Semistrict Tamsamani's n-groupoids and connected n-types

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Modelling homotopy types: overview

- Motivation: Higher order version of fundamental group (Grothendieck)

- Parallel developments:

Homotopy Theory	Higher categories
Cat^n-groups Hypercrossed complexes ...	"conjecture test": weak n-groupoids model n-types e.g. Tamsamani, Batanin model

- Low dimensions

  $n = 2$ strict 2-groupoids model 2-types.

  $n = 3$ Gray groupoids model 3-types [Joyal -Tierney; Leroy]

- Semistrictification hypothesis
  Weak n-groupoids suitably equivalent to "semistrict" ones.

- Main result Every Tamsamani weak n-groupoid representing a connected n-type is suitably equivalent to a "semistrict" one.

- Method Comparison between cat^{n-1}-groups and Tamsamani's weak n-groupoids.

Internal categories and simplicial objects.

$\mathcal{C}$ category with finite limits

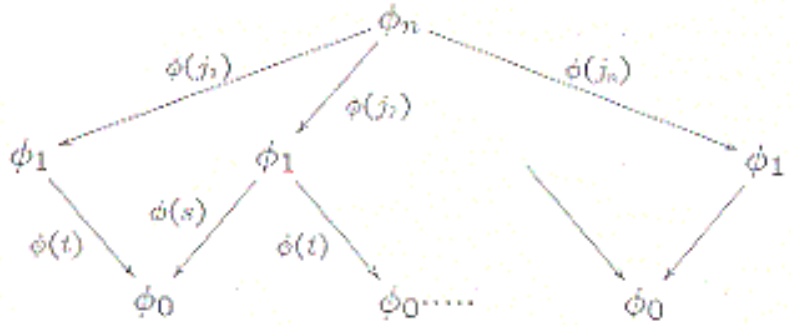
- The category $\mathrm{Cat}\,\mathcal{C}$:

Objects:

$$C_1 \times_{C_0} C_1 \xrightarrow{c} C_1 \underset{i}{\overset{d_0}{\underset{d_1}{\rightrightarrows}}} C_0 \qquad + \text{ axioms}$$

- Nerve functor $Ner : \mathrm{Cat}\,\mathcal{C} \to [\Delta^{op}, \mathcal{C}]$

- Segal maps $\phi \in [\Delta^{op}, \mathcal{C}] \quad \phi[n] = \phi_n$



$t(0) = 1 \quad s(0) = 0 \quad j_r(0) = r - 1 \quad j_r(1) = r$

Hence maps

$$\eta_n : \phi_n \to \phi_1 \times_{\phi_0} \overset{n}{\cdots} \times_{\phi_0} \phi_1$$

- Fact: ϕ nerve of $\mathrm{Cat}\,\mathcal{C}$ iff for all $n \geq 2$ η_n isomorphism

Cat^n-groups: Definitions

- Definition: $\mathrm{Cat}^0(\mathrm{Gp}) = \mathrm{Gp}$
  $\mathrm{Cat}^n(\mathrm{Gp}) = \mathrm{Cat}(\mathrm{Cat}^{n-1}(\mathrm{Gp}))$

- Multinerve $\mathcal{N} : \mathrm{Cat}^n(\mathrm{Gp}) \to [\Delta^{n^{op}}, \mathrm{Gp}]$
  $\mathrm{Cat}^n(\mathrm{Gp}) \overset{Ner}{\to} [\Delta^{op}, \mathrm{Cat}^{n-1}(\mathrm{Gp})] \overset{Ner}{\to}$
  $[\Delta^{op}, [\Delta^{op}, \mathrm{Cat}^{n-2}(\mathrm{Gp})]] = [\Delta^{2^{op}}, \mathrm{Cat}^{n-2}(\mathrm{Gp})] \to \cdots$

- Equivalent definition of $\mathrm{Cat}^n(\mathrm{Gp}) \subset [\Delta^{n^{op}}, \mathrm{Gp}]$
  $n = 1$ $\mathcal{G} \in [\Delta^{op}, \mathrm{Gp}]$, Segal maps isos
  Given $\mathrm{Cat}^{n-1}(\mathrm{Gp}) \subset [\Delta^{n-1^{op}}, \mathrm{Gp}]$
  Define $\mathcal{G} \in \mathrm{Cat}^n(\mathrm{Gp}) \subset [\Delta^{n^{op}}, \mathrm{Gp}]$ if
  - $\mathcal{G}_k \in \mathrm{Cat}^{n-1}(\mathrm{Gp})$ for all $k \geq 0$
  - Segal maps isomorphisms

- Classifying space B
  $\mathrm{Cat}^n(\mathrm{Gp}) \overset{\mathcal{N}}{\to} [\Delta^{n^{op}}, \mathrm{Gp}] \overset{diag}{\to} [\Delta^{op}, \mathrm{Gp}] \overset{\overline{W}}{\to}$
  $[\Delta^{op}, \mathrm{Set}]_0 \overset{|\cdot|}{\to} \mathrm{Top}_*$

Cat^n-groups as homotopy models.

- Fact: $\mathcal{G} \in \mathrm{Cat}^n(\mathrm{Gp})$
  Then $B\mathcal{G}$ is connected $(n+1)$-type.

- Weak equivalence f in $\mathrm{Cat}^n(\mathrm{Gp})$ if Bf weak homotopy equivalence.

- Theorem

  [MacLane-Whitehead $n = 1$]
  [Loday; Bullejos-Cegarra-Duskin; Porter; $n > 1$]

$$\overline{B} : \frac{\mathrm{Cat}^n(\mathrm{Gp})}{\sim} \simeq \mathcal{H}o\begin{pmatrix}\text{connected}\\ n+1\text{-types}\end{pmatrix} : \overline{\mathcal{P}_n}$$

Tamsamani's model: strict case

- Strict n-categories: Inductive definition
  $1\text{-Cat} = \text{Cat}$
  $n\text{-Cat} = ((n-1)\text{-Cat})\text{-Cat}$

- Fact: $\mathcal{C}$ category with commuting coproducts
  $\mathcal{C}\text{-Cat} \simeq (\text{Cat}\,\mathcal{C})_{disc} =$
  $= \{\phi \in [\Delta^{op}, \mathcal{C}] \mid \phi_0 \text{ discrete, Segal maps isom.}\}$

- Multi-simplicial definition of n-Cat:
  $n = 1$ $\mathcal{G} \in [\Delta^{op}, \mathbf{Set}]$, Segal maps isomorphisms
  Given $(n-1)\text{-Cat} \subset [\Delta^{n-1^{op}}, \mathbf{Set}]$
  Define $\mathcal{G} \in n\text{-Cat} \subset [\Delta^{n^{op}}, \mathbf{Set}]$ if
  - $\mathcal{G}_k \in (n-1)\text{-Cat}$ for all $k \geq 0$
  - $\mathcal{G}_0$ constant
  - Segal maps isomorphisms.

- Note: $\mathcal{G}(0, -)$, $\mathcal{G}(1, \cdots^k, 1, 0, -)$ discrete
  $1 \leq k \leq n-2$.

- Functor: $\tau_1^{(n)} : n\text{-Cat} \to \text{Cat}$, inductively
  $\tau_0^{(1)} : \text{Cat} \to \mathbf{Set}$ iso class of objects
  $\tau_1^{(1)} = \text{id} \qquad \tau_1^{(n)} = \overline{\tau}_0^{(n-1)} \qquad \tau_0^{(n)} = \tau_0^{(1)} \tau_1^{(n)}$

- Strict n-groupoids:
  $1\text{-Gpd} = \text{Gpd}$
  $\mathcal{G} \in n\text{-Cat}$ is in n-Gpd if $\mathcal{G}_k \in (n-1)\text{-Gpd}$ for all $k \geq 0$ and $\tau_1^{(n)} \mathcal{G} \in \text{Gpd}$.

Tamsamani's model: general case

- Idea: Weaken associativity of composition and unit laws by requiring Segal maps to be "equivalences" rather than isomorphisms.

- Inductive definition: [Tamsamani; Toen]
  $n = 1 \quad \mathcal{W}_1 = \mathrm{Cat}$
  1-equivalence = equivalence of Cat
  $\tau_0^{(1)} : \mathrm{Cat} \to \mathbf{Set}$ iso classes of objects
  $\delta^{(1)} : \mathbf{Set} \to \mathcal{W}_1$ discrete category
  $\tau_1^{(1)} : \mathcal{W}_1 \to \mathrm{Cat}$ identity.

- Note: these satisfy
  i) $\delta^{(1)}$ fully faithful, finite product preserving
  $\tau_0^{(1)}\delta^{(1)} = \mathrm{id}$.
  ii) $\tau_0^{(1)}$ sends 1-equiv. to bijections.
  iii) $\tau_0^{(1)}$ preserves fibre products over discrete objects.
  iv) $\mathcal{C} \to \mathcal{D}$ morphism in $\mathcal{W}_1$, $\mathcal{D}$ discrete
  then $\mathcal{C} \cong \coprod_{x \in \mathcal{D}} \mathcal{C}_x$.

Tamsamani's model: general case, cont.

Inductive step:

- Given $\mathcal{W}_{n-1}$, $(n-1)$-equivalences
  $\tau_0^{(n-1)} : \mathcal{W}_{n-1} \to \mathbf{Set}$
  $\delta^{(n-1)} : \mathbf{Set} \to \mathcal{W}_{n-1}$ image "discrete"
  + axioms

- Define $\phi \in \mathcal{W}_n \subset [\Delta^{op}, \mathcal{W}_{n-1}]$ s.t.
  ϕ_0 discrete
  Segal maps $(n-1)$-equivalences.

- Note $\tau_1^{(n)} : \mathcal{W}_n \to$ Cat restriction of
  $\overline{\tau}_0^{(n-1)} : [\Delta^{op}, \mathcal{W}_{n-1}] \to [\Delta^{op}, \mathbf{Set}]$
  $\phi_1 \cong \coprod_{x,y \in \phi_0} \phi_{(x,y)}$

- Define $f : \phi \to \psi$ in $\mathcal{W}_n$ n-equivalence if
  - $\phi_{(x,y)} \to \psi_{(fx,fy)}$ $(n-1)$-equiv.
  - $\tau_1^{(n)}\phi \to \tau_1^{(n)}\psi$ equiv. of Cat
  - $\tau_0^{(n)} = \tau_0^{(1)}\tau_1^{(n)}$

Weak n-groupoids as homotopy models.

- Tamsamani's weak n-groupoids $\mathcal{T}_n \subset \mathcal{W}_n$

  $n = 1$ $\quad \mathcal{T}_1 = \mathsf{Gpd}$

  Given $\quad \mathcal{T}_{(n-1)}$

  Define $\quad \phi \in \mathcal{T}_n \subset \mathcal{W}_n$ if

  $\phi_{(x,y)} \in \mathcal{T}_{n-1}$ for all $x, y \in \phi_0$

  $\tau_1^{(n)} \phi \in \mathcal{T}_1$

- Theorem [Tamsamani]

  Equivalence of categories

$$\overline{B} : \frac{\mathcal{T}_n}{\sim^n} \simeq \mathcal{H}o\,(n\text{-types}) : \overline{\Pi}_n$$

$\mathbf{Cat}^n$-groups and $\mathcal{T}_{n+1}$: summary

$\mathrm{Cat}^n(\mathrm{Gp})$	$\mathcal{T}_{n+1}$
• $\mathcal{G} \in [\Delta^{n^{op}}, \mathrm{Gp}]$ $\mathcal{G}_k$ multinerve of $\mathrm{Cat}^{n-1}(\mathrm{Gp})$ Segal maps iso.	• $\phi \in [\Delta^{n+1^{op}}, \mathrm{Set}]$ $\phi_n \in \mathcal{T}_n$ ϕ_0 constant Segal maps n-equivalences $\tau_1^{(n+1)}\phi \in \mathrm{Gpd}$
• multisimplicial inductive definition based on Gp strict structure "cubical"	• multisimplicial inductive definition based on Set weak structure "globular"

- Main issues in the comparison:

  i) From strict cubical structure to weak globular one while preserving homotopy type.

  ii) From group-based structure to set-based structure.

Comparison problem: main ideas.

- Internal weak n-groupoids $\mathbb{D}_n \subset [\Delta^{n^{op}}, \mathrm{Gp}]$
  Weak globular structures internal to Gp.

- Functors $\quad \mathrm{Cat}^n(\mathrm{Gp}) \overset{disc}{\to} \mathbb{D}_n \overset{V}{\to} \mathcal{H}_{n+1} \subset \mathcal{T}_{n+1}$
  preserving the homotopy type.

- Special cat^n-groups $\mathrm{Cat}^n(\mathrm{Gp})_s \subset \mathrm{Cat}^n(\mathrm{Gp})$
  The "faces" which in an object of n-Cat (Gp) are discrete are now "strongly contractible".
  In particular n-Cat (Gp) $\subset \mathrm{Cat}^n(\mathrm{Gp})_s$

- Discretization functor $disc$
  $\mathrm{Cat}^n(\mathrm{Gp}) \overset{Sp}{\to} \mathrm{Cat}^n(\mathrm{Gp})_s \overset{\mathcal{D}_n}{\to} \mathbb{D}_n$
  Sp cofibrant replacement functor
  $\mathcal{D}_n$ "squeezes contractible faces to discrete ones".

- Semistrict $(n+1)$-groupoids $\mathcal{H}_{n+1}$
  $V : \mathbb{D}_n \to \mathcal{H}_{n+1}$
  induced by nerve Gp $\to [\Delta^{op}, \mathrm{Set}]$

$\mathbf{Cat}^n$-groups: further properties.

- $\mathrm{Cat}^n(\mathrm{Gp})$ as algebraic category

  $\mathcal{U}_n : \mathrm{Cat}^n(\mathrm{Gp}) \to \mathbf{Set} : \mathcal{F}_n \qquad \mathcal{F}_n \dashv \mathcal{U}_n$

  $\mathcal{U}_n$ monadic.

- Regular epis

  $\mathrm{Cat}^n(\mathrm{Gp})$ has enough regular epi projectives

  $\mathcal{F}_n\mathcal{U}_n\mathcal{G} \xrightarrow{\varepsilon_{\mathcal{G}}} \mathcal{G}$

- Strongly contractible (s.c.) cat^n-groups

  $n = 1$ $\quad d : \mathcal{G} \rightleftarrows \mathcal{G}^d : t \quad dt = \mathrm{id}$

  $\mathcal{G}^d$ discrete, $\quad d$ w.e.

  Given s.c. $\mathrm{Cat}^{n-1}(\mathrm{Gp})$

  Define $\mathcal{G} \in \mathrm{Cat}^n(\mathrm{Gp})$ s.c. if

  $d : \mathcal{G} \rightleftarrows \mathcal{G}^d : t \quad dt = \mathrm{id}$

  $\mathcal{G}^d$ discrete, $\quad d$ w.e.

  $\mathcal{G}_0^{(k)} \quad \mathcal{G}_1^{(k)}$ s.c. $\mathrm{Cat}^{n-1}(\mathrm{Gp})$

  for all directions $1 \leq k \leq n$.

- Theorem: $\quad \mathcal{F}_n X$ is strongly contractible.

A model structure on $\mathrm{Cat}^n(\mathrm{Gp})$.

$\mathrm{Cat}^n(\mathrm{Gp}) = \mathrm{Cat}(\mathrm{Cat}^{n-1}(\mathrm{Gp}))$

Apply T. Everaert, R.W. Kieboom,
T. Van der Linden - TAC 2005

- Weak equiv. = homology isomorphisms
- Every object is fibrant
- Functorial cofibrant replacement

$\mathcal{G} \in \mathrm{Cat}^n(\mathrm{Gp})$ pullback in $\mathrm{Cat}^{n-1}(\mathrm{Gp})$

$$\begin{array}{ccc} \mathcal{P}_1 & \longrightarrow & \mathcal{F}_{n-1}\mathcal{U}_{n-1}\mathcal{G}_0 \times \mathcal{F}_{n-1}\mathcal{U}_{n-1}\mathcal{G}_0 \\ \downarrow & & \downarrow{\scriptstyle \varepsilon \times \varepsilon} \\ \mathcal{G}_1 & \xrightarrow{(d_0, d_1)} & \mathcal{G}_0 \times \mathcal{G}_0. \end{array}$$

$c(\mathcal{G}) \in \mathrm{Cat}^n(\mathrm{Gp})$

$c(\mathcal{G})_1 = \mathcal{P}_1 \qquad c(\mathcal{G})_0 = \mathcal{F}_{n-1}\mathcal{U}_{n-1}\mathcal{G}_0$

$c(\mathcal{G})$ cofibrant

$c(\mathcal{G}) \to \mathcal{G}$ trivial fibration $\quad Bc(\mathcal{G}) \simeq B\mathcal{G}$

The discretization functor, n=2.

- Special cat^2-groups $\mathcal{G} \in \mathrm{Cat}^2(\mathrm{Gp})_s$

  $\mathcal{G}_0$ is s.c. cat^1-group, $d : \mathcal{G}_0 \rightleftarrows \mathcal{G}_0^d : t \quad dt = \mathrm{id}$

- Functor $c : \mathrm{Cat}^2(\mathrm{Gp}) \to \mathrm{Cat}^2(\mathrm{Gp})_s$

  $c(\mathcal{G}) \to \mathcal{G}$ weak equivalence.

- Internal weak 2-groupoids $\mathbb{D}_2 \subset [\Delta^{op}, \mathrm{Cat}^1(\mathrm{Gp})]$

  $\mathcal{G} \in \mathbb{D}_2$ if - $\mathcal{G}_0$ discrete

  - Segal maps w.e.

- Discrete multinerve $ds\mathcal{N}\mathcal{G}$

  $ds\mathcal{N} : \mathrm{Cat}^2(\mathrm{Gp})_s \to \mathbb{D}_2$

  $$\cdots \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \rightrightarrows \mathcal{G}_1 \underset{\sigma_0 t}{\overset{d\partial_0}{\underset{d\partial_1}{\rightleftarrows}}} \mathcal{G}_0^d$$

  $\mathcal{D}_2 = ds\mathcal{N}$ preserves homotopy type

- Define $disc = \mathcal{D}_2 \circ c$

Semistrictification, n=2.

- Semistrict 3-groupoids $\mathcal{H}_3 \subset \mathcal{T}_3$

  $\mathcal{G} \in \mathcal{H}_3 \subset [\,\Delta^{op}, \mathcal{T}_2\,]$ if $\mathcal{G}_0 = \{*\}$

  $$\mathcal{G}_n \cong \mathcal{G}_1 \times \overset{n}{\cdots} \times \mathcal{G}_1$$

- Functor $V : \mathbb{D}_2 \to \mathcal{H}_3$

  induced by nerve $\mathrm{Gp} \to [\Delta^{op}, \mathbf{Set}]$.

  V preserves homotopy type.

- Theorem [P.] Commutative diagram

$$\begin{array}{ccc} \mathrm{Cat}^2(\mathrm{Gp})/\!\sim & \xrightarrow{V \circ disc} & \mathcal{H}_3/\!\sim^3 \\ & {\scriptstyle B} \searrow \quad \swarrow {\scriptstyle B} & \\ & \mathcal{H}o\left(\begin{array}{c}\text{connected}\\ \text{3-types}\end{array}\right) & \end{array}$$

  Further, every object of $\mathcal{T}_3$ representing a connected 3-type is equivalent to an object of $\mathcal{H}_3$ through a zig-zag of 3-equivalences in $\mathcal{T}_3$.

The connection with Gray groupoids.

- Gray groupoids.

  Gray =(2-cat, $\otimes_{gray}$).

  Gray-enriched category with invertible cells.

  $(\text{Gray-Gpd})_0$ = Gray groupoids with 1 object.

- Theorem [P.] Commutative diagram

$$\begin{array}{ccc} \mathcal{H}_3/\!\sim^3 & \xrightarrow{\quad S \quad} & (\text{Gray-Gpd})_0/\!\sim \\ & {\scriptstyle B}\searrow \quad \swarrow{\scriptstyle B} & \\ & \mathcal{H}o\left(\begin{array}{c}\text{connected}\\ \text{3-types}\end{array}\right) & \end{array}$$

- idea of proof

  - Monoidal functor

    $(\mathcal{T}_2, \times) \xrightarrow{G} (Bigpd, \times) \xrightarrow{st} (2\text{-}gpd, \otimes_{gray})$

  - note $\mathcal{H}_3 \subset \text{Mon}(\mathcal{T}_2, \times)$

  - $\mathcal{G} \in \mathcal{H}_3 \Rightarrow st\,G\,\mathcal{G} \in (\text{Gray-Gpd})_0$

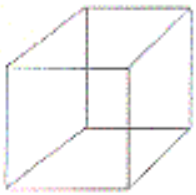
    Let $S(\mathcal{G}) = st\,G\,\mathcal{G}$.

Special cat^3-groups.

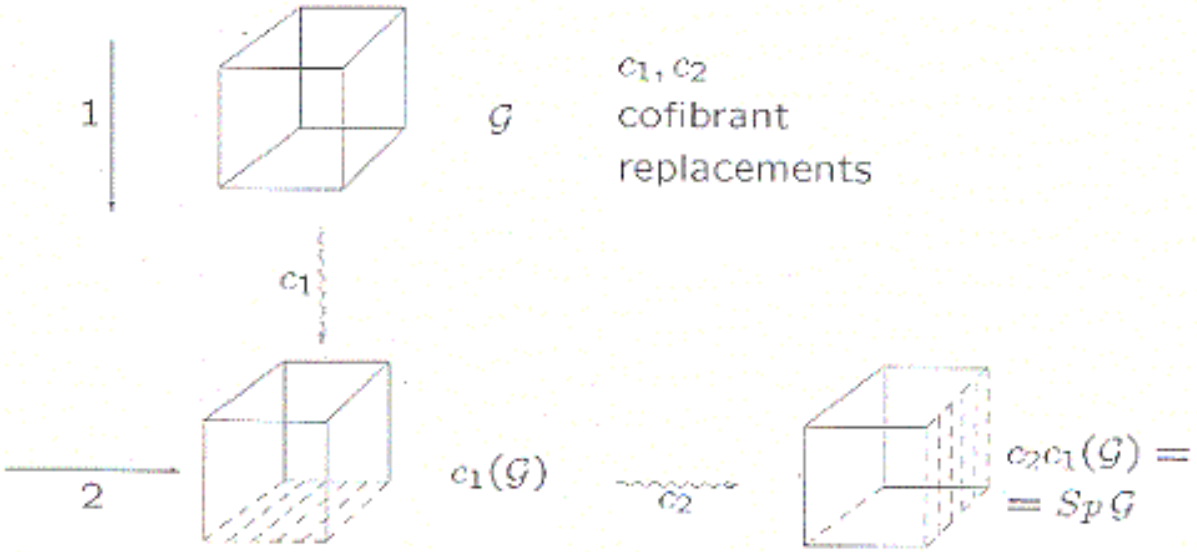
- Notation

  $\mathrm{Cat}^1(\mathrm{Gp})$ ——

  $\mathrm{Cat}^2(\mathrm{Gp})$ internal category in $\mathrm{Cat}\,(\mathrm{Gp})$ in 2 ways

  $\mathrm{Cat}^3(\mathrm{Gp})$  internal category in $\mathrm{Cat}^2(\mathrm{Gp})$ in 3 ways

- Functor



$(Sp\,\mathcal{G})_0$ s.c. $\mathrm{Cat}^2(\mathrm{Gp})$

$(Sp\,\mathcal{G})_1$ special $\mathrm{Cat}^2(\mathrm{Gp})$

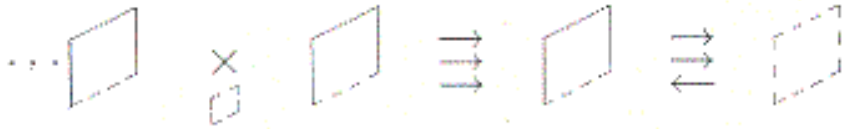
$\Big\}$ $Sp\,\mathcal{G}$ special $\mathrm{Cat}^3(\mathrm{Gp})$

$Sp\,\mathcal{G} \to c_1(\mathcal{G}) \to \mathcal{G}$ w.e.

Discretization functor, $n = 3$.

- Nerve of special cat^3-groups

  $Ner : \mathrm{Cat}^3(\mathrm{Gp})_s \to [\Delta^{op}, \mathrm{Cat}^2(\mathrm{Gp})_s]$



- Discrete multinerve

  $ds\mathcal{N} : \mathrm{Cat}^3(\mathrm{Gp}) \to [\Delta^{op}, \mathrm{Cat}^2(\mathrm{Gp})_s]$

- Internal weak 3-groupoids $\mathcal{G} \in \mathbb{D}^3 \subset [\Delta^{op}, \mathbb{D}^2]$

  $\mathcal{G}_0$ discrete, Segal maps w.e.

- Functor $\mathcal{D}_3 : \mathrm{Cat}^3(\mathrm{Gp})_s \to \mathbb{D}_3$

  $$\mathcal{D}_3 = \overline{\mathcal{D}}_2 \circ ds\mathcal{N}$$

  $\mathcal{G}$ $\quad \cdots \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \overset{\to}{\underset{\to}{\rightrightarrows}} \mathcal{G}_1 \overset{\to}{\underset{\leftarrow}{\rightleftarrows}} \mathcal{G}_0$

  $\mathcal{D}_3\mathcal{G}$ $\quad \cdots \mathcal{D}_2(\mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1) \overset{\to}{\underset{\to}{\rightrightarrows}} \mathcal{D}_2(\mathcal{G}_1) \overset{\to}{\underset{\leftarrow}{\rightleftarrows}} \mathcal{G}_0^d = \mathcal{D}_2(\mathcal{G}_0^d)$

  preserves homotopy type

- Discretization $disc : \mathrm{Cat}^3(\mathrm{Gp}) \to \mathbb{D}_3$

  $$disc = \mathcal{D}_3 \circ Sp$$

Semistrictification, general n.

- Semistrict Tamsamani's $n+1$-groupoids $\mathcal{H}_{n+1}$
  $\phi \in \mathcal{H}_{n+1} \subset \mathcal{T}_{n+1} \subset [\Delta^{op}, \mathcal{T}_n]$ if
  $\phi_0 = \{*\}, \quad \phi_n \cong \phi_1 \times \overset{n}{\cdots} \times \phi_1$

- Functor $V : \mathbb{D}_n \to \mathcal{H}_{n+1}$
  induced by nerve $\mathrm{Gp} \to [\Delta^{op}, \mathbf{Set}]$
  V preserves homotopy type.

- Theorem [P.] Commutative diagram

$$\begin{array}{ccc} \mathrm{Cat}^n(\mathrm{Gp})/\!\sim & \xrightarrow{V \circ disc} & \mathcal{H}_{n+1}/\!\sim^{n+1} \\ & {\scriptstyle B} \searrow \quad \swarrow {\scriptstyle B} & \\ & \mathcal{H}o\left(\begin{array}{c}\text{connected} \\ n+1\text{-types}\end{array}\right) & \end{array}$$

  Further, every object of $\mathcal{T}_{n+1}$ representing a connected $(n+1)$-type is equivalent to an object of $\mathcal{H}_{n+1}$ through a zig-zag of $(n+1)$-equivalences in $\mathcal{T}_{n+1}$.

Discretization functor, general n.

- Special $\mathrm{Cat}^n(\mathrm{Gp})$ inductive definition.
  $\mathcal{G} \in \mathrm{Cat}^n(\mathrm{Gp})_s$ if $\mathcal{G}_0^{(k)}$ is s.c. and $\mathcal{G}_1^{(k)}$ is special for some direction k, $1 \leq k \leq n$.

- Functor $Sp : \mathrm{Cat}^n(\mathrm{Gp}) \to \mathrm{Cat}^n(\mathrm{Gp})_s$
  $Sp = c_{n-1} \circ c_{n-2} \circ \cdots \circ c_2 \circ c_1$
  $Sp\,\mathcal{G} \to \mathcal{G}$ w.e.

- Nerve of $\mathrm{Cat}^n(\mathrm{Gp})_s$
  $Ner : \mathrm{Cat}^n(\mathrm{Gp})_s \to [\Delta^{op}, \mathrm{Cat}^{n-1}(\mathrm{Gp})_s]$

- Discrete multinerve
  $ds\mathcal{N} : \mathrm{Cat}^n(\mathrm{Gp})_s \to [\Delta^{op}, \mathrm{Cat}^{n-1}(\mathrm{Gp})_s]$

- Internal weak n-groupoids $\mathcal{G} \in \mathbb{D}_n \subset [\Delta^{op}, \mathbb{D}_{n-1}]$
  $\mathcal{G}_0$ discrete, Segal maps w.e.

- Functor $\mathcal{D}_n : \mathrm{Cat}^n(\mathrm{Gp})_s \to \mathbb{D}_n$
  $\mathcal{D}_n = \overline{\mathcal{D}}_{n-1} \circ ds\mathcal{N}$
  preserves homotopy type
  $\mathcal{D}_n(\mathcal{G}) = \mathcal{G}$ if $\mathcal{G} \in n\text{-}\mathrm{Cat}\,(\mathrm{Gp})$

- Discretization $disc : \mathrm{Cat}^n(\mathrm{Gp}) \to \mathbb{D}_n$
  $disc = \mathcal{D}_n \circ Sp$