

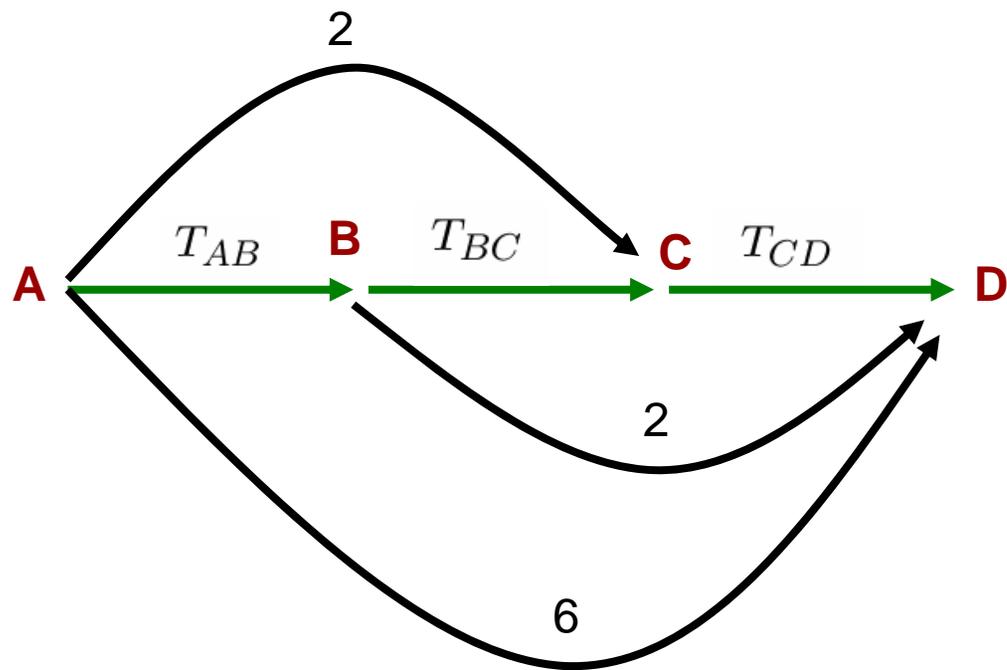
# PRICING A SEGMENTED MARKET SUBJECT TO CONGESTION

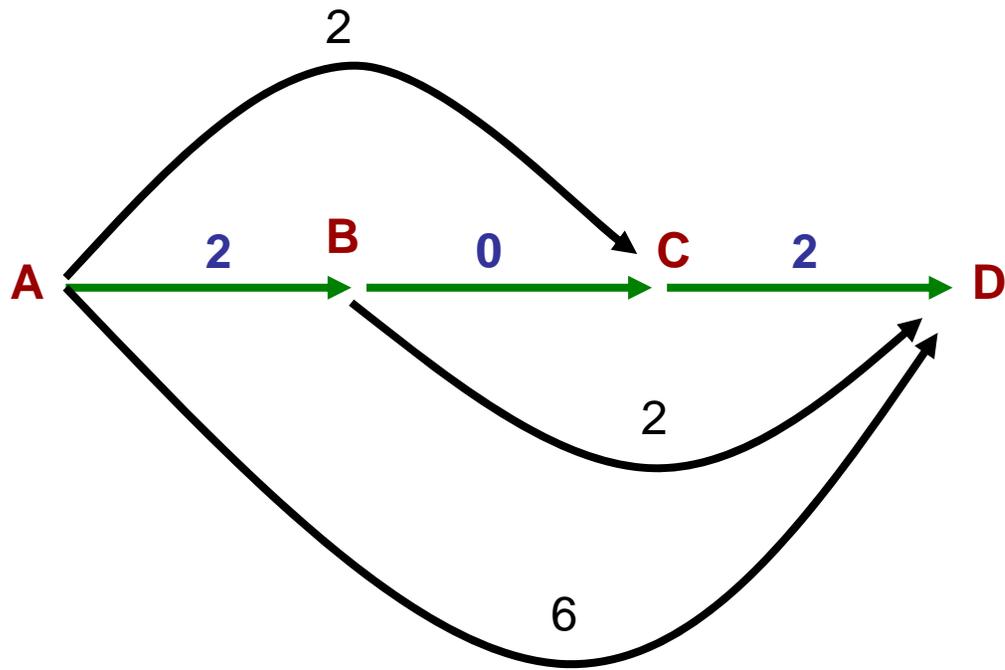
Maxime Fortin, Lehman Brothers  
Patrice Marcotte, Université de Montréal  
Gilles Savard, École Polytechnique  
Alexandre Schoeb, ExPretio Technologies

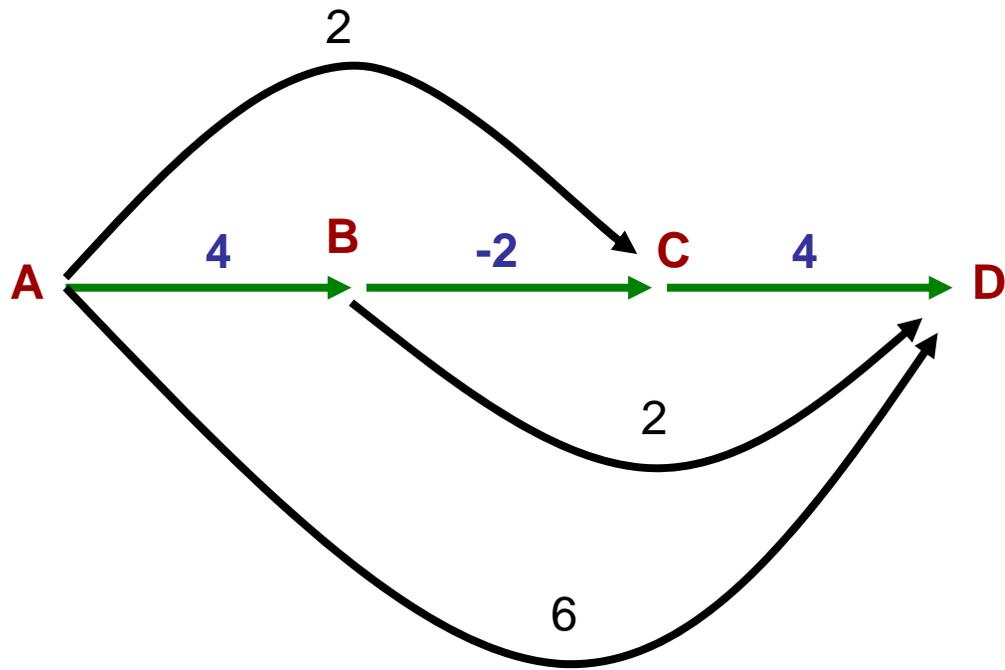
Fields Institute

Toronto

March 6, 2007

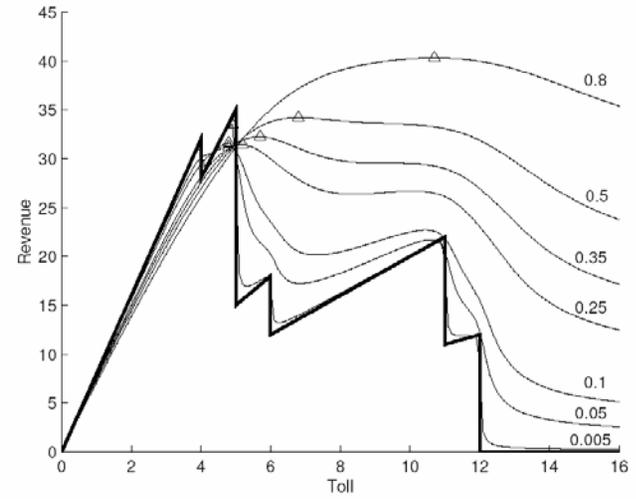






$$\max_{T,x,y} Tx$$

$$\min_{x,y} (c + T)x + dy$$
$$Ax + By = b$$
$$x, y \geq 0$$

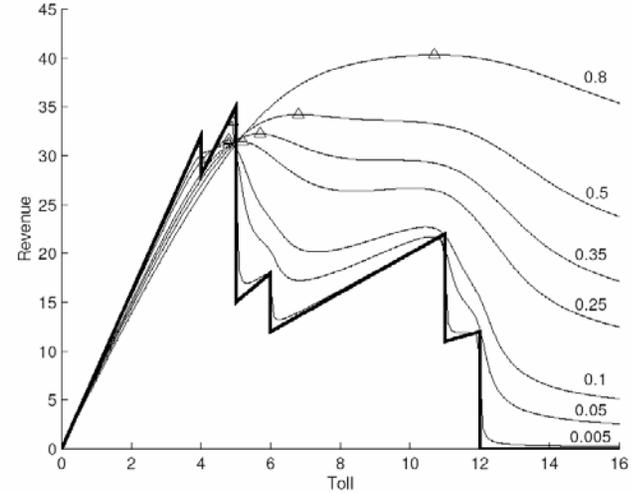


$$\max_{T,x,y} Tx$$

$$\min_{x,y} (c + T)x + dy$$

$$Ax + By = b$$

$$x, y \geq 0$$



Naive reaction model (linear assignment), **but** theoretically rich

strongly NP-Hard, TSP, solvable for global solution, ...

and the base for **realistic** models

multi-commodity, multi-criteria, multi-class, congestion, capacity, design, stochasticity, dynamic, ...

# Outline

- Bilevel framework and RM background
- The pricing model
- An ascent approach
- A hybrid approach
- Numerical results
- SNCF: a real life application

# Bilevel programming and RM

- Pricing (and seat inventory) optimization requires a comprehensive representation of both the seller(s) and the market (reaction).
- The prices (fare basis codes) and inventory control are part of the seller model
- Need for a detailed and implicit representation of the market reaction to price, products and competition
  - Assignment model, discrete choice models, equilibrium model , ...
  - Segmentation of market: how many classes (user groups)
- In airline and rail industries, the capacities are fixed
- In telecommunication and road transportation, congestion (or QoS) has to be considered

# The Bilevel Framework

$$\begin{aligned} \max_{x,y} \quad & F(x, y) \\ & G(x, y) \leq 0 \\ & y \in \arg \min_{y'} f(x, y') \\ & g(x, y') \leq 0 \end{aligned}$$

- Generally non convex
- May be disconnected
- Strongly NP-hard problem (even local)

# Pricing Model

## Upper level

- The firm seeks for a price vector  $t$  (on a subset of arcs) that maximizes its revenue (or that induces *system optimum*)

## Lower level

- Demand is segmented w.r.t. the inverse of value of time  $\alpha$
- Value of time characterized by a continuous density function  $h$
- Customers minimize their travel perceived cost

$$\pi_p(\alpha, \bar{v}, t) = D_p(\bar{v}) + \alpha(C_p + \sum_{a \in A'(p)} t_a)$$

Total path flow:  $\bar{v} = \left\{ \int_0^{\alpha_{\max}} v_p(\alpha) d\alpha \right\}_{p \in P}$

# Pricing Model

For fixed  $t$ , a flow density vector is an equilibrium (almost everywhere) if and only if it satisfies:

$$\text{VI: } v \in \bar{Y} = \left\{ v \in \{ \ell^2(0, \alpha_{\max}) \}^{|P|} : v(\alpha) \in Y(\alpha), \quad \forall \alpha \in [0, \alpha_{\max}] \right\}$$
$$\langle D(\bar{v}) + \alpha C, v - y \rangle \leq 0, \quad \forall y \in \bar{Y}$$

$$C = \{ C_p + \sum_{a \in A_1(p)} t_a \}_{p \in P}$$

$Y(\alpha)$  = set of feasible flow density path vector

$\bar{Y}$  = set of feasible total path flow vectors (compact polyhedron)

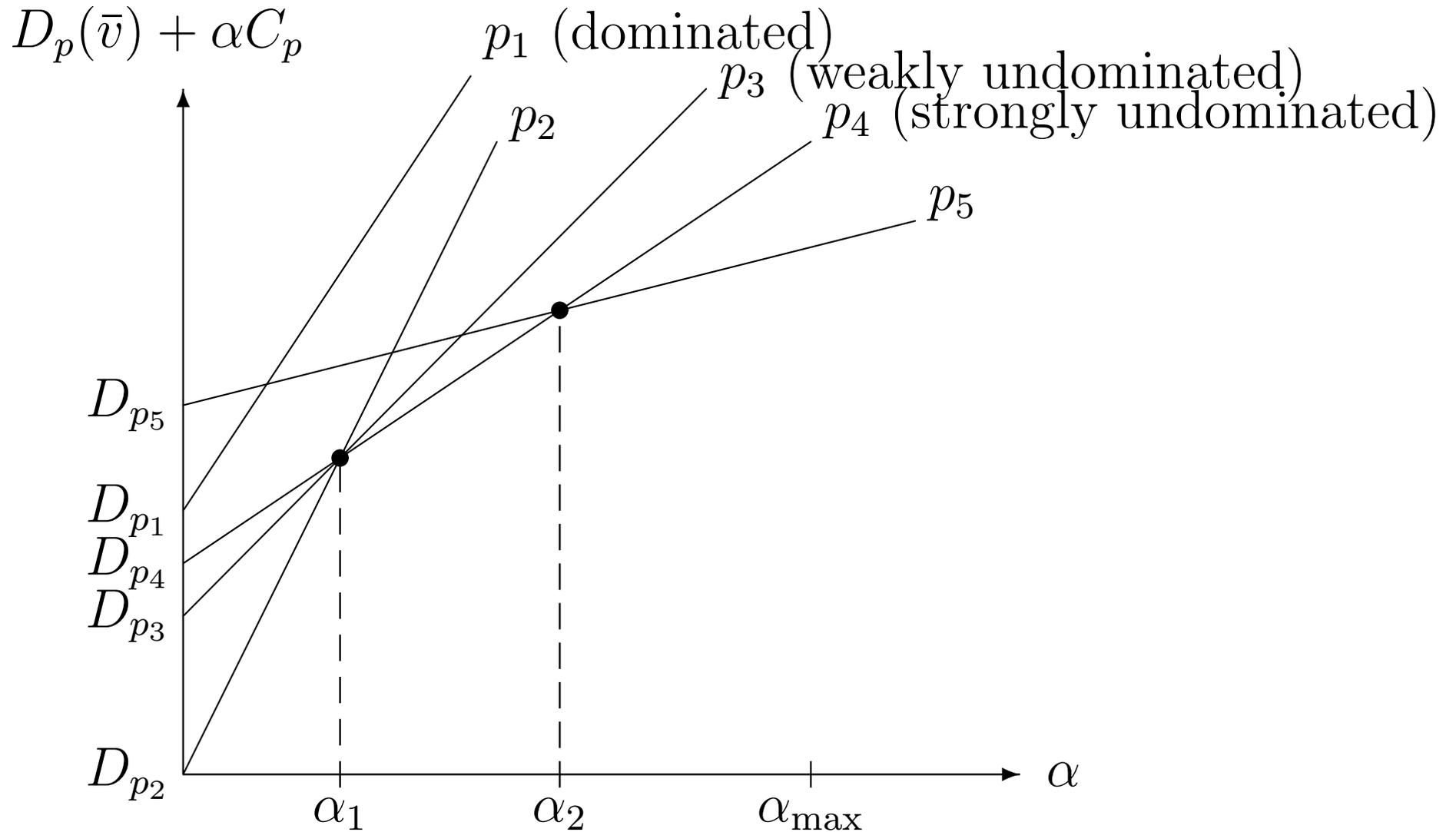
$$\langle \Phi, \Psi \rangle = \int_0^{\alpha_{\max}} \langle \Phi(\alpha), \Psi(\alpha) \rangle d\alpha.$$

# Mathematical formulation

If  $D = (D_p)_{p \in P}$  is the gradient of some function  $d$ , we obtain

$$\begin{aligned} \max_{t, x, v(\alpha)} \quad & \sum_{a \in A_1} t_a x_a \\ x_a = \quad & \sum_{p|a \in A(p)} \int_0^{\alpha_{\max}} v_p(\alpha) d\alpha, \quad \forall a \in A \\ \min_{x, v(\alpha)} \quad & d(\bar{v}) + \int_0^{\alpha_{\max}} \alpha \langle C, v(\alpha) \rangle d\alpha \\ & v(\alpha) \in Y(\alpha), \quad \forall \alpha \in [0, \alpha_{\max}]. \end{aligned}$$

# Basic diagram



## Assumptions and results

**Nondegeneracy assumption:**  $C_i \neq C_j$  for all distinct extreme points  $Y_i$  and  $Y_j$  of  $\bar{Y}$ .

$\Rightarrow$  If the delay function  $D$  is strictly monotone on  $\bar{Y}$  then the equilibrium solution is unique (almost everywhere).

$\Rightarrow$  If the delay function  $D$  is strictly monotone on the compact polyhedron  $\bar{Y}$  then the function  $\bar{v}(t)$  is continuous in  $t$ .

## Lower level solution

$[0, \alpha_{\max}]$  can be partitioned into  $\bigcup_{j=0, \dots, M} [\alpha_{i_j}, \alpha_{i_{j+1}}]$  with  $0 = \alpha_{i_0} < \alpha_{i_1} < \dots < \alpha_{i_M} < \alpha_{i_{M+1}}$  such that  $Y_{i_j}$  is strongly undominated on  $[\alpha_{i_{j-1}}, \alpha_{i_j}]$

$$y(\bar{v}, \alpha) = Y_{i_j} h(\alpha) \quad \text{if } \alpha \in (\alpha_{i_{j-1}}, \alpha_{i_j}]$$

- **Critical points:**

$$\alpha_{i_j} = \frac{D_{i_{j+1}}(\bar{v}) - D_{i_j}(\bar{v})}{C_{i_j} - C_{i_{j+1}}}$$

- Unique solution a.e. (F-W linearization approach)

# Revenue function

$$R(t) = \sum_{a \in A_1} t_a x_a(t)$$

$$x_a(t) = \sum_{j|a \in A(Y_{i_j})} d(H(\alpha_{i_j}(t)) - H(\alpha_{i_{j-1}}(t)))$$

- Continuous
- Nondifferentiable (differentiable a.e.)
- Nonconcave
- Gradient easy to compute (where diff.)

# The gradient

Introducing O-D pair  $k$ , we associate with the optimal paths the critical points:

$$0 = \alpha_0^k, \alpha_1^k, \dots, \alpha_{M_k}^k, \alpha_{M_k+1}^k = \alpha_{\max}$$

If the set  $\mathcal{WU}$  is empty and the VOT density  $h$  is continuous, the profit function  $R$  is differentiable at  $t$

$$\frac{\partial R(\bar{t})}{\partial t_a} = x_a(\bar{t}) + \sum_{a \in A_1} \bar{t}_a \frac{\partial x_a(\bar{t})}{\partial t_a}$$

$$\frac{\partial \alpha_i^k}{\partial t_a} = \frac{1}{C_i^k - C_{i+1}^k} \cdot \frac{\partial (D_{i+1}^k - D_i^k)}{\partial t_a} - \frac{D_{i+1}^k - D_i^k}{(C_i^k - C_{i+1}^k)^2} \cdot \frac{\partial (C_i^k - C_{i+1}^k)}{\partial t_a}$$

$$\frac{\partial C_i^k}{\partial t_a} = \mathbb{1} \{a \in A(Y_i^k)\}, \quad \frac{\partial D_i^k(\bar{x})}{\partial t_a} = \sum_{a \in A(p_i^k)} d'_a(\bar{x}_a) \frac{\partial \bar{x}_a}{\partial t_a}$$

$$\frac{\partial \bar{x}_a}{\partial t_a} = \sum_{k \in K} \sum_{i=1}^{M_k} \mathbb{1} \{a \in A(Y_i^k)\} \left( h(\alpha_i^k) \frac{\partial \alpha_i^k}{\partial t_a} - h(\alpha_{i-1}^k) \frac{\partial \alpha_{i-1}^k}{\partial t_a} \right).$$

# An ascent algorithm

## ALGORITHM Toller

**Phase 1 Step 0:** find an initial point  $t_0$  and set  $k = 0$ ;

**Phase 2 Step 1:** compute  $g_k = \nabla R(t_k)$ ; if  $\|g_k\| \leq \epsilon$  then **stop**;

**Step 2:** is the linesearch successful at  $t_k$  ?

**Yes:**  $\lambda_k \in \arg \min_{\lambda \geq 0} R(t_k + \lambda g_k)$ ;

**No:**  $\lambda_k = 1/(k + 1)$ ;

**Step 3:** set  $t_{k+1} = t_k + \lambda_k g_k$ ,  $k \leftarrow k + 1$  and go to step 1.

# Resolution: Hybrid approach

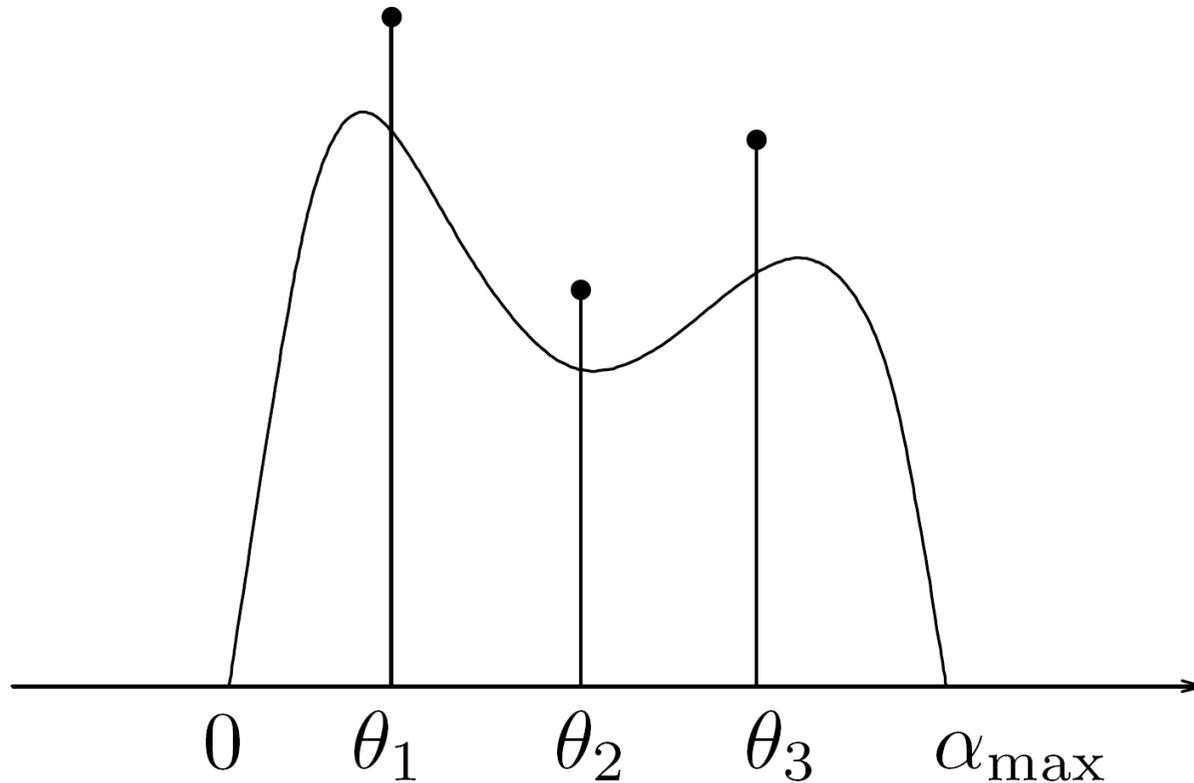
## Phase I: starting point from discretized problem

- Discretize the density and congestion functions  $\rightarrow$  Pd- $n$
- Formulate problem Pd- $n$  as MIP
- Solve problem with branch-and-cut algorithm (CPLEX)

## Phase II: local ascent method

- Use `Toller` with prices obtained at phase I as starting point

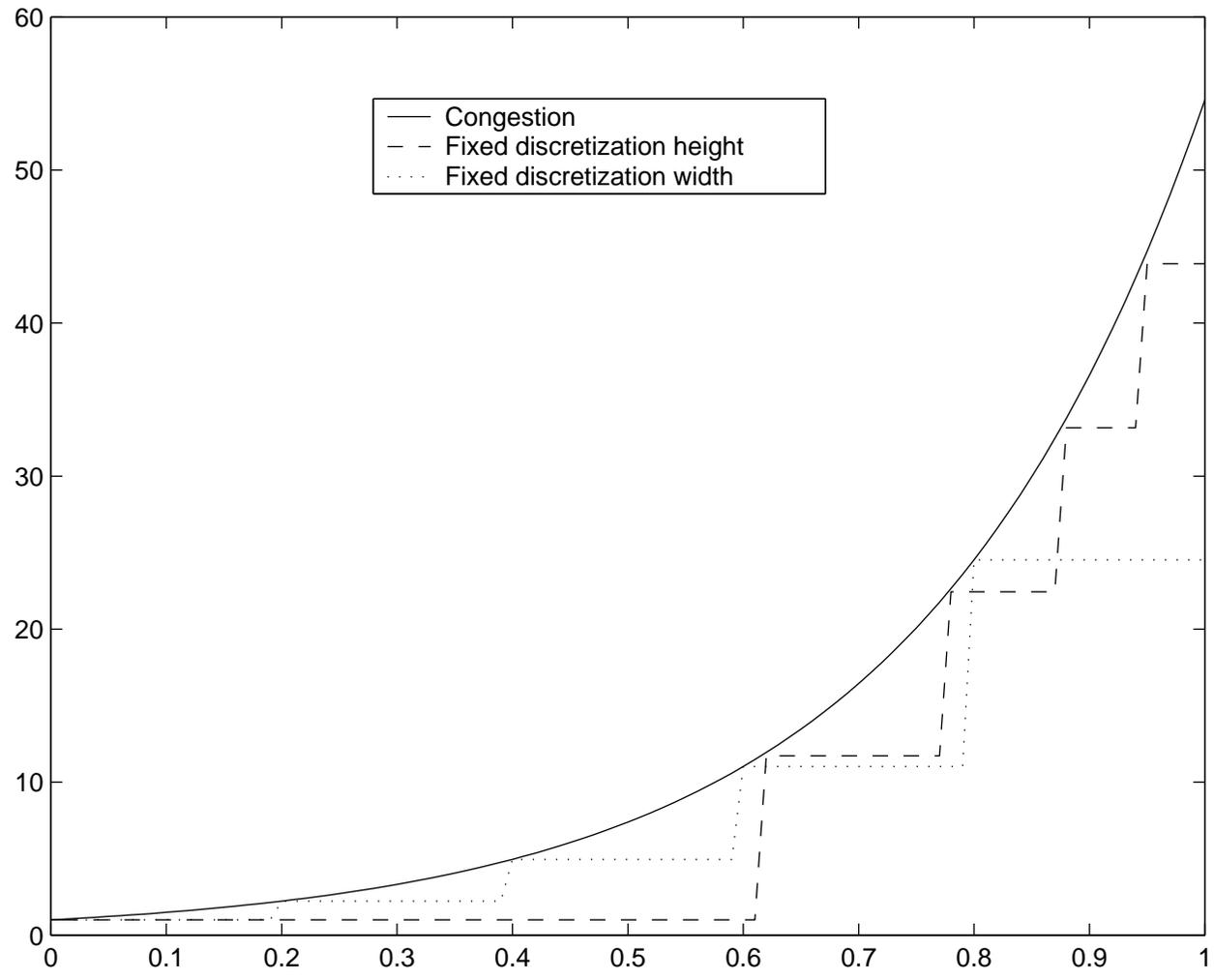
# Discretization of the density $h$



$$\theta_i = \left(\frac{i}{n}\right) \alpha_{\max}$$

$$P(\theta_i) = \frac{h(\theta_i)}{\sum_{l=1}^n h(\theta_l)}$$

# Discretization of the congestion $D$



$$\max \sum_{i=1}^n \sum_{k \in K} \frac{1}{\alpha_i} \zeta_{i,k} d_k h_i - \sum_{i=1}^n \sum_{a \in A} \sum_{l=1}^L \frac{1}{\alpha_i} D_a^l x_{a,i,l} - \sum_{i=1}^n \sum_{a \in A} \sum_{l=1}^L C_a x_{a,i,l}$$

$$\sum_{p \in P_k} v_{p,i} = d_k h_i \quad \forall k \in K, \forall i \in [1, n]$$

$$v_{p,i} \geq 0 \quad \forall p \in P, \forall i \in [1, n]$$

$$\sum_{l=1}^L x_{a,i,l} = \sum_{p|a \in A(p)} v_{p,i} \quad \forall a \in A, \forall i \in [1, n]$$

$$\sum_{a \in A(p)} \sum_{l=1}^L D_a^l z_{a,l} + \alpha_i \left( \sum_{a \in A(p)} C_a + \sum_{a \in A'(p)} T_a \right) - \zeta_{i,k} = \mu_{p,i} \quad \forall k \in K, \forall p \in P_k, \forall i \in [1, n]$$

$$\mu_{p,i} \geq 0 \quad \forall p \in P, \forall i \in [1, n]$$

$$\mu_{p,i} \leq M y_{p,i} \quad \forall p \in P, \forall i \in [1, n]$$

$$v_{p,i} \leq M(1 - y_{p,i}) \quad \forall p \in P, \forall i \in [1, n]$$

$$y_{p,i} \in \{0, 1\} \quad \forall p \in P, \forall i \in [1, n]$$

$$\sum_{i=1}^n \sum_{l=1}^L x_{a,i,l} \geq s^{l-1} - M(1 - z_{a,l}) \quad \forall a \in A, \forall l \in [1, L]$$

$$\sum_{i=1}^n \sum_{l=1}^L x_{a,i,l} \leq s^l + M(1 - z_{a,l}) \quad \forall a \in A, \forall l \in [1, L]$$

$$\sum_{l=1}^L z_{a,l} = 1 \quad \forall a \in A$$

$$z_{a,l} \in \{0, 1\} \quad \forall a \in A, \forall l \in [1, L]$$

$$x_{a,i,l} \leq M z_{a,l} \quad \forall a \in A, \forall i \in [1, n], \forall l \in [1, L]$$

# Lower level discrete solution

## Solution

$$v_{p,i}^* = \begin{cases} dP(\theta_i) & \text{if } \theta_i \in (\alpha_{j-1}, \alpha_j] \text{ and } p = p_j \\ 0 & \text{otherwise} \end{cases}$$

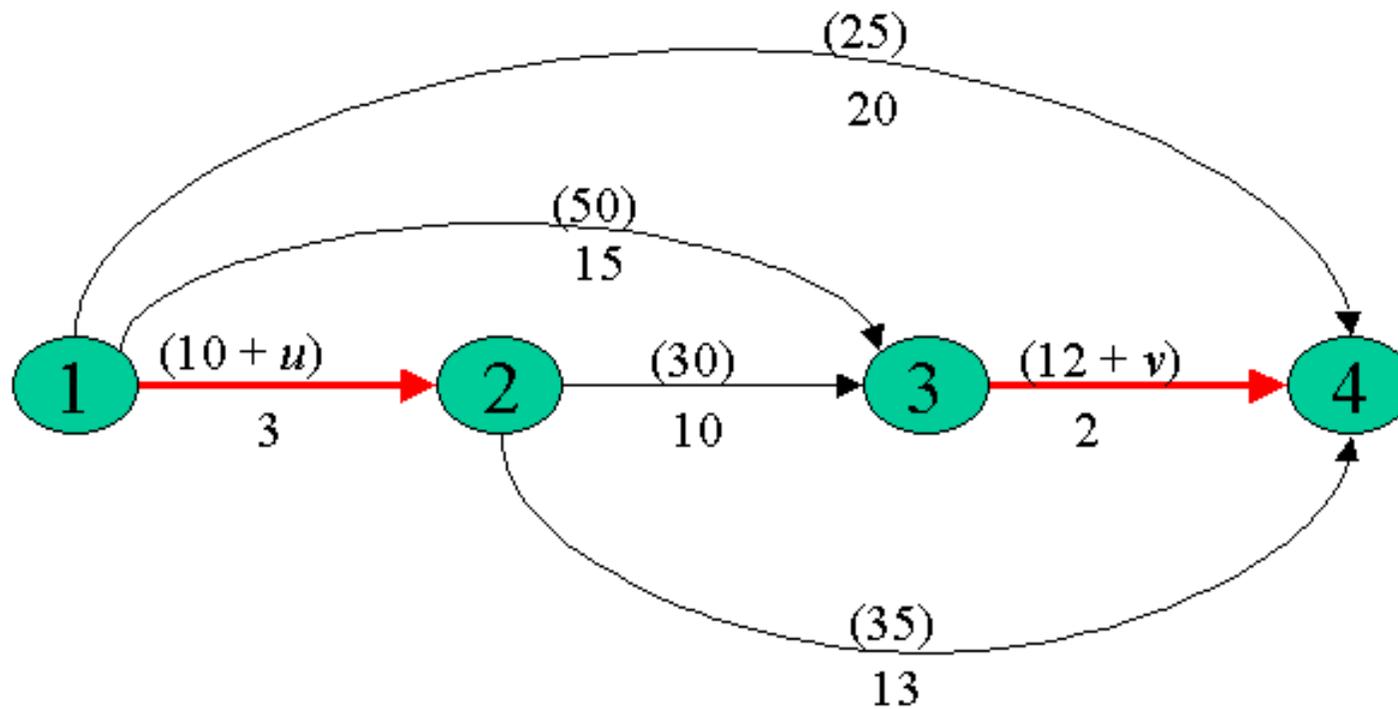
For fixed congestion we have:

$\Rightarrow$  As  $n \rightarrow \infty$

$$\sum_{i=1}^n v_{p,i}^* \rightarrow \int_0^{\alpha_{\max}} y(\alpha) d\alpha$$

# Congestion versus fixed delay

Network



$h(\alpha)$ : triangular distribution over  $[0, 10]$

## Congestion versus fixed delay

Paths	Arcs	Delay	Fixed costs	Tolls	PPC
A	1-2-3-4	15	52	$u + v$	$15 + \alpha(52 + u + v)$
B	1-2-4	16	45	$u$	$16 + \alpha(45 + u)$
C	1-3-4	17	62	$v$	$17 + \alpha(62 + v)$
D	1-4	20	25		$20 + \alpha 25$

## Ascent algorithm only: fixed delay

Iterations	$u$	$v$	Revenue
0	0.00	0.00	0.00
1	10.0	3.60	125.00
2	5.00	3.60	140.63
3	6.88	3.60	148.04
4	6.67	3.60	148.15
5	6.67	3.60	148.15

Table 1: Example with fixed delay (non optimal)

## Ascent algorithm only: fixed delay

Iterations	$u$	$v$	Revenue
0	5.00	0.80	148.37
1	6.88	1.33	155.47
2	6.77	1.00	156.12
3	6.67	1.00	156.15
4	6.67	1.00	156.15

Table 2: Example with fixed delay (optimal)

## Ascent algorithm only: with congestion

Iterations	$u$	$v$	Revenue
0	0.00	0.00	0.00
1	88.90	62.97	274.69
2	90.12	87.82	296.87
3	121.54	92.69	304.80
4	111.70	110.87	313.18
5	122.33	106.66	314.49
6	120.27	111.09	316.31
7	129.62	120.51	317.91
8	129.74	120.46	317.91

Table 3: Example with congestion:  $d(\bar{v}_a) = d_a \exp(0.1\bar{v}_a)$

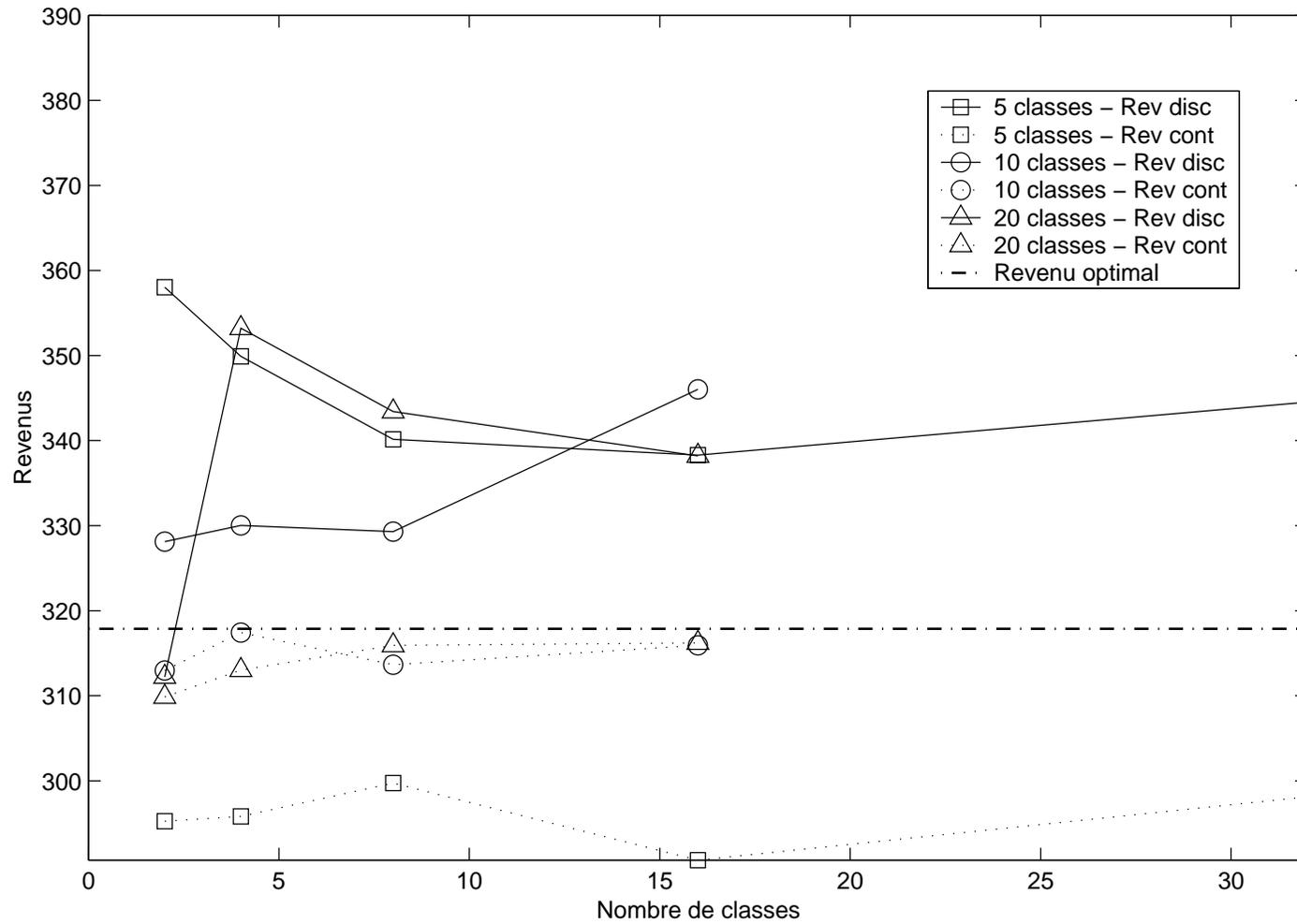
## Discretization phase: equal width

n	L	Revenue disc.	Revenue cont.	time (s)
2	5	358.03	295.23	0.56
4	5	349.89	295.80	0.54
8	5	340.14	299.73	3.32
16	5	338.28	290.67	17.52
32	5	344.51	298.11	473.03
2	10	328.11	312.96	1.04
4	10	330.03	317.43	2.18
8	10	329.28	313.63	21.37
16	10	346.02	315.93	788.87
2	20	312.22	309.86	1.94
4	20	353.20	313.00	6.38
8	20	343.41	315.93	208.16
16	20	338.19	316.21	7352.31

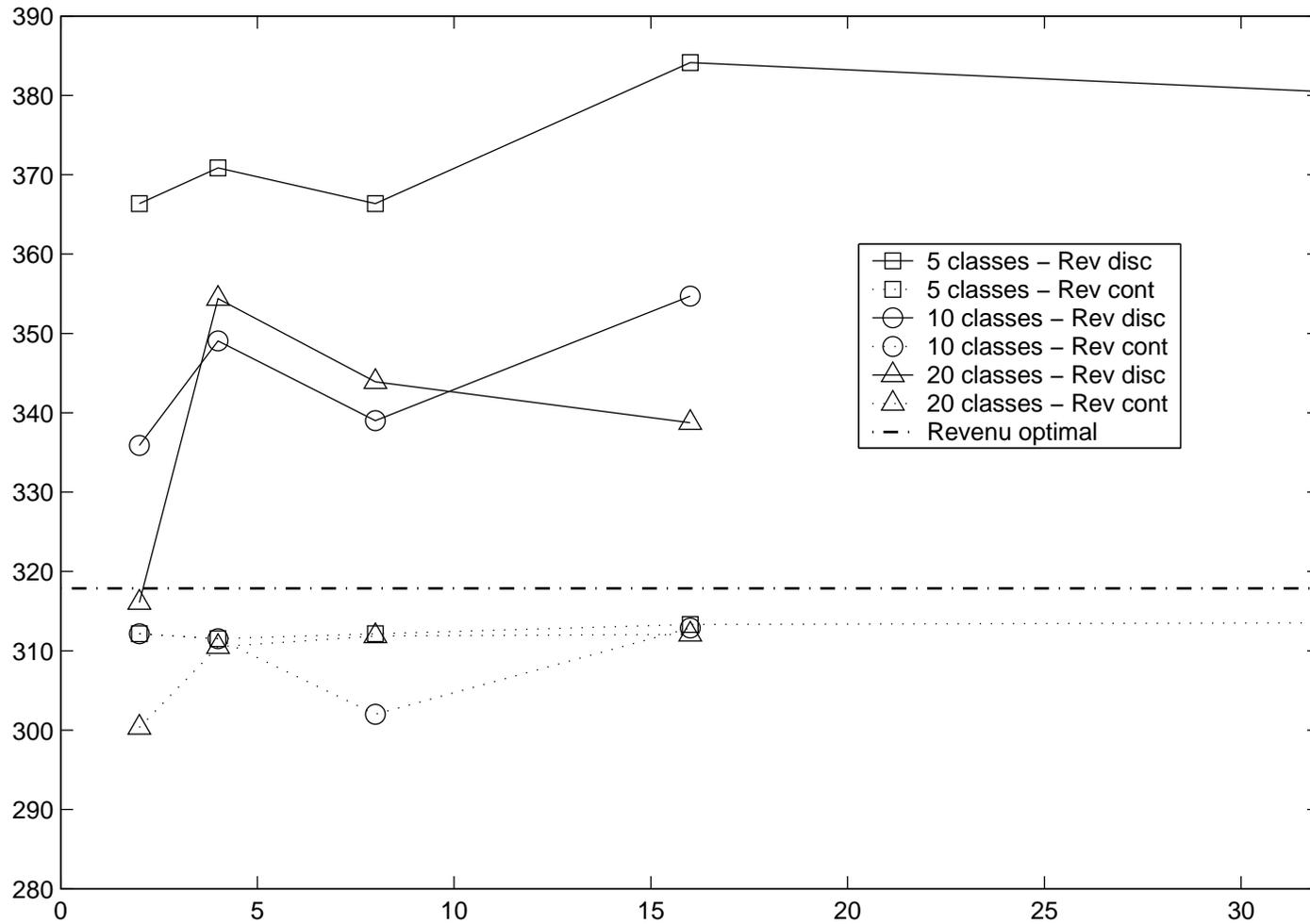
## Discretization phase: equal height

n	L	Revenue disc.	Revenue cont.	time (s)
2	5	366.35	312.16	0.70
4	5	370.85	311.53	1.53
8	5	366.35	312.16	3.83
16	5	384.12	313.32	13.51
32	5	380.47	313.52	31.47
2	10	335.89	312.15	1.49
4	10	349.05	311.53	3.45
8	10	339.00	302.00	20.09
16	10	354.70	312.86	184.14
2	20	316.07	300.34	5.91
4	20	354.39	310.52	14.75
8	20	343.88	311.87	704.71
16	20	338.74	312.07	2032.32

# Discretization phase: equal width



# Discretization phase: equal height



## Test problems

- a 0-tax starting point
- b 2 classes, 2 congestion-steps
- c 3 classes, 2 congestion-steps
- d 4 classes, 2 congestion-steps
- e 2 classes, 4 congestion-steps
- f 4 classes, 4 congestion-steps

	Nodes	Arcs	Tolls	O-D	Dem.	Delay (fol.)	Cost (lea.)
A	30	200	20	10	1	20-50 (*30)	15-35 (/2)
B	30	200	20	10	4-8	20-50 (*10)	15-35 (/2)

MaxRev: maximizing the revenue

MinCong: minimizing the total congestion

## Numerical results: MaxRev-A

Pb	a	b	c	d	e	f
1	449.7	446.9	444.2	444.0	444.1	443.9
2	223.4	233.2	233.5	233.5	233.3	233.0
3	619.3	620.6	620.8	619.4	619.3	619.5
4	337.5	339.3	339.3	338.7	339.2	338.6
5	371.6	407.1	392.4	383.3	407.0	383.3
6	390.2	406.5	408.0	407.8	459.7	461.9
7	632.3	640.4	640.6	640.6	640.4	640.6
8	124.8	125.0	124.9	125.0	125.0	125.0
9	159.3	160.2	160.0	159.8	161.2	161.3
10	450.0	455.9	455.7	456.0	455.9	456.0
Avg. improv.	-	2.1%	1.7%	1.4%	3.5%	2.9%
Std. dev.	-	2.9%	2.2%	1.9%	5.6%	5.4%

## Numerical results: MaxRev-B

Pb	a	b	c	d	e
1	1110.8	1117.5	1116.0	1117.4	1120.9
2	1510.3	1523.3	1532.2	1535.0	1442.0
3	659.4	739.2	739.2	739.8	737.1
4	737.5	909.6	928.0	877.8	942.2
5	368.7	369.6	370.5	368.8	360.5
Avg. improv.	-	7.4%	8.1%	6.7%	6.8%
Std. dev.	-	9.1%	9.9%	7.6%	11.9%

# Numerical results: MinCong-A

Pb	T=0	a	b	c	d	e	f
1	7163.4	7009.1	7009.1	7009.1	7009.1	7090.9	7115.5
2	7519.8	7446.6	7467.8	7457.6	7465.5	7467.9	7522.5
3	6829.0	6484.8	6457.5	6457.5	6454.8	6481.9	6457.8
4	7474.1	7342.9	7342.9	7342.9	7342.9	7316.9	7279.4
5	8512.9	8421.7	8421.1	8422.5	8422.8	8463.9	8422.9
6	8179.9	7926.5	8350.3	8531.4	8513.6	7957.2	7921.5
7	6862.0	6722.6	6722.6	6722.6	6722.6	6648.2	6646.2
8	9393.1	9364.9	9364.9	9364.9	9364.9	9364.9	9364.9
9	7387.5	7181.4	7211.0	7176.0	7210.1	7184.7	7215.2
10	7291.3	7075.4	7075.4	7132.0	7082.5	7083.0	7082.5
Avg. improv.	-	2.2%	1.7%	1.4%	1.5%	2.1%	2.2%
Std. dev.	-	1.3%	1.8%	2.3%	2.3%	1.4%	1.6%

# Numerical results: MinCong-B

Pb	T=0	a	b	c	d	e
1	7633.1	7360.7	7379.1	7360.7	7360.7	7353.3
2	9645.2	9005.1	9005.6	9005.1	9005.1	9004.2
3	10292.6	10145.5	10167.0	10145.5	10145.5	10167.03
4	9165.4	8884.0	8891.0	8884.0	8884.0	8803.2
5	8563.7	8423.0	8424.1	8423.0	8423.0	8423.1
Avg. improv.	-	3.3%	3.2%	3.3%	3.3%	3.4%
Std. dev.	-	1.9%	1.9%	1.9%	1.9%	1.9%

# Application at SNCF



# Bilevel Revenue Optimization Model

- Revenue Optimization modeled as a hierarchical (bilevel) optimization process

## Carrier

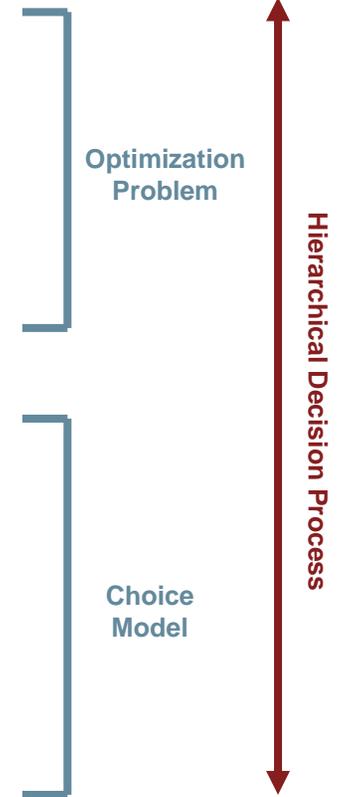
Seeks to maximize revenue over whole network while taking into account

- Commercial constraints (policies, fare structures, rules, etc.)
- Network constraints (schedules, network capacities)
- Availability constraints
- Competitors' offerings
- And, most importantly, **customer behavior**

## Customers (passengers)

Seek to satisfy their travel needs while maximizing the utility associated with their purchase decision

- They consider price and other product attributes (tangible or abstract)
- They have access to detailed competitive information
- They make informed decisions

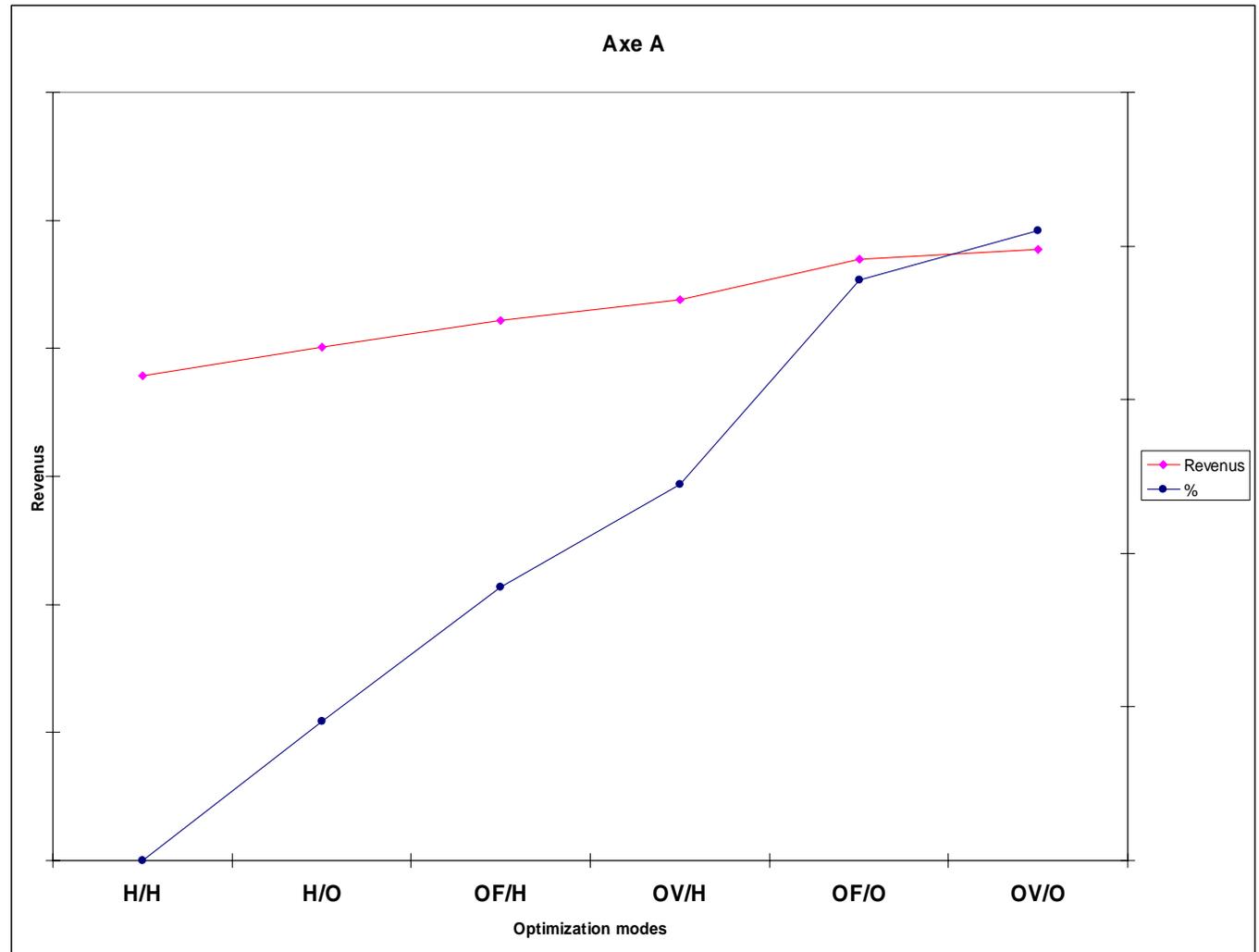


# Optimization Modes

<b>Allocations</b>	Historical (H)	Optimal (O)
<b>Pricing</b>		
Historical (H)	√	√
Optimal Static (OS)	√	√ <b>(jointly)</b>
Optimal Dynamic (OD)	√	√ <b>(jointly)</b>

- Choice model based on 13 millions of observations (over 1 year)
- Cross-nested logit model

# Optimization Modes



Pricing / Inventory Management

H=Historical S=Static  
O=Optimal D=Dynamic

# Application at SNCF: design of rebate cards

A0	A1	A2	A2bis	A3	A4
<b>AP</b>					
Plein Tarif					
<b>cartes à 25%</b>					
Carte Escapade 25%	<b>AC</b>				
Business Pass					
Groupe de 0 à 15%	<b>cartes à 30%</b>				
	Congrès				
	Rail Plus				
<b>AS</b>					
Abonnement EEA	EuroDomino	<b>AD</b>	<b>AD bis</b>	<b>AL</b>	<b>AI</b>
Congé annuel 25%	Eurail Pass				
Retraité pension. 25%	France Pass	<b>cartes à 40%</b>	<b>cartes à 50%</b>	Groupe jeune 50%	
Famille nombreuse	Brit Pass			Promenade enfants	
Militaires		Congé annuel 50%			
Réformés de guerre	<b>AO</b>	Retraité pension. 50%			
Abonnements	Groupes 30%	InterRail			<b>cartes à 60%</b>
	Groupes jeunes 30%	Guide Handicapé 50%			
<b>AE</b>					
SNCF & Ayants droits					
Voyages de service					
Bons SAG	<b>AA</b>	<b>AH</b>		<b>AU</b>	<b>AJ</b>
Gratuits ministères	<b>Prem's N3</b>	<b>Prem's N2</b>		<b>Prem's N1</b>	<b>Prem's N0</b>
B0	B1	B2	B2bis	B3	B4
<b>BP</b>					
Plein Tarif					
<b>cartes à 25%</b>					
Carte Escapade 25%	<b>BC</b>				
Business Pass					
Groupes de 0 à 15%	<b>cartes à 30%</b>				
	Congrès				
	Rail Plus				
<b>BS</b>					
Abonnement EEA	EuroDomino	<b>BD</b>	<b>BD bis</b>	<b>BL</b>	<b>BI</b>
Congé annuel 25%	Eurail Pass				
Retraité pension. 25%	France Pass	<b>cartes à 40%</b>	<b>cartes à 50%</b>	Groupe jeune 50%	
Familles nombreuses	Brit Pass			Promenade enfants	
Militaires		Congé annuel 50%			
Réformés de guerre	<b>BO</b>	Retraité pension. 50%			
Abonnements	Groupes 30%	InterRail			<b>cartes à 60%</b>
	Groupes jeunes 30%	Guide Handicapé 50%			
<b>BE</b>					
SNCF & Ayants droits					
Voyages de service					
Bons SAG	<b>BA</b>	<b>BH</b>		<b>BU</b>	<b>BJ</b>
Gratuits ministères	<b>Prem's N3</b>	<b>Prem's N2</b>		<b>Prem's N1</b>	<b>Prem's N0</b>

**Pax: -0,70%**

**Revenue: -2,84%**

# Conclusion

- Phase I finds a ‘good’ solution with a small number of groups
- Discretization of congestion is more sensible
- Hybrid algorithm appears to be efficient in max pricing
- Bilevel programming provides a global picture
- Future work: inverse optimization, exploiting the structure of the discretized problem, stochastic issues