Mean-reverting market model: Novikov condition, speculative opportunities, and non-arbitrage *

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Summary

Let us consider 3 facts:

(A) $\mathbf{E}\exp(T\xi^2) = +\infty$ for any Gaussian random variable ξ and any $T > 1/(2\mathrm{Var}\,\xi)$.

(B) Let $R(t)$ be a Gaussian Ornsten-Uhlenbek process,

$$dR(t) = -\lambda R(t)dt + \sigma dw(t).$$

The existence of an equivalent measure such that $R(t)$ is a martingale for $t \in [0, T]$ is usually ensured by Novikov condition:

$$\mathbf{E}\exp\left(\frac{\lambda^2}{2\sigma^2}\int_0^T \tilde{R}(t)^2 dt\right) < +\infty. \qquad (0.1)$$

By (A), it is unclear if this condition holds for large T. The hypothesis that Novikov condition holds for large T is non-trivial, and it is even counterintuitive. [1]

(C) A market model with the stock price $S(t) = e^{R(t)}$ is arbitrage free if Novikov condition is satisfied (in sligtly different form than (0.1)). However, there is common sense that a mean-reverting market model may allow some speculative opportunities.

We study these facts and their connections.

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[1] the similar question arises for the process $\exp(R(t))$: can it be a martingale on a large time interval?

1 The mean-reverting model

We assume the so-called mean-reverting model, when the discounted stock price evolves as

$$\begin{aligned} \tilde{S}(t) &= s_0 e^{\tilde{R}(t)}, \\ d\tilde{R}(t) &= (\alpha - \lambda \tilde{R}(t))dt + \sigma dw(t), \end{aligned} \tag{1.1}$$

where $\sigma > 0$, α, $\lambda > 0$ are deterministic. We assume that $\tilde{R}(0)$ is non-random. The process $\tilde{R}(t)$ is Gaussian. (If $\alpha = 0$, then $\tilde{R}(t)$ is an Ornstein-Uhlenbek process)

Proposition 1.1 *There exists a stationary Gaussian process $\tilde{R}_0(t)$ is Gaussian, and $\mathbf{E}|R(t) - R_0(t)|^2 \to 0$ and $\tilde{R}(t) \to \tilde{R}_0(t)$ a.s. as $t \to +\infty$.*

Clearly,

$$d\tilde{S}(t) = \tilde{S}(t)\left(\tilde{a}(t)dt + \sigma(t)dw(t)\right), \quad t > 0, \tag{1.2}$$

where

$$\tilde{a}(t) = \alpha - \lambda \tilde{R}(t) + \frac{\sigma^2}{2}, \quad \sigma(t) = \sigma.$$

Definition 1.2 *We say that the Novikov condition is satisfied for a time interval $[0, T]$ if*

$$\mathbf{E} \exp \frac{1}{2} \int_0^T \tilde{a}(t)^2 \sigma(t)^{-2} dt < +\infty. \tag{1.3}$$

It is well known that if the Novikov condition is satisfied, then By Girsanov's Theorem, one can define the (equivalent) probability measure $\mathbf{P}_{*,T}$ such that $w_*(t) \triangleq w(t) + \int_0^t \sigma^{-1} \tilde{a}(s) ds$ is a Wiener process under $\mathbf{P}_{*,T}$ for $t \in [0, T]$.

The following result is well known.

Theorem 1.3 *Let a market model be such that Novikov condition is satisfied for some $T > 0$. Then the market model does not allow arbitrage for the time interval $[0, T]$.*

2 Absence of arbitrage for the mean-reverting model

Theorem 2.1 *For any $\kappa \in \mathbf{R}$, $T > 0$,*

$$\mathbf{E} \exp\left(\frac{1}{2\sigma^2} \int_0^T [\kappa - \lambda \tilde{R}(t)]^2 dt \right) < +\infty. \tag{2.1}$$

In particular, Novikov condition (1.3) holds for any $T > 0$ for the mean-reverting stock price model with $\tilde{a}(t) = \alpha - \lambda \tilde{R}(t) + \sigma^2/2$.

As is known, $\mathbf{E}\exp(T\xi^2) = +\infty$ for any Gaussian random variable ξ and any $T > 1/(2\mathrm{Var}\,\xi)$. For the mean-reverting model, $\tilde{a}(t)$ is Gaussian, and $\tilde{a}(t) \to \mathrm{const} - \lambda\tilde{R}_0(t)$ as $t \to +\infty$, where $\tilde{R}_0(t)$ is a stationary Gaussian process. Therefore, the fact that the Novikov condition holds for mean-reverting model for large T is non-trivial, and it is even counterintuitive. The proof uses certain properties of the mean-reverting process; in particular, constant in time Gaussian processes are excluded.

Corollary 2.2 *For any $T > 0$, there exists an equivalent probability measure $\mathbf{P}_* = \mathbf{P}_{*,T}$ such that the process $\tilde{R}(t)$ is a martingale under $\mathbf{P}_*$ in $t \in [0,T]$ with respect to the filtration $\mathcal{F}_t$. If $\tilde{R}(0) = 0$, then the process $\tilde{R}(t)/\sigma$ is a Wiener process under $\mathbf{P}_*$ in $t \in [0,T]$.*

It will be useful to consider the log-normal processes.

Corollary 2.3 *For any $T > 0$, there exists an equivalent probability measure $\widehat{\mathbf{P}}_* = \widehat{\mathbf{P}}_{*,T}$ such that the process $\tilde{S}(t) \triangleq s_0 e^{\tilde{R}(t)}$ is a martingale under $\widehat{\mathbf{P}}_*$ in $t \in [0,T]$ with respect to the filtration $\mathcal{F}_t$.*

Existence of an equivalent martingale measure is a sufficient condition of absence of arbitrage for a given finite time interval $[0,T]$.

Corollary 2.4 *The mean-reverting model does not allow arbitrage for the time interval $[0,T]$ for any $T > 0$.*

Discussion and some applications

Non-robustness of Novikov condition for the mean-reverting model for large time intervals

Lemma 2.5 *Let $\tilde{R}(0)$ be such as described in Remark **??**. Let $\alpha = 0$. Then, for any $\varepsilon > 0$, there exist $T > 0$ such that*

$$\mathbf{E}\exp\left(\left[\frac{\lambda^2}{2\sigma^2} + \varepsilon\right]\int_0^T \tilde{R}(t)^2 dt\right) = \infty. \tag{2.2}$$

Corollary 2.6 *Let the stock price evolution be described by equation (**??**) with $(a(t), \sigma(t))$ such that $\tilde{a}(t) \equiv \lambda_1\tilde{R}(t)$, $\sigma(t) \equiv \sigma$, where $\lambda_1, \sigma \in \mathbf{R}$ are given, and where $\tilde{R}(t)$ is defined by (1.1) with the same σ, with $\alpha = 0$, and with some $\lambda > 0$. It follows from the results above that if $|\lambda_1| \le \lambda$, then Novikov condition holds for any finite time interval $[0,T]$. If $|\lambda_1| > \lambda$, then there exists time $T > 0$ such that Novikov condition does not hold for time interval $[0,T]$. (Note that the case when $\lambda_1 = \lambda$ corresponds to the mean reverting model).*

Lemma 2.5 and Corollary 2.6 mean that the mean-reverting model is on the "edge" of the area where Novikov condition holds for all time intervals.

On the sets of trajectories for Wiener and Ornstein-Uhlenbek processes

In fact, a Wiener process can be transformed to an Ornstein-Uhlenbek process on unlimited time interval by some exponential time change (see, e.g., Cox and Miller (1965), p. 224). Furthermore, it is easy to see that a Wiener process is an Ornstein-Uhlenbek process under a changed probability measure on any small enough time interval $[0, T]$ without time change, i.e., with the same time scale (it suffices to notice that Novikov condition holds for small enough T and apply Girsanov Theorem). Corollary 2.2 shows that this is also true for any finite arbitrarily large time interval $[0, T]$. In particular, it follows that a Wiener process has the same set of trajectories as an Ornstein-Uhlenbek process in the same time scale for any finite time interval $[0, T]$. It follows that an Ornstein-Uhlenbek processes cannot be distinguished surely from a Wiener process using observations of a sample on any arbitrarily large finite time interval.

Applications to optimal portfolio selection problems

There is one more application of Theorem 2.1 to mathematical finance: it can be used for analysis of diffusion market models such that the appreciation rate $a(t)$ is a Gaussian process satisfying linear Ito equations. Lakner (1995), (1998), and Dokuchaev (2005), studied optimal portfolio selection problems in this setting. To ensure that there exists a equivalent martingale measure, some restrictive conditions on upper bound for admissible terminal time T were imposed in these three papers. Now these restrictions can be lifted for the special case when the appreciation rate is a process defined by (1.1), where $\tilde{R}(0)$ is such as described in Remark **??**, and for the models of prices from Corollary 2.6.

3 Speculative opportunities for the mean-reverting model

3.1 On wealth and strategies

We consider the diffusion model of a securities market consisting of a risk free bond or bank account with the price $B(t)$, $t \geq 0$, and a risky stock with price $S(t)$, $t \geq 0$. The price of the bond evolves as

$$B(t) = e^{rt}B(0), \tag{3.1}$$

where $r \geq 0$ is a risk free interest rate.

Let $X(0) > 0$ be the initial wealth at time $t = 0$, and let $X(t)$ be the wealth at time $t > 0$. We assume that the wealth $X(t)$ at time $t \geq 0$ is

$$X(t) = \beta(t)B(t) + \gamma(t)S(t). \tag{3.2}$$

Here $\beta(t)$ is the quantity of the bond portfolio, $\gamma(t)$ is the quantity of the stock portfolio, $t \geq 0$. The pair $(\beta(t), \gamma(t))$ describes the state of the bond-stocks securities portfolio at time t. Each of these pairs is called a strategy.

3.2 The discounted wealth and stock prices

It is natural to estimate the loss and gain by comparing it with the results for the "keep-only-bonds" strategy.

Definition 3.1 *The process* $\tilde{X}(t) \triangleq e^{-rt}X(t)$*. The process* $\tilde{S}(t) \triangleq e^{-rt}S(t)$, $\tilde{S}(0) = S(0)$, *is called the discounted stock prices.*

Let $\tilde{a}(t) \triangleq a(t) - r(t)$. Clearly,

$$d\tilde{S}(t) = \tilde{S}(t)(\tilde{a}(t)dt + \sigma(t)dw(t)).$$

Proposition 3.2 *An admissible straregy is self-financing if and only if*

$$d\tilde{X}(t) = \gamma(t)d\tilde{S}(t), \tag{3.3}$$

i.e.,

$$\tilde{X}(t) = X(0) + \int_0^t \gamma(s)d\tilde{S}(s). \tag{3.4}$$

4 Existence of speculative opportunities

In this section, we assume the mean reverting model for stock prices, with deterministic $\tilde{R}(0)$.

Let $\mathcal{A}$ denote the class of functions $\Psi : (0, +\infty) \to \mathbf{R}$ such that

- $\Psi(S(0)) = X(0)$;
- $\Psi(x)$ is concave, continuous, and twice differentiable in $x > 0$;
- $\Psi(x)$ is not an affine function on $(0, +\infty)$, i.e., $\Psi(x)$ cannot be represented as $Cx + c$ with $C, c \in \mathbf{R}$.
- there exists $C > 0$, $c > 0$ such that

$$\left| \frac{d^k\Psi}{dx^k}(x) \right| \le C(1 + |x|^c + |x|^{-c}), \quad k = 0, 1, 2.$$

Proposition 4.1 *For the mean-reverting market model, the self-financing strategy $(\beta(\cdot), \gamma(\cdot))$ is admissible, if $\gamma(t) = \frac{d\Psi}{dx}(\tilde{S}(t))$, where $\Psi \in \mathcal{A}$. In that case, $\mathbf{E}X(T)^2 < +\infty$ for all T.*

Let $\Psi \in \mathcal{A}$ be given. Set $\psi_- \triangleq \inf_{x \ge 0} \Psi(x)$. Let $\Gamma \triangleq [\psi_-, +\infty)$, if $\psi_- > -\infty$, and let $\Gamma \triangleq \mathbf{R}$, if $\psi_- = -\infty$.

Let $\mathcal{U}_0$ be the class of all functions $U(x) = \delta^{-1}x^\delta$, $\delta < 1$. We assume that this class includes also $U(x) = \ln x$, which corresponds to $\delta = 0$.

Let $\widehat{\mathcal{U}}$ denotes the class of all monotonic non-decreasing functions $U : \Gamma \to \mathbf{R}$ such that there exists $C = C_U$ such that $U(x) < C(|x| + 1)$ $(\forall x \in \Gamma)$.

Note that if $U \in \widehat{\mathcal{U}}$, then $\mathbf{E}U^+(\tilde{X}(T)) < +\infty$ for all T and for admissible strategies, where $\tilde{X}(\cdot)$ is the corresponding discounted wealth, $U^+(x) \triangleq \max(0, U(x))$. Hence the expectation $\mathbf{E}U(\tilde{X}(T))$ is well defined (the case when $\mathbf{E}U(\tilde{X}(T)) = -\infty$ is not excluded).

Let $\mathcal{U}_\Psi^+$ denotes the set of all functions $U \in \widehat{\mathcal{U}}$ such that $\inf_{x \in \Gamma} U(x) > -\infty$.

Let $\mathcal{U}_\Psi$ denotes the set of all functions $U \in \widehat{\mathcal{U}}$ such that $\sup_{x \in \Gamma} U(x) = +\infty$, and there exists $\nu > \sigma^2/(2\lambda)$ such that $\int_{-\infty}^{+\infty} \exp\left(-\frac{x^2}{\nu}\right) u^-(x)^2 dx < +\infty$, where $u(x) \triangleq U(\Psi(s_0 e^x))$, $u^-(x) \triangleq \max(0, -u(x))$.

Remark 4.2 *The classes of utilities $\mathcal{U}_\Psi^+$ and $\mathcal{U}_\Psi$ can be reasonably wide, with an appropriate choice of Ψ. For instance, let $\Psi(x) = Cx^p + c$, $p \in (0, 1)$, $C > 0$, $c \ge 0$. Clearly, $\psi_- = c$. If $c > 0$, then $\mathcal{U}_0 \subset \mathcal{U}_\Psi^+$. If $c \ge 0$, then $\mathcal{U}_\Psi$ contains all functions $U(x) = \delta^{-1}x^\delta$, $\delta > 0$, and $U(x) = \ln x$.*

Let

$$\tilde{S}_0(t) \triangleq s_0 e^{\tilde{R}_0(t)}.$$

Theorem 4.3 *Assume the mean-reverting model for the stock prices. Let $\Psi \in \mathcal{A}$ be given, and let a self-financing strategy $(\beta(\cdot), \gamma(\cdot))$ be such that*

$$\gamma(t) = \frac{d\Psi}{dx}(\tilde{S}(t)).$$

Then this strategy has the following properties:

(i) The corresponding discounted wealth $\tilde{X}(T)$ is such that

$$\tilde{X}(T) = \Psi(\tilde{S}(T)) + \zeta(T), \tag{4.1}$$

where $\Psi(\tilde{S}(T))$ converges to the stationary process $\Psi(\tilde{S}_0(T))$ as $T \to +\infty$ with probability 1, and $\zeta(T)$ is a monotonically increasing in T process such that $\zeta(T) \geq 0$, $\zeta(T) \to +\infty$ a.s.

(ii) If $\psi_- > -\infty$ and $U \in \mathcal{U}_\Psi^+$, then $\tilde{X}(T) \to +\infty$ as $T \to +\infty$ with probability 1, and $\mathbf{E}U(\tilde{X}(T)) \to \sup_{x\in\Gamma} U(x)$ as $T \to +\infty$.

(iii) If $U \in \mathcal{U}_\Psi$, then $\mathbf{E}U(\tilde{X}(T)) \to +\infty$ as $T \to +\infty$.

Remark 4.4 *The strategy does not use any knowledge about the value of $(\alpha, \lambda, \sigma)$; this makes it similar to the strategies from technical analysis. Moreover, the strategy is risk bounded with an appropriate Ψ for all $(\alpha, \lambda, \sigma)$ or even uniformly over all $(\alpha, \lambda, \sigma)$. For instance, if $\Psi(x) = x^p + c$, $p \in (0, 1)$, $c \geq 0$, and let $X(0) = S(0)^p + c$, then*

$$\tilde{X}(t) = c + \tilde{S}(t)^p + \frac{p(1-p)}{2} \int_0^t \sigma(s)^2 \tilde{S}(s)^p ds \geq c + \tilde{S}(t)^p. \tag{4.2}$$

Note that (4.2) holds even if the model is not mean-reverting; in fact, it holds for any process $(r(t), a(t), \sigma(t))$ such as described in Section 3.1.

Remark 4.5 *For a discrete time market model, strategies that explore oscillating of prices or stationarity can also be found (examples of these strategies were given in Dokuchaev (2006) for price series oscillating in a given interval).*

5 Speculative opportunities: formal definitions and examples

Frittelli (2004) introduced a utility based "market free lunch" as an alternative description of arbitrage opportunities for financial markets, in addition to "free lunch". Further details can be found in Klein (2006). We explore below some similar utility-based definitions; our purpose is to underline the features of the mean-reverting model. However, the exact definitions given in the cited papers cannot separate the mean-reverting model from other models. We saw that the mean-reverting model is such that a martingale measure exists for any finite time interval, and it is the same situation as for the model with bounded risk premium process $\sigma(t)^{-1}\tilde{a}(t)$: there is no arbitrage for any finite time interval. However, Theorem 4.3 shows that the mean-reverting model allows some special opportunities that cannot be expressed via the terms of arbitrage, asymptotic arbitrage, "free lunch", or "market free lunch". To describe the situation more clearly, we introduce the following definition (that uses a class of utilities, similarly to Frittelli (2004)).

Definition 5.1 *Let $\bar{\Sigma}$ be a class of admissible self-financing strategies $\{(\beta(\cdot),\gamma(\cdot))\}$. Let $\mathcal{U}$ be a class of functions $U:\mathbf{R}\to\mathbf{R}$.*

(i) We say that a market allows speculative opportunities with respect to $(\bar{\Sigma},\mathcal{U})$, if there exists a strategy $(\beta(\cdot),\gamma(\cdot))\in\bar{\Sigma}$ such that

$$\inf_{T>0}\mathbf{E}U(\tilde{X}(T))>-\infty\quad\forall U\in\mathcal{U},$$

and that there exists $U\in\mathcal{U}$ such that

$$\mathbf{E}U(\tilde{X}(T))\to\sup_x U(x)\quad as\quad T\to+\infty. \tag{5.1}$$

Here $\tilde{X}(t)$ is the corresponding discounted wealth.

(ii) We say that a market allows strong speculative opportunities with respect to $(\bar{\Sigma},\mathcal{U})$, if there exists a strategy from $\bar{\Sigma}$ such that (5.1) holds for all $U\in\mathcal{U}$.

(Note that we do not exclude the case when $\sup_x U(x)=+\infty$ in (5.1)).

At first sight, speculative opportunities from Definition 5.1 (i) are easy to find for a typical model, since it requires maximization of just one expected utility for $T\to+\infty$ only, with mild restrictions on other utilities. However, it can be seen from the following example that a very generic market model does not allow these opportunities for popular Merton's strategies and for the defined above set $\mathcal{U}_0$ of utilities. It is also true if $\tilde{a}(t)=\text{const}\neq 0$ and $\sigma(t)=\text{const}>0$ are known constants, i.e., for the prime model for Merton's strategies.

Example 5.2 Let $\tilde{a}(\cdot)$ be a bounded random process independent from $w(\cdot)$, and let σ be constant, $X(0)=1$. Let $\mathcal{F}_t$ be the filtration generated by $(S(t),r(t),\tilde{a}(t))$ (i.e., we assume that $\tilde{a}(t)$ is observable). Let $\bar{\Sigma}_M$ be the class of all strategies such that $\gamma(t)=\nu(t)\tilde{a}(t)\tilde{X}(t)\tilde{S}(t)^{-1}\sigma^{-2}$, where $\nu(t)$ is a positive bounded deterministic function. This class includes the so-called Merton's strategies. Let $(\nu(\cdot),\delta)$ be such that

$$\mathbf{E}\left\{\delta^{-1}\tilde{X}(T)^{\delta}|\tilde{a}(\cdot)\right\}\to\sup_{x>0}\delta^{-1}x^{\delta}\quad\text{as}\quad T\to+\infty,$$

then

$$\int_0^T \nu(t)\tilde{a}(t)^2 dt \to +\infty \quad \text{as} \quad T \to +\infty.$$

It follows that there exists $\widehat{\delta} < 1$ such that $\mathbf{E}\left\{\widehat{\delta}^{-1}\tilde{X}(T)^{\widehat{\delta}}|\tilde{a}(\cdot)\right\} \to -\infty \quad \text{as} \quad T \to +\infty$. Therefore, this market does not allow speculative opportunities with respect to $(\bar{\Sigma}_M, \mathcal{U}_0)$ (Definition 5.1 (i)). Note that we do not exclude the case case when $\tilde{a}(t) = \text{const} \neq 0$ is known, constant in time, and deterministic; in this case, any Merton's strategy with constant ν is the optimal myopic strategy for some $U \in \mathcal{U}_0$; it maximizes $\mathbf{E}U(\tilde{X}(T))$ for all $T > 0$.

The class $\bar{\Sigma}_M$ looks special and narrow, but it is not the reason why speculative opportunities in Example 5.2 are absent in this class. In fact, this class is even too wide for practical applications, because it includes the strategies that use direct observations of $\tilde{a}(t)$. This is not realistic, because in practice the process $\tilde{a}(t)$ is unknown and need to be estimated. At first sight, it is not a problem, since $\tilde{a}(t)$ can be estimated using observations of historical prices. For instance, the corresponding Merton's strategy with $\tilde{a}(t)$ replaced by its estimation is optimal for the utility function $U(x) = \ln x$. Unfortunately, calculation of this estimate requires a prior distribution of $\tilde{a}(t)$, and a wrong hypothesis about this distribution leads to losses (see Dokuchaev and Savkin (2004), p. 417).

In contrast, the mean-reverting market model gives an example of a model that allows strong speculative opportunities in the sense of Definition 5.1 (ii). Moreover, the corresponding strategy does not use observation of market parameters, and it does not require a prior distribution of the market parameters.

Example 5.3 *The mean-reverting market model allows strong speculative opportunities with respect to* $(\bar{\Sigma}_\Psi, \mathcal{U}_\Psi^+)$ *and* $(\bar{\Sigma}_\Psi, \mathcal{U}_\Psi)$*, where* $\bar{\Sigma}_\Psi$ *is a singleton consisting of the strategy defined in Theorem 4.3.*

Note that the classes $\mathcal{U}_\Psi^+$ and $\mathcal{U}_\Psi$ of utilities are quite wide for a right choice of Ψ.

The question arises if there are speculative opportunities for a generic market model with constant $(\sigma, \tilde{a}, r)$ such that $\tilde{a} = a - r \neq 0$ for other classes of strategies. So far, we don't know the answer; the proof for the mean-reverting model is based on the stationarity properties, and cannot be extended on this case. For instance, it can be seen that the strategy from Remark 4.4 that gives (4.2) is less risky than the "buy-and-hold" strategy, and it may give a good performance when $\tilde{a} > 0$; however, the possibility of the scenario when $\tilde{S}(T)$ stays near zero may decrease the expected utilities.

6 Appendix: Proofs

Proof of Proposition 3.2 and Theorem 1.3 is well known and will be omitted.

Proof of Proposition 1.1. We have that

$$d\tilde{R}_0(t) = (\alpha - \lambda \tilde{R}_0(t))dt + \sigma dw(t), \tag{A.1}$$

$$\tilde{R}_0(0) = \frac{\alpha}{\lambda} + \int_{-\infty}^{0} e^{\lambda s}\sigma dw(s) = \frac{\alpha}{\lambda} + \int_0^{\infty} e^{-\lambda s}\sigma d\tilde{w}(s), \tag{A.2}$$

and $Y(t) \triangleq \tilde{R}_0(t) - \tilde{R}(t)$ satisfies

$$dY(t) = -\lambda Y(t)dt, \quad Y(0) = \tilde{R}_0(0) - \tilde{R}(0), \tag{A.3}$$

i.e., $Y(t) = \tilde{R}_0(t) - \tilde{R}(t) = e^{-\lambda t}[\tilde{R}_0(0) - \tilde{R}(0)]$. Clearly, this process converges to zero in mean square and with probability 1. $\square$

Proof of Theorem 2.1. Let $e_m(\cdot) : \mathbf{R}^n \to [0,1]$ be continuous functions such that $e_m(x) = 1$ if $|x| \leq m$, $e_m(x) = 0$ if $|x| > m$, and $e_m(x) \leq e_{m+1}(x)$ for all x, $m = 1, 2, 3, ...$ Let $u_m(x,t,T)$ be the solution of the Cauchy problem for the following backward parabolic equation

$$\begin{aligned} &\frac{\partial u_m}{\partial t}(x,t,T) + (\alpha - \lambda x)\frac{\partial u_m}{\partial x}(x,t,T) + \frac{\sigma^2}{2}\frac{\partial^2 u_m}{\partial x^2}(x,t,T) \\ &\qquad\qquad + \frac{1}{2\sigma^2}(\kappa - \lambda x)^2 e_m(x) u_m(x,t,T) = 0, \quad t < T, \ x \in \mathbf{R}, \\ &u_m(x,T,T) = 1. \end{aligned} \tag{A.4}$$

Clearly, (A.4) is the Kolmogorov's equation for the process $R(t)$, and

$$u_m(x,t,T) = \mathbf{E}\left\{ \exp \frac{1}{2\sigma^2} \int_t^T (\kappa - \lambda \tilde{R}(s))^2 e_m(\tilde{R}(s)) ds \,\Big|\, \tilde{R}(t) = x \right\}, \quad t \in (0,T].$$

Let $\tilde{b}(t) \triangleq \kappa - \lambda \tilde{R}(s)$,

$$\zeta_m \triangleq \exp \frac{1}{2\sigma^2} \int_0^T \tilde{b}(s)^2 e_m(\tilde{R}(s)) ds, \quad \zeta \triangleq \exp \frac{1}{2\sigma^2} \int_0^T \tilde{b}(s)^2 ds.$$

By the definitions, it follows that

$$u_m(\tilde{R}(0), 0, T) = \mathbf{E}\zeta_m.$$

Clearly, $\zeta_m \to \zeta$ as $m \to +\infty$ a.s., and the convergence is monotonic, i.e., $\zeta_m > 0$ is non-decreasing in m a.s. It follows that the function $u_m(\tilde{R}(0), 0, T)$ is non-negative, and it is monotonic and non-decreasing in m for any T. In addition, $\mathbf{E}\zeta_m \to \mathbf{E}\zeta$ as $m \to +\infty$ (even if $\mathbf{E}\zeta = +\infty$), i.e., $u_m(\tilde{R}(0), 0, T) = \mathbf{E}\zeta_m \to \mathbf{E}\zeta$.

To prove the theorem, it suffices to show that

$$\sup_{m>0} u_m(x,0,T) < +\infty \quad \forall T>0, \forall x\in\mathbf{R}. \tag{A.5}$$

Let us prove (A.5). Let $y \triangleq \frac{\lambda}{2\sigma^2}$, and let $v_m(x,t) \triangleq u_m(x,t,T)e^{-x^2y}$. We have that

$$\begin{aligned}
u_m(x,t,T) &= v_m(x,t)e^{x^2y}, \qquad \frac{\partial u_m}{\partial t} = \frac{\partial v_m}{\partial t}e^{x^2y},\\
\frac{\partial u_m}{\partial x} &= \frac{\partial v_m}{\partial x}e^{x^2y} + v_m e^{x^2y}\cdot 2xy = e^{x^2y}\left[\frac{\partial v_m}{\partial x} + 2xyv_m\right],\\
\frac{\partial^2 u_m}{\partial x^2} &= \frac{\partial^2 v_m}{\partial x^2}e^{x^2y} + \frac{\partial v_m}{\partial x}e^{x^2y}\cdot 4xy + v_m e^{x^2y}(2xy)^2 + v_m e^{x^2y}2y\\
&= e^{x^2y}\left[\frac{\partial^2 v_m}{\partial x^2} + 4xy\frac{\partial v_m}{\partial x} + (2xy)^2 v_m + 2yv_m\right].
\end{aligned}$$

Using these formulas, equation (A.4) can be transformed to a equation for v_m:

$$\begin{aligned}
&\frac{\partial v_m}{\partial t}(x,t) + (\alpha-\lambda x)\left[\frac{\partial v_m}{\partial x}(x,t) + 2xyv_m(x,t)\right]\\
&\qquad + \frac{\sigma^2}{2}\left[\frac{\partial^2 v_m}{\partial x^2}(x,t) + 4xy\frac{\partial v_m}{\partial x}(x,t) + (2xy)^2 v_m(x,t) + 2yv_m(x,t)\right]\\
&\qquad + \frac{1}{2\sigma^2}(\kappa-\lambda x)^2 e_m(x)v_m(x,t) = 0, \qquad t<T,\ x\in\mathbf{R},\\
&v_m(x,T) = e^{-x^2y}.
\end{aligned} \tag{A.6}$$

We have that

$$-\lambda x + \frac{\sigma^2}{2}\cdot 4xy = 0, \quad -2\lambda y + 2\sigma^2 y^2 + \frac{\lambda^2}{2\sigma^2} = 2\sigma^2\left(y - \frac{\lambda}{2\sigma^2}\right)^2 = 0. \tag{A.7}$$

By (A.7), equation (A.6) can be rewritten as

$$\begin{aligned}
&\frac{\partial v_m}{\partial t}(x,t) + \alpha\frac{\partial v_m}{\partial x}(x,t) + 2\alpha xyv_m(x,t) + \frac{\sigma^2}{2}\left[\frac{\partial^2 v_m}{\partial x^2}(x,t) + 2yv_m(x,t)\right]\\
&\quad + \frac{1}{2\sigma^2}(\kappa^2 - 2\kappa\lambda x)e_m(x)v_m(x,t) - yx^2(1-e_m(x))v_m(x,t) = 0, \qquad t<T,\ x\in\mathbf{R},\\
&v_m(x,T) = e^{-x^2y}.
\end{aligned} \tag{A.8}$$

Clearly, this equation has an unique solution that can be presented as

$$\begin{aligned}
v_m(x,t) = \mathbf{E}e^{-y\xi(T)^2}\exp\int_t^T \biggl(&\sigma^2 y + 2\alpha y\xi(s) + \frac{1}{2\sigma^2}[\kappa^2 - 2\kappa\lambda\xi(s)]e_m(\xi(s))\\
&- y[1-e_m(\xi(s))]\xi(s)^2\biggr)ds,
\end{aligned}$$

and where $\xi(s) = \xi^{x,t}(s)$ is the solution of the following linear Ito equation:

$$d\xi(s) = \alpha ds + \sigma dw(s), \quad s>t, \qquad \xi(t) = x.$$

Note that

$$0 \le v_m(x,t) \le \mathbf{E}\exp\int_t^T (c_1 + c_2|\xi(s)|)ds \le c_3\mathbf{E}\exp\int_t^T c_2|\xi(s)|ds,$$

where $c_i > 0$ are constants that do not depend on m. By Jensen's inequality, it follows from convexity of exponent that

$$\mathbf{E}\exp\int_t^T c_2|\xi(s)|ds = \mathbf{E}\exp\Big(\frac{1}{T-t}\int_t^T (T-t)c_2|\xi(s)|ds\Big)$$
$$\le \mathbf{E}\frac{1}{T-t}\int_t^T \exp((T-t)c_2|\xi(s)|)ds = \frac{1}{T-t}\int_t^T \mathbf{E}\exp((T-t)c_2|\xi(s)|)ds.$$

Hence

$$v_m(x,t) \le \frac{c_3}{T-t}\int_t^T \mathbf{E}\exp((T-t)c_2|\xi(s)|)ds.$$

The process $\xi(s) = \xi^{x,t}(s)$ is Gaussian, and its mean and variance are bounded on $[t,T]$ for any given (x,t). In addition, $\xi(\cdot)$, c_2, and c_3, do not depend on m. It follows that $\sup_{m>0} v_m(x,0) < +\infty$ for any $x \in \mathbf{R}$. We have that $u_m(x,t,T) = v_m(x,t)e^{x^2 y}$ is the solution of (A.4). Hence

$$\sup_{m>0} u_m(x,0,T) = \sup_{m>0} v_m(x,0) < +\infty \quad \forall x \in \mathbf{R}.$$

By (A.5), the proof follows. □

Proof of Corollary 2.2 and 2.3 is given under the assumptions of Remark **??**.

Proof of Corollary 2.2. It is well known that if the Novikov condition is satisfied, then $\mathbf{E}\mathcal{Z}(T)^{-1} = 1$, where $\mathcal{Z}(T)$ is defined by (**??**). It was proven above that the Novikov condition is satisfied on the conditional probability space given $\tilde{R}(0)$, i.e., under the measure $\mathbf{P}(\cdot|\,\tilde{R}(0))$. Hence $\mathbf{E}\{\mathcal{Z}(T)^{-1}|\tilde{R}(0)\} = 1$ a.s., where $\mathcal{Z}(T)$ is defined by (**??**). It follows that $\mathbf{E}\mathcal{Z}(T)^{-1} = \mathbf{E}(\mathbf{E}\{\mathcal{Z}(T)^{-1}|\tilde{R}(0)\}) = 1$. By Girsanov Theorem, it follows that $w_*(t) \triangleq w(t) + \int_0^t \sigma^{-1}\tilde{b}(s)ds$ is a Wiener process under $\mathbf{P}_{*,T}$ for $t \in [0,T]$, where $\tilde{b}(t) \triangleq \alpha - \lambda\tilde{R}(t)$, and where $\mathbf{P}_{*,T}$ is a measure defined by $d\mathbf{P}_{*,T}/d\mathbf{P} = \mathcal{Z}(T)^{-1}$. Hence it is a equivalent martingale measure. This completes the proof. □

Proof of Corollary 2.3. By Ito formula,

$$d\tilde{S}(t) = \tilde{S}(t)\Big[\frac{\sigma^2}{2}dt + d\tilde{R}(t)\Big] = \tilde{S}(t)[\tilde{a}(t)dt + \sigma dw(t)],$$

where $\tilde{a}(t) = \alpha - \lambda\tilde{R}(t) + \sigma^2/2$. By Theorem 2.1, (2.1) holds for $\kappa = \alpha + \sigma^2/2$ on the conditional probability space given $\tilde{R}(0)$, i.e., under the measure $\mathbf{P}(\cdot|\,\tilde{R}(0))$. Hence $\mathbf{E}\{\widehat{\mathcal{Z}}(T)^{-1}|\tilde{R}(0)\} = 1$ a.s., where $\widehat{\mathcal{Z}}(T)$ is defined by

$$\widehat{Z}(T) \triangleq \exp\left(\int_0^T \widehat{a}(t)\sigma(t)^{-1}dw(t) + \frac{1}{2}\int_0^T \widehat{a}(t)^2\sigma(t)^{-2}dt\right). \qquad \text{(A.9)}$$

(**??**). It follows that $\mathbf{E}\widehat{\mathcal{Z}}(T)^{-1} = \mathbf{E}(\mathbf{E}\{\widehat{\mathcal{Z}}(T)^{-1}|\tilde{R}(0)\}) = 1$. By Girsanov Theorem again, it follows that $w_*(t) \triangleq w(t) + \int_0^t \sigma^{-1}\tilde{a}(s)ds$ is a Wiener process under $\widehat{\mathbf{P}}_{*,T}$ for $t \in [0,T]$, where $\widehat{\mathbf{P}}_{*,T}$ is a measure defined by $d\widehat{\mathbf{P}}_{*,T}/d\mathbf{P} = \widehat{\mathcal{Z}}(T)^{-1}$. This completes the proof. □

Proof of Lemma 2.5. It suffices to consider non-random $\tilde{R}(0)$. (It is the same as to prove (2.2) for the conditional space given $\tilde{R}(0)$). Let $k \triangleq \lambda^2/\sigma^2 + 2\varepsilon$. We have that

$$\mathbf{E}\exp\Big(\frac{k}{2}\int_0^T \tilde{R}(t)^2dt\Big) \geq \mathbf{E}\exp\frac{k}{2T}\Big(\int_0^T \tilde{R}(t)dt\Big)^2 = \mathbf{E}e^{\frac{1}{2}\xi^2},$$

where $\xi \triangleq \sqrt{\frac{k}{T}}\eta$, $\eta \triangleq \int_0^T \tilde{R}(t)dt$. It suffices to show that, for large T,

$$\operatorname{Var}\xi > 1, \quad \text{i.e.,} \quad \operatorname{Var}\eta > \frac{T}{k}. \tag{A.10}$$

We have that

$$\begin{aligned}
\operatorname{Var}\eta &= \mathbf{E}\Big(\int_0^T \tilde{R}(t)dt - \mathbf{E}\int_0^T \tilde{R}(t)dt\Big)^2 = \mathbf{E}\Big(\int_0^T [\tilde{R}(t) - \mathbf{E}\tilde{R}(t)]dt\Big)^2 \\
&= \mathbf{E}\Big(\int_0^T dt\int_0^t e^{-\lambda(t-s)}\sigma dw(s)\Big)^2 \\
&= \mathbf{E}\int_0^T\int_0^T dtdq\int_0^t e^{-\lambda(t-s)}\sigma dw(s)\int_0^q e^{-\lambda(q-p)}\sigma dw(p) \\
&= 2\mathbf{E}\int_0^T dt\int_t^T dq\int_0^t e^{-\lambda(t-s)}\sigma dw(s)\int_0^q e^{-\lambda(q-p)}\sigma dw(p) \\
&= 2\int_0^T dt\int_t^T dq\int_0^t e^{-\lambda(t-s)}e^{-\lambda(q-s)}\sigma^2 ds = 2\sigma^2\int_0^T dt\int_t^T dq\, e^{-\lambda(q-t)}\int_0^t e^{-2\lambda(t-s)}ds \\
&= 2\sigma^2\int_0^T dt\int_t^T dq\, e^{-\lambda(q-t)}\frac{1-e^{-2\lambda t}}{2\lambda} = 2\sigma^2\int_0^T dt\int_t^T dq\,\frac{e^{-\lambda(q-t)} - e^{-\lambda(q-t)}e^{-2\lambda t}}{2\lambda} \\
&= \sigma^2\int_0^T dt\int_t^T dq\,\frac{e^{-\lambda(q-t)} - e^{-\lambda q-\lambda t}}{\lambda} \\
&= \frac{\sigma^2}{\lambda^2}\int_0^T dt\Big[e^{\lambda t}(e^{-\lambda t} - e^{-\lambda T}) - e^{-\lambda t}(e^{-\lambda t} - e^{-\lambda T})\Big] \\
&= \frac{\sigma^2}{\lambda^2}\Big[T - \frac{e^{\lambda T}-1}{\lambda}e^{-\lambda T} - \frac{1-e^{-2\lambda T}}{2\lambda} + \frac{1-e^{-\lambda T}}{\lambda}e^{-\lambda T}\Big] \geq \frac{1}{k-2\varepsilon}(T-c),
\end{aligned}$$

where

$$c \triangleq \max_{T>0}\Big(\frac{e^{\lambda T}-1}{\lambda}e^{-\lambda T} + \frac{1-e^{-2\lambda T}}{2\lambda} - \frac{1-e^{-\lambda T}}{\lambda}e^{-\lambda T}\Big)$$

does not depend on T. (Remember that $k - 2\varepsilon = \lambda^2/\sigma^2$). Then (A.10) follows for large T. This completes the proof. □

Proof of Proposition 4.1. It suffices to prove that (**??**) holds. Remind that $\tilde{a}(t) = \alpha + \sigma^2/2 - \lambda\tilde{R}(t)$. We have that there exists $M_i > 0$, μ_i such that

$$|\gamma(t)S(t)|^2 = S(t)^2\left|\frac{d\Psi}{dx}(\tilde{S}(t))\right|^2 \leq M_1(\tilde{S}(t)^{\mu_1} + \tilde{S}(t)^{\mu_2})^2 \leq M_2\left(e^{\mu_3\tilde{R}(t)} + e^{\mu_4\tilde{R}(t)}\right),$$

and

$$|\tilde{a}(t)^2\gamma(t)S(t)|^2 = \left(\alpha + \frac{\sigma^2}{2} - \lambda\tilde{R}(t)\right)^2 S(t)^2 \left|\frac{d\Psi}{dx}(\tilde{S}(t))\right|^2 \le M_3(1+\tilde{R}(t)^2)\left(e^{\mu_5\tilde{R}(t)} + e^{\mu_6\tilde{R}(t)}\right).$$

It follows that, for any $T > 0$,

$$\mathbf{E}\int_0^T \gamma(t)^2 S(t)^2 dt < +\infty,$$

and

$$\sup_{t\in[0,T]} \mathbf{E}\gamma(t)^2(a(t)^2S(t)^2 + \sigma^2 S(t)^2) < +\infty.$$

Set

$$\tilde{X}(t) \triangleq X(0) + \int_0^t \gamma(s)d\tilde{S}(s), \quad X(t) \triangleq \tilde{X}(t)\exp\left(\int_0^t r(s)ds\right), \quad \beta(t) \triangleq \frac{X(t) - \gamma(t)S(t)}{B(t)}.$$

Then (**??**) holds, and $\mathbf{E}X(T)^2 < +\infty$. By Proposition 3.2, $(\beta(\cdot),\gamma(\cdot))$ is an admissible self-financing strategy with the total wealth $X(t)$ and the discounted wealth $\tilde{X}(t)$. □

Proof of Theorem 4.3. By Ito formula, it follows immediately that if

$$\tilde{X}(t) = \Psi(\tilde{S}(t)) - \frac{1}{2}\int_0^t \frac{d^2\Psi}{dx^2}(\tilde{S}(s))\tilde{S}(s)^2\sigma(s)^2 ds,$$

then

$$d\tilde{X}(t) = \gamma(t)d\tilde{S}(t).$$

By Proposition 3.2, it follows that $\tilde{X}(t)$ is the corresponding discounted wealth.

Let us prove (i). We have that (4.1) is satisfied with

$$\zeta(T) \triangleq -\frac{1}{2}\int_0^T \frac{d^2\Psi}{dx^2}(\tilde{S}(t))\tilde{S}(t)^2\sigma^2 dt.$$

Clearly, $\zeta(T) \ge 0$, and the process $\zeta(T)$ is monotonic and non-decreasing.

Since Ψ is twice differentiable and it is not an affine function on $(0,+\infty)$, we have that $\mathrm{mes}\,\{x > 0 : \frac{d^2\Psi}{dx^2}(x) < 0\} > 0$. (Here mes denotes the length of an interval, or the Lebesgue measure of a subset of $\mathbf{R}$). Let $d > 0$, $M > 0$, and an interval $D \subset [d,+\infty)$ be such that $\mathrm{mes}\,(D) > 0$ and $\frac{d^2\Psi}{dx^2}(x) < -M$ for all $x \in D$. Clearly,

$$\zeta(T) \ge \frac{1}{2}\sigma^2 d^2 M \int_0^T \mathbb{I}_{\{\tilde{S}(t)\in D\}} dt. \tag{A.11}$$

Here $\mathbb{I}$ denotes the indicator function. Let D_0 be a subset of the interior of D such that $D_0 \ne D$ (i.e., D_0 has a positive distance from the endpoints of D). Clearly, $\tilde{R}_0(t)$ is a

Gaussian stationary process with continuous spectral distribution function, hence the process $\tilde{R}_0(t)$ is ergodic. It follows that

$$\frac{1}{T}\int_0^T \mathbb{I}_{\{\tilde{S}_0(t)\in D_0\}}dt \to \mathbf{P}(\tilde{S}_0(T)\in D_0) \quad \text{a.s.} \quad \text{as} \quad T\to+\infty.$$

(See, e.g., Rogers and Williams (2000), p.300). Clearly, $\mathbf{P}(\tilde{S}_0(T)\in D_0)>0$, and

$$\mathbb{I}_{\{\tilde{S}(t)\in D\}} = \mathbb{I}_{\{\tilde{S}_0(t)\in D_0\}} + \mathbb{I}_{\{\tilde{S}(t)\in D\}} - \mathbb{I}_{\{\tilde{S}_0(t)\in D_0\}} \ge \mathbb{I}_{\{\tilde{S}_0(t)\in D_0\}} - \mathbb{I}_{\{\tilde{S}(t)\notin D,\tilde{S}_0(t)\in D_0\}}.$$

Hence

$$\frac{1}{T}\int_0^T \mathbb{I}_{\{\tilde{S}(t)\in D\}}dt \ge \frac{1}{T}\int_0^T \mathbb{I}_{\{\tilde{S}_0(t)\in D_0\}}dt - \eta(T), \tag{A.12}$$

where

$$\eta(T) = \frac{1}{T}\int_0^T \mathbb{I}_{\{\tilde{S}(t)\notin D,\tilde{S}_0(t)\in D_0\}}dt.$$

Remember that $\tilde{R}_0(t)-\tilde{R}(t)=e^{-\lambda t}[\tilde{R}_0(0)-\tilde{R}(0)]$. Hence $\eta(T)\to 0$ a.s.. By (A.11)-(A.12), it follows that $\zeta(T)\to+\infty$ as $T\to+\infty$ a.s.. By Proposition 1.1, $\Psi(\tilde{S}(T))-\Psi(\tilde{S}_0(T))\to 0$ as $T\to+\infty$ a.s. Then (i) follows.

Let us prove (ii). By the assumptions, it follows that $U(x)\to\sup_{y\in\Gamma}U(y)$ as $x\to+\infty$. We have that $\Psi(\tilde{S}(T))\ge\psi_->-\infty$. It follows from statement (i) that

$$\tilde{X}(T)\to+\infty \quad \text{as} \quad T\to+\infty \quad \text{a.s..} \tag{A.13}$$

We have that $U\in\mathcal{U}_\Psi^+$. Hence $U(\tilde{X}(T))\to\sup_{y\in\Gamma}U(y)$ a.s. as $T\to+\infty$. If $\sup_{y\in\Gamma}U(y)<+\infty$, then the function U is bounded on $[\psi,+\infty)$. Then statement (ii) follows from (A.13) and from Lebesgue's Dominated Convergence Theorem. If $\sup_{y\in\Gamma}U(y)=+\infty$, then statement (ii) follows from (A.13) and from Fatou's Lemma. Therefore, statement (ii) is proved.

Let us prove statement (iii). We have that $U(x) = U^+(x) - U^-(x)$, where $U^+(x) \triangleq \max(0, U(x)$, $U^-(x) \triangleq \max(0, -U(x)$,

Let us show first that

$$\mathbf{E}U^+(\tilde{X}(T)) \to +\infty \quad \text{as} \quad k \to +\infty. \tag{A.14}$$

Let $c \in \mathbf{R}$ be given such that $\mathbf{P}(\Psi(\tilde{S}_0(T)) \ge c) = p > 0$. Let $I_T \triangleq \mathbb{I}_{\{\Psi(\tilde{S}(T)) \ge c\}}$. Since U^+ is non-negative and non-decreasing, we have that

$$\mathbf{E}U^+(\tilde{X}(T)) \ge \mathbf{E}I_T U^+(\tilde{X}(T)) \ge \mathbf{E}I_T U^+(c + \zeta(T)).$$

Therefore, it suffices to show that $\mathbf{E}I_T U^+(c + \zeta(T)) \to +\infty$.

Let us assume that it is not true, i.e., there exists $C > 0$ such that for any $t > 0$ there exists $T_t \in [t, +\infty)$ such that $\mathbf{E}I_{T_t} U^+(c + \zeta(T_t)) \le C$. Then

$$\mathbf{P}(I_{T_t} U^+(c + \zeta(T_t)) \ge K) \le \frac{C}{K} \quad \forall t > 0, K > 0. \tag{A.15}$$

Remind that $\sup U^+(x) = +\infty$. If $t \to +\infty$, then $\zeta(T_t) \to +\infty$ a.s., hence $U^+(c + \zeta(T_t)) \to +\infty$ a.s.. Further, $\mathbf{P}(I_{T_t} = 1) = \mathbf{P}(\Psi(\tilde{S}(T_t)) \ge c) \to p > 0$, where $p \triangleq \mathbf{P}(\Psi(\tilde{S}_0(T)) \ge c)$ (this value does not depend on T). Thus, (A.15) does not hold for large T_t and $K \ge 2C/p$. Hence (A.14) follows.

To complete the proof, it suffices to show that $\mathbf{E}U^-(\tilde{X}(T))$ is bounded as $T \to +\infty$.

Remember that the processes $\tilde{R}(t)$ and $\tilde{R}_0(t)$ are Gaussian. It is known that

$$\tilde{R}_0(T) = \frac{\alpha}{\lambda}, \quad \operatorname{Var} R_0(T)^2 = \frac{\sigma^2}{2\lambda}, \quad \mathbf{E}\tilde{R}(T) = m, \quad \operatorname{Var} \tilde{R}^2(T) = v,$$

where

$$m = m(T) = (1 - e^{-\lambda T})\frac{\alpha}{\lambda}, \quad v = v(T) = (1 - e^{-2\lambda T})\frac{\sigma^2}{2\lambda}. \tag{A.16}$$

We have that

$$\begin{aligned} &\mathbf{E}U^-(\tilde{X}(T))) \le \mathbf{E}U^-(\Psi(\tilde{S}(T))) = \mathbf{E}u^-(\tilde{R}(T)) \\ &= \frac{1}{\sqrt{2\pi v}} \int_{\mathbf{R}} e^{-\frac{(x-m)^2}{2v}} u^-(x) dx = \frac{1}{\sqrt{2\pi v}} \int_{\mathbf{R}} e^{\frac{1}{2}(-\kappa x^2 + k_1 x + k_0)} e^{-\frac{x^2}{2\nu}} u^-(x) dx. \end{aligned}$$

Here $\kappa = \kappa(T)$ and $k_i = k_i(T)$ are reals uniquely defined from the equality

$$-\frac{(x-m)^2}{v} \equiv -\kappa x^2 + k_1 x + k_0 - \frac{x^2}{\nu}.$$

For instance, $\kappa = 1/v - 1/\nu$. It follows that $\kappa > 0$, and

$$\mathbf{E}U^-(\tilde{X}(T)) \le \frac{1}{\sqrt{2\pi v}} \left[\int_{\mathbf{R}} e^{-\kappa x^2 + k_1 x + k_0} dx\right]^{1/2} \left[\int_{\mathbf{R}} e^{-\frac{x^2}{\nu}} u^-(x) dx\right]^{1/2}.$$

By (A.16), $v \to \sigma^2/(2\lambda)$ as $T \to +\infty$. Hence there exist $\varepsilon > 0$ and $T_1 > 0$, such that $\kappa \geq \varepsilon$ for all $T > T_1$. In addition, the fact that a is bounded in T implies that k_0 and k_1 are bounded in T. It follows that the value of $\int_{\mathbf{R}} e^{-\kappa x^2 + k_1 x + k_0} dx$ is bounded in T. Therefore, $\mathbf{E}U^-(\tilde{X}(T))$ is bounded as $T \to +\infty$. This completes the proof. $\square$

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