The Discrete Logarithm Problem on Algebraic Curves

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Workshop on Computational challenges arising in algorithmic number theory and Cryptography
Fields Institute, Toronto

November 2, 2006

Outline

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1. Discrete Logarithms
   - Definitions and notation
   - Relationship between different groups; DLOG reduction
2. Known facts about DLOG reduction
   - Reductions between elliptic curves
   - Reductions from elliptic curves to hyperelliptic Jacobians

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   - Relationship between different groups; DLOG reduction
2. Known facts about DLOG reduction
   - Reductions between elliptic curves
   - Reductions from elliptic curves to hyperelliptic Jacobians
3. Open problems in DLOG reduction
   - Elliptic curves not admitting reductions
   - Reductions between hyperelliptic Jacobians
   - Reductions from elliptic curves to non-hyperelliptic Jacobians

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The Discrete Logarithm Problem

- Let G be a cyclic group of order n, with generator g.
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- Many cryptographic protocols require a group for which computing $\mathrm{DLOG}_g(h)$ is hard.
- What determines the difficulty of computing discrete logarithms?

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  - except for rare exceptions such as $\mathrm{DLOG}_g(g)$, $\mathrm{DLOG}_g(e)$, ...

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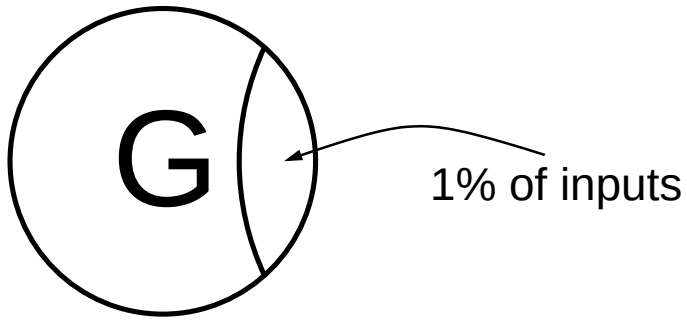

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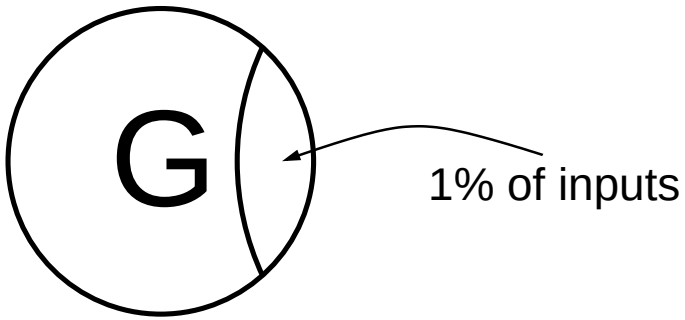

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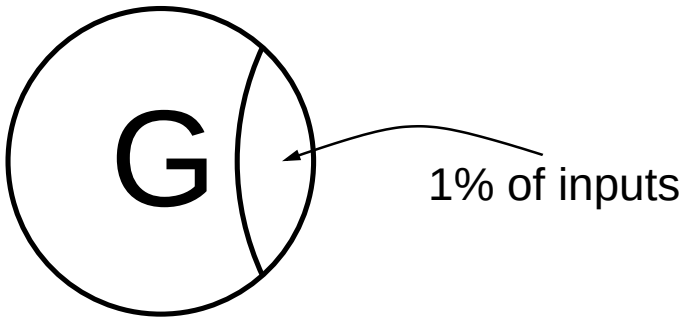

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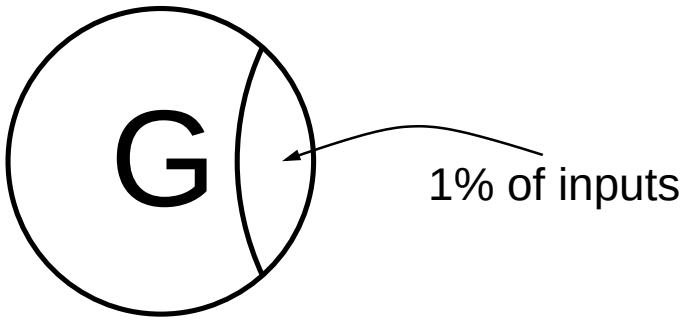

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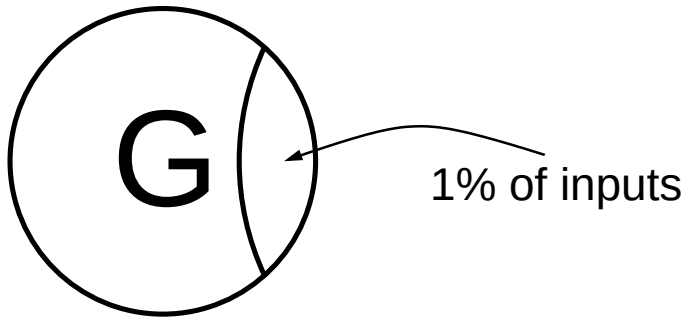

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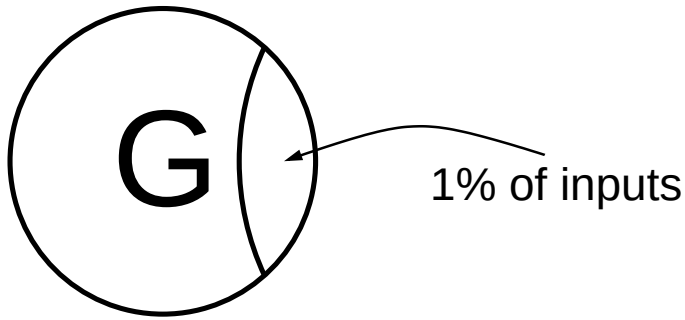

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- Eventually $\mathcal{A}(g^r g^k)$ will return $(r+k)$. We can then find k since we know r and $(r+k)$.
- Therefore, **on average** the discrete log problem is equivalent for all $h \in G$.

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  Note that by size you mean isomorphism class, since G is cyclic.

Effect of $|G|$ on DLOG

Size of G does have some effect on DLOG:

- All else being equal, a larger group size makes DLOG harder
- Certain group sizes are insecure no matter what the group
  - e.g. if $|G|$ is smooth (that is, all prime divisors of $|G|$ are small) then DLOG on G is easy.

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  - Divison is easy by Euclid's algorithm.

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- After correcting for the above issues, it is widely believed that DLOG difficulty is a function of group size (within a single family of groups, bit representations, smoothness constraints, etc.)

Current status of DLOG in various groups

Any group of order n:

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Jacobians of hyperelliptic curves of genus g over a finite field $\mathbb{F}_q$:

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In all cases, DLOG difficulty is a function of group size.

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The concept of DLOG reduction

- Goal: To establish relationships between discrete logarithms on group A and group B.
- The basic tool for this reduction is group homomorphisms.
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- In other words, if you can easily compute DLOG in G' (Step 2), then you can easily compute DLOG in G.
- However, you also need to be able to easily compute the homomorphism ϕ (Step 1).

Elliptic curves

- A group homomorphism between elliptic curves is called an *isogeny*.
- An isogeny is a rational function — it is given by a quotient of polynomials.
- The *degree* of an isogeny is the degree of the polynomial.
- **Theorem (Tate, 1966):** Two elliptic curves over a finite field have the same size if and only if they are *isogenous* (i.e. there exists an isogeny between them).
- Isogenous is an equivalence relation. We will call the equivalence classes *isogeny classes*.

Example of an isogeny

- $p = 7925599076663155737601$
- $E_1 : y^2 = x^3 + 12046162683058694734 * x + 7901506751297038348133$ in GF(p)
- $E_2 : y^2 = x^3 + (3021319262486407622796 * u + 4101162511412606196442) * x + (7040333493178698383420 * u + 174577275676632103431)$ in GF(p^2)
- ϕ: $E_1 \longrightarrow E_2$ given by $\phi(x, y) = ((x^7 + (2646061772402770501474 * u + 28775605307889315926 5) * x^6 + (13293530722861505653 8 * u + 3530390499615039152484) * x^5 + (463794718376492302 73 * u + 1073811655050424931224) * x^4 + (2474785317056152334847 * u + 1839199255709390890698) * x^3 + (4285381276738035289332 * u + 2268033696082534919907) * x^2 + (1160928171089162069604 * u + 447867418402154326079 3) * x + (3220829138361157238167 * u + 4664892256879213165649))/(x^6 + (2646061772402770501474 * u + 28775605307889315926 5) * x^5 + (1945985508507744496834 * u + 648093055215868995 31) * x^4 + (4591727489633569666202 * u + 1570102870983786495532) * x^3 + (1500460390828721967700 * u + 692170444361451309763 5) * x^2 + (1297386801518789580736 * u + 285069874090833393640 0) * x + (3945372319876153578002 * u + 361974201101530900968)), (x^9 * y + (396909265860415575221 1 * u + 439443361794991760769 8) * x^8 * y + (653503558986201519334 8 * u + 779053291492004982110 9) * x^7 * y + (142198737502751098509 1 * u + 47681237267235708636) * x^6 * y + (230396899509609634966 1 * u + 3345680927799022267788) * x^5 * y + (243327773580243744178 9 * u + 335179462792558750055 3) * x^4 * y + (151602679570769848004 6 * u + 818260455738162732467) * x^3 * y + (102705817773763612561 4 * u + 3693613550368489401398) * x^2 * y + (450864584106502597890 9 * u + 491859307018303225658 5) * x * y + (8333818603777677580 * u + 6166744817175250513803) * y)/(x^9 + (396909265860415575221 1 * u + 439443361794991760769 8) * x^8 + (472198538858288575305 2 * u + 333051503235034633646 1) * x^7 + (355977212666782882640 97 * u + 615342200698874578176 5) * x^6 + (190294095199030591345 2 * u + 832145497772529583998) * x^5 + (255389155365196737883 3 * u + 549429624397957274232) * x^4 + (582104136352814424328 1 * u + 489551452715872062891 8) * x^3 + (746557228296674389403 4 * u + 123645603788466192332) * x^2 + (475221656789097062097 8 * u + 497829871306819801522) * x + (6192295778031003334018 * u + 4253951270570522301 94)))

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- **Theorem (Tate, 1966):** Two elliptic curves over a finite field have the same size if and only if they are isogenous.
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- Endomorphisms are not useful for reductions between **different** curves, so for reduction we must use isogenies of low degree.

Outline

Using isogenies to relate DLOG on elliptic curves

- Form a graph whose vertices are elliptic curves E and whose edges are low degree isogenies $\phi\colon E_1 \to E_2$.
- Galbraith (1999) observed that random walks on this graph produce efficiently computable isogenies which can be used for DLOG reduction.

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   - Definition: Two elliptic curves over a finite field are *endomorphous* (resp., *near-endomorphous*) if their endomorphism rings are equal (resp., nearly equal).
   - Endomorphous is an equivalence relation. We will call the equivalence classes *endomorphism classes*.
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2. Requires the heuristic assumption that *short* random walks have *roughly* uniform probability of reaching every vertex.

Using isogenies to relate DLOG on elliptic curves (cont'd)

- **Theorem:** (Jao, Miller, Venkatesan) Assuming the generalized Riemann hypothesis, there exists an absolute constant c such that random walks of length $(\log n)^c$ deviate from uniform probability by no more than a factor of 2, for isogenies of degree less than $c(\log n)^2$.

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- **Corollary:** All near-endomorphous elliptic curves over the same field have equivalent discrete logarithm problems **on average**.

Outline

Weil Descent

- Discovered by [Gaudry, Hess, Smart]
- Let E be an elliptic curve over $\mathbb{F}_{q^k}$. There exists a computable group homomorphism from E to a hyperelliptic Jacobian over $\mathbb{F}_q$.

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- For some values of E and q^k, the genus of the hyperelliptic curve is large enough to make this attack practical.
- If E' is isogenous to a curve E which is vulnerable to Weil descent, then E' can be attacked too [Galbraith, Hess, Smart]
  - Construction relies on random walks of isogenies
  - Requires uniform mixing of random walks

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- Subexponential for $g = 2[, 4, 5, 7, 8, 10, \ldots]$ implies elliptic curve DLOG is subexponential.

Outline

1. Discrete Logarithms
   - Definitions and notation
   - Relationship between different groups; DLOG reduction
2. Known facts about DLOG reduction
   - Reductions between elliptic curves
   - Reductions from elliptic curves to hyperelliptic Jacobians
3. Open problems in DLOG reduction
   - Elliptic curves not admitting reductions
   - Reductions between hyperelliptic Jacobians
   - Reductions from elliptic curves to non-hyperelliptic Jacobians

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  - Remember, bit representation matters for DLOG!

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- Can we prove equivalence results for other fields?

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   - Stay tuned ...