nu, .)$.

Same process repeats for these new particles and so on.

**Creation of mass:**

$\widehat{N}_x$: Lévy law of a super-diffusion whose spatial motion is Brownian motion killed at rate $u$, i.e. a super-diffusion conditioned to become extinct at the boundary.

Let each particle $x$ in the back-bone emit a super-diffusion at a constant rate during its life-time, according to the law $\widehat{N}_{x(t)}$.

**Theorem**: *$N_x^\nu$ is the law of the cumulative contribution of each of these emissions as $t$ goes to $\infty$.*

**How to get $P_\mu^\nu$?**

Take a Poisson sample of points $x_1, \ldots, x_n$.

Start a particle from $x_i$, assigning it a measure $\nu_i$, a random fragment of $\nu$, and

$$\nu_1 + \nu_2 + \ldots + \nu_n = \nu.$$

Let each individual particle evolve independently according to $N_{x_i}^{\nu_i}$.

**Related work**

Salisbury and Verzani (1999) considered a family of $X$-harmonic functions, characterized by a function $g$, a solution of

$$Lu = 2u^2$$

and harmonic functions $v_1, \ldots, v_k$.

They showed if $g = 0$, and $v_i = k(., z_i)$ where $z_1, \ldots, z_k \in \partial D$, and $k(x, z)$ is the Poisson kernel, then the resulting $H$ transform corresponds to conditioning the super-Brownian motion to hit the points $z_i$.

This family of $H$-transforms also has a branching backbone construction similar to ours.

• We should get our system from theirs in the scaling limit. Why?

$H$-transform for conditioning the super-Brownian motion on the outcome of a Poisson sample from the boundary with intensity $nX_D$ is in their class of $X$-harmonic functions.

Note that as $n$ goes to infinity, this conditioning "approaches" to ours.

Dynkin (2005): Finding the elements of Martin boundary of a super-diffusion.

(Martin boundary for a super-diffusion: The set of extreme elements of the convex set of all X-harmonic functions.)

• So far, no probabilistic construction of the elements of Martin boundary is known.

Dynkin showed if $H$ is extreme then for every $\mu$, and for every sequence $D_k$ exhausting $D$

$$H(\mu) = \lim_{k \to \infty} H_{D_k}^{X_{D_k}}(\mu)$$

$P^H$ almost surely.

- $H_D^\nu$ can be interpreted as the analogue of the Poisson kernel.

Our next goal is to answer:

Is $H_D^\nu$ is extreme?