

Conference in honor of Spencer Bloch

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On Gabber's recent work in étale cohomology

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- [G1] O. Gabber, *A finiteness theorem for non abelian H^1 of excellent schemes*, Conférence en l'honneur de L. I., Orsay, June 2005.
- [G2] O. Gabber, *Finiteness theorems for étale cohomology of excellent schemes*, Conference in honor of P. Deligne on the occasion of his 61st birthday, IAS, Princeton, October 2005.

L. I., Y. Laszlo, and F. Orgogozo, Seminar on Gabber's work in étale cohomology, Ecole Polytechnique, spring 2006, notes in preparation.

PLAN

1. Absolute purity
2. Finiteness
3. Local uniformization
4. Affine Lefschetz
5. Local duality
6. Glimpses on proof of finiteness

1. ABSOLUTE PURITY

THEOREM 1.1 [Gabber, 1994]

X regular, locally noetherian scheme,

$i : Y \subset X$ regular divisor,

$j : U = X - Y \rightarrow X$

$\Lambda = \mathbb{Z}/n\mathbb{Z}$, $n > 0$ invertible on X .

Then :

$$R^q j_* \Lambda = \begin{cases} \Lambda & \text{if } q = 0 \\ \Lambda_Y(-1) & \text{if } q = 1 \\ 0 & \text{if } q > 1. \end{cases}$$

$$\Lambda_Y \xrightarrow{\sim} R^1 j_* \Lambda(1) \xrightarrow{\sim} \mathcal{H}_Y^2(\Lambda(1))$$

$$1 \mapsto c(Y) \in H^0(Y, \mathcal{H}_Y^2(\Lambda(1))) = H_Y^2(X, \Lambda(1))$$

Equivalent formulation :

$$Ri^! \Lambda = \Lambda_Y(-1)[-2]$$

Historical sketch

Grothendieck's conjecture SGA 5 I

X, Y smooth / k : SGA 4 XVI

X excellent of char. 0 : SGA 4 XIX

$\dim X = 2$: Gabber (1976)

X/\mathbb{Z} finite type, $\ell|n \Rightarrow \ell \geq \dim X + 2$:

Thomason (1984)

general case : Gabber (1994)

see Fujiwara, Azumino conference (2000)

[G2] Gabber (2005) : no K -theory

COROLLARY 1.2

X regular, locally noetherian

$i : Y \subset X$ regular, closed subscheme,

$\text{codim}(i) = d$

Then :

$$Ri^!(\Lambda) = \Lambda_Y(-d)[-2d],$$

i. e.

$$\mathcal{H}_Y^q(\Lambda) = \begin{cases} 0 & \text{if } q \neq 2d \\ \Lambda_Y(-d) & \text{if } q = 2d \end{cases}$$

$$\Lambda_Y \xrightarrow{\sim} \mathcal{H}_Y^{2d}(\Lambda)(d)$$

$$1 \mapsto c(Y) \in H^0(Y, \mathcal{H}_Y^{2d}(\Lambda)(d)) = H_Y^{2d}(X, \Lambda(d))$$

If $Y = \cap Y_i$ ($1 \leq i \leq d$) (transverse divisors), then

$$c(Y) = \prod c(Y_i)$$

COROLLARY 1.3

X regular, locally noetherian

$D = \sum_{i \in I} D_i \subset X$ snc (= strict normal crossings) divisor

$j : U = X - D \rightarrow X$

Then :

$$R^q j_* \Lambda = \begin{cases} \Lambda & \text{if } q = 0 \\ \oplus \Lambda_{D_i}(-1) & \text{if } q = 1 \\ \Lambda^q R^1 j_* \Lambda & \text{if } q > 1. \end{cases}$$

$$\oplus \Lambda_{D_i} \xrightarrow{\sim} R^1 j_* \Lambda(1)$$

$1 \in H^0(D_i, \Lambda) \mapsto$ image of $c(D_i) \in H^0(X, R^1 j_{i*} \Lambda(1))$

into $H^0(X, R^1 j_* \Lambda(1))$

$(j_i : X - D_i \rightarrow X)$

cup-product induces $\Lambda^q R^1 j_* \Lambda \xrightarrow{\sim} R^q j_* \Lambda$

$$R^q j_* \Lambda \xrightarrow{\sim} \oplus_{|J|=q} \Lambda_{D_J}(-|J|)$$

X strictly semistable /trait, $D =$ special fiber :
Rapoport-Zink (1982)

Relative variants : SGA 4
(X/S smooth, smooth couples, relative dnc, ...)

New proof of 1.1 [G2] uses :

- material from old proof
(generalized Gysin morphisms)
- canonical, simultaneous desingularizations
of finite families of affine toric varieties / \mathbb{Q} (\Leftarrow Bierstone-Milman, Encinas-Villamayor)
de Jong, log geometry (log regularity (Kato),
very tame actions of finite groups)

2. FINITENESS

Recall [EGA IV 7.8] : a ring A is

- **quasi-excellent** if :
 - (i) noetherian
 - (ii) fibers of $\text{Spec } \widehat{\mathcal{O}}_{X,x} \rightarrow \text{Spec } \mathcal{O}_{X,x}$ geometrically regular
 - ($X = \text{Spec } A$)
 - (iii) openness of $\text{Reg}(\text{Spec } A')$ (A' finite type / A)
- (NB. for A local, (i) + (ii) \Rightarrow (iii))

- excellent if :

quasi-excellent + universally catenary

a scheme X is quasi-excellent (qe, for short) (resp.
excellent) if

X = union of open affine $\text{Spec } A_i$,

A_i quasi-excellent (resp. excellent)

X/Y f. t. + Y quasi-excellent (resp. excellent)
 $\Rightarrow X$ quasi-excellent (resp. excellent)

A complete local noetherian, or
Dedekind and $\text{Frac } A$ of char. 0
 \Rightarrow excellent

THEOREM 2.1 [G2] :

Y noetherian, qe, $f : X \rightarrow Y$ f. t.,

$n \geq 1$ invertible on Y , $\Lambda = \mathbb{Z}/n\mathbb{Z}$,

F = constructible Λ -module on X

Then :

- (a) $R^q f_* F$ constructible $\forall q$,
- (b) $\exists N$ s. t. $R^q f_* F = 0$ for $q \geq N$.

Remarks :

- (a) + (b) $\Leftrightarrow Rf_* : D_c^b(X, \Lambda) \rightarrow D_c^b(Y, \Lambda)$
- f **proper** : Y qe, n invertible on Y superfluous
(finiteness th. [SGA 4 XIV])
- $\text{char}(Y) = 0$: Artin [SGA 4 XIX]

- $f = S$ -morphism, X, Y f. t. / S regular, $\dim \leq 1$: Deligne [SGA 4 1/2, Th. Finitude]
- $f = S$ -morphism, X, Y f. t. / S noetherian
 \Rightarrow generic constructibility of $R^q f_* F$: Deligne [SGA 4 1/2, Th. Finitude]
- qe not needed for $q = 0$, needed for $q > 0$

THEOREM 2.2 [G1]

$f : X \rightarrow Y$ f. t., Y noetherian

(1) F constructible (sheaf of sets) on X

$\Rightarrow f_*F$ constructible

(2) \mathbb{L} = set of primes invertible on Y ,

Y qe,

F constructible sheaf of groups of \mathbb{L} -torsion on X ,

$\Rightarrow R^1f_*F$ constructible

Remarks :

- f **proper** : [SGA 4 XIV] (qe, \mathbb{L} invertible on Y) superfluous
- no noncommutative analogue of [Deligne, SGA 4 1/2, Th. finitude] (/ trait) was known, but generic constructibility OK (Orgogozo)

Proof of 2.1 (a) : local uniformization

Proof of 2.1 (b) : same new techniques used
for the purity th.

Proof of 2.2 ;

- [G1] : ultraproducts
- alternate (in preparation) : local uniformization

3. LOCAL UNIFORMIZATION

S a scheme

pspf topology on (schemes loc. f. p. / S) :

generated by :

- proper surjective f. p. morphisms
- Zariski open covers

(pspf = propre, surjectif, présentation finie)

pspf finer than étale

S noetherian : pspf $/S =$ Voevodsky's h-topology
 $=$ Goodwillie-Lichtenbaum's ph-topology

S pspf local $\Leftrightarrow S = \text{Spec } V$
 V valuation ring, $\text{Frac}(V)$ alg. closed

THEOREM 3.1 [G2]

X noetherian, qe, $Y \subset X$ nowhere dense closed subset

Then :

\exists finite family $(f_i : X_i \rightarrow X) (i \in I)$ s. t. :

- (f_i) pspf covering
- $\forall i, X_i$ regular, connected
- $Y_i = f_i^{-1}(Y) =$ support of strict dnc (or \emptyset)

- $\forall i, f_i$ generically quasi-finite and

sends maximal pts to maximal pts

NB. f_i not necessarily proper

3.1 = local uniformization theorem

compare with

- Hironaka ($/\mathbb{Q}$)
- de Jong (f. t. $/S$ regular, dim. ≤ 1)

which are both global

Rough outline of proof

- reduction to X local henselian
- reduction to X local complete :
uses : Artin-Popescu's th.
+ Gabber's new formal approximation technique
- by induction on $\dim(X)$, proof in local complete case
relies on :
 - Gabber's fibration th.
 - de Jong's th. on nodal curves
 - log regularity and resolution of toric singularities
(Kato)

Gabber's fibration th. :

THEOREM 3.2 [G2]

X local, normal, complete, excellent

$\dim(X) \geq 2$, $Y \subset X$ closed

$\Rightarrow (X, Y) \sim$ completion of (X', Y')

X' fibered in curves / S local, complete, regular of dim.

$= \dim(X) - 1$

$\sim =$ up to finite, surjective, local $X_1 \rightarrow X$

Proof of 3.2 uses :

- Gabber's improvement of Cohen's th.
- Elkik's approximation th.
- Epp's potential reducedness th.

Starting point :

LEMMA [G1] :

A noetherian, local, complete, reduced,
equidim. of dim. r , equichar.

$\Rightarrow A$ finite, generically étale / B ,

$B = k[[t_1, \dots, t_r]]$, k = res. field of A

(refinement of [EGA 0_{IV} 19.8.8] (Cohen th.))

3.2 \Rightarrow 3.1

- de Jong \Rightarrow make curve **nodal**
- induction \Rightarrow uniformize base
- Kato \Rightarrow solve toric singularities of top

Remark :

method of proof \Rightarrow solution of Kato's conjecture
on p -dimension of local fields :

THEOREM 3.3 (Gabber-Orgogozo)

A local, noetherian, henselian, excellent, integral,
 $K = \text{Frac}(A)$, k = residue field,

$\text{char}(k) = p$

Then :

$$(*) \quad \dim_p(K) = \dim_p(k) + \dim(A)$$

Here, for a field F :

$$\dim_p(F) = \text{cd}_p(F) \text{ if } \text{char}(F) \neq p$$

and

$$= \inf\{d \text{ s. t. } [F : F^p] \leq p^d \text{ and } H^1(F', \Omega_{F'/\mathbb{F}_p, \log}^d) = 0 \\ \forall F'/F \text{ finite}\}$$

if $\text{char}(F) = p > 0$

Remarks :

- $A = \text{dvr}$: Kato
- if A strictly local, $\text{char}(k) \neq p$
(*) still OK
([SGA 4 X] + Gabber's affine Lefschetz)

4. AFFINE LEFSCHETZ

Recall [SGA 4 XIV] :

Y f. t. /field k , $f : X \rightarrow Y$ affine,

$\Lambda = \mathbb{Z}/n\mathbb{Z}$, n prime to $\text{char}(k)$

F = sheaf of Λ -modules on X

$\Rightarrow \dim \text{supp}(R^q f_* F) \leq \dim \text{supp}(F) - q$

generalized dim

X locally noetherian ; $x, y \in X$

$y = (\text{étale})$ immediate specialization of x if :

$y \in \overline{\{x\}}$ and

strict henselization of $\overline{\{x\}}$ at \bar{y}

(geom. pt / y)

has an irr. component of dim. 1

$\delta : X \rightarrow \mathbb{Z}$ = (étale) dimension function on X if :

$$\delta(y) = \delta(x) - 1$$

\forall (étale) immediate specialization y of x

- (dim. function exists Zariski loc. on X)
 \Leftrightarrow (X universally catenary)
- X qe \Rightarrow a dim. function exists étale loc.

Examples :

- X f. t. $/k$ (a field) ; $\delta(x) := \text{tr.deg}_k k(x)$ is a dim. function
- X irreducible, univ. catenary (e. g. closed subscheme of regular scheme) ; $\delta(x) = -\dim \mathcal{O}_{X,x}$ is a dim. function

- $f : X \rightarrow S$ f. t., $\delta_S = \text{dim. function on } S$
 $\Rightarrow \delta_X(x) = \delta_S(f(x)) + \deg.\text{tr}(k(x)/k(f(x)))$
is a dim. function on X
(Artin's rectified dimension)

THEOREM 4.1 [G2]

$f : X \rightarrow Y$ affine of f. t.

Y qe, with dim. function δ_Y

δ_X associated rectified dimension on X

$\Lambda = \mathbb{Z}/n\mathbb{Z}$, $n > 0$ invertible on Y

Then :

\forall sheaf of Λ -modules F on X , $\forall q$

$$\delta_Y(R^q f_* F) \leq \delta_X(F) - q.$$

Here :

$$\delta(G) = \sup_{G_x \neq 0} \delta(x).$$

Remarks :

- Y of f. t. /field : standard affine Lefschetz [SGA 4 XIV]
- $\text{char}(Y) = 0$: [SGA 4 XIX]
- Y of f. t. / trait : Gabber (1994)

4.1 \Leftrightarrow

COROLLARY 4.2

$Y = \text{Spec } R$ strictly local, excellent, $\dim(Y) = d$

$f \in R$, $U = \text{Spec } R[f^{-1}]$

Then :

$$H^q(U, \Lambda) = 0 \quad \forall q > d.$$

Note : **excellent** needed for $d > 1$

Proof of 4.1 :

techniques similar to those used in
proof of finiteness th.

(approximation, uniformization th.)

5. LOCAL DUALITY

DEFINITION 5.1

X noetherian, $\Lambda = \mathbb{Z}/n\mathbb{Z}$, $n > 0$ invertible on X

$K \in D_c^b(X, \Lambda)$ called **dualizing complex** if

- K is of finite tor-dimension
- $D_K := R\mathcal{H}om(-, K) : D_c^b(X, \Lambda) \rightarrow D_c^b(X, \Lambda)$
- $\forall F \in D_c^b(X, \Lambda), F \xrightarrow{\sim} D_K D_K F.$

Remarks

(a) 5.1 weaker than [SGA 5 I],

where further imposed :

$\exists N$ s. t. \forall constructible F , $\mathcal{E}xt^i(F, K) = 0$ for $i > N$

(b) K dualizing $\Rightarrow K$ unique up to $K \mapsto K \otimes L[r]$,

L invertible, $r \in \mathbb{Z}$

THEOREM 5.2 [G2]

X excellent, noetherian, having dimension function

$\Lambda = \mathbb{Z}/n\mathbb{Z}$, $n > 0$ invertible on X

Then :

- (1) X admits a dualizing complex K
([SGA 5] dualizing if $\dim(X) < \infty$)
- (2) $f : X' \rightarrow X$ of f. t. $\Rightarrow Rf^!K$ dualizing
- (3) X regular $\Rightarrow \Lambda_X$ dualizing

Historical sketch

(3) : Grothendieck's conjecture [SGA 5 I]

(proved in *loc. cit.* for :

- $\dim(X) \leq 1$ (excellent superfluous)
- modulo purity, strong resolution,
in particular for $\text{char}(X) = 0$)

Deligne [SGA 4 1/2, Th. finitude] :

S regular, $\dim(S) \leq 1$ (non nec. excellent),

$f : X \rightarrow S$ f. t. $\Rightarrow Rf^! \Lambda_S$ dualizing

Proof of 5.2

uses :

- theory of local traces $H_x^{2d}(X, \Lambda(d)) \xrightarrow{\sim} \Lambda$
(X normal, excellent, dim. d , x closed pt.)
- construction of
candidate dualizing complexes $K \in D_c^+(X, \Lambda)$,
satisfying

$$R\Gamma_{\bar{x}}(K) \xrightarrow{\sim} \Lambda(\delta(x))[2\delta(x)]$$

\forall pt x and geom. pt. \bar{x} above x

6. GLIMPSES ON PROOF OF FINITENESS

- enough to show : $Rj_*\Lambda \in D_c^+(X, \Lambda)$ for
 $j : U \rightarrow X$ dense open immersion, X qe
- if de Jong available,
easy reduction to absolute purity :

construct cartesian diagram :

$$(*) \quad \begin{array}{ccc} U_{\cdot} & \xrightarrow{j_{\cdot}} & X_{\cdot} \\ \downarrow & & \downarrow \varepsilon_{\cdot} \\ U & \xrightarrow{j} & X \end{array}$$

with

- ε_{\cdot} proper hypercovering
- X_n regular $\forall n$
- $j_n : U_n \rightarrow X_n =$ inclusion of
complement of strict dcn $\forall n$

cohomological descent for $\varepsilon_* \Rightarrow$

$$Rj_*\Lambda = R\varepsilon_{.*}Rj_{.*}\Lambda$$

$Rj_{p*}\Lambda$ in D_c^b : absolute purity

$R^q\varepsilon_{p*}Rj_{p*}\Lambda$ constructible : ε_p proper

$R^i j_*\Lambda$ constructible : spectral sequence

$$R^q\varepsilon_{p*}Rj_{p*}\Lambda \Rightarrow R^{p+q}j_*\Lambda$$

- instead of de Jong (not available), use **uniformization theorem** to construct (*) with $\varepsilon_\cdot = \text{pspf}$ hypercovering
(and X_n, j_n as above)
pb : ε_n no longer proper
- circumvent this by :
 - Deligne's generic constructibility th. ([SGA 4 1/2 Th. fin.])
 - Gabber's hyper base change th. [G2]

- by standard criterion of constructibility,

have to show :

(P) $\forall i \geq 0, \forall g : X' \rightarrow X$ closed irreducible subset,
 \exists dense open $V \subset X'$ s. t. $g^* R^i j_* \Lambda|V$ constructible

- by Gabber's hyper base change th. [G2]

$$g^* Rj_* \Lambda = R\varepsilon'_* g.^*(Rj_* \Lambda)$$

where $g.$, ε' defined by cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g.} & X. \\ \downarrow \varepsilon' & & \downarrow \varepsilon. \\ X' & \xrightarrow{g} & X \end{array}$$

Remark

base change by g for ε_n not OK

as ε_n non proper

only **hyper** base change works

- by absolute purity,

$$K_p := g_p^*(Rj_{p*}\Lambda) \in D_c^b(X'_p, \Lambda)$$

- by Deligne's generic constructibility th.

\exists dense open $V_{pq} \subset X'$ s. t.

$R^q \varepsilon'_{p*} K_p|V_{pq}$ constructible

- spectral sequence

$$R^q \varepsilon'_{p*} K_p \Rightarrow g^* R^{p+q} j_* \Lambda$$

implies $\exists V$ satisfying (P)

Main ingredient for hyper base change :

Deligne's oriented products

(also used in Orgogozo's work on
nearby cycles over general bases)

Idea :

If $Rj_*\Lambda$ commutes with b. c. by g

(and same for $Rj_{p*}\Lambda$)

hyper base change OK

(\Leftarrow cohomological descent for psp hypercoverings

$U_\cdot \rightarrow U$ and $U'_\cdot \rightarrow U'$)

where

$U' = X' \times_X U$, $U'_\cdot = X' \times_X U_\cdot$.

However, $Rj_*\Lambda$ may not commute with b. c. by g

e. g. $g : X - U \rightarrow X$, then $U' = X' \times_X U = \emptyset$

Remedy : instead of $U' = X' \times_X U$, $U'_\cdot = X'_\cdot \times_{X_\cdot} U_\cdot$.

use Deligne's oriented products

$$\overset{\leftarrow}{U'} = X' \overset{\leftarrow}{\times}_X U \quad , \quad \overset{\leftarrow}{U'_\cdot} = X'_\cdot \overset{\leftarrow}{\times}_{X_\cdot} U_\cdot,$$

kind of punctured tubular neighborhoods
(of X' (resp. X'_\cdot) in X (resp. X_\cdot))

for which

base change and cohomological descent hold
(Gabber's oriented descent th.)

Here $\overset{\leftarrow}{U'}$ is a topos, with morphisms

$p_1 : \overset{\leftarrow}{U'} \rightarrow X'$, $p_2 : \overset{\leftarrow}{U'} \rightarrow U$, $\tau : jp_2 \rightarrow gp_1$

satisfying universal property