Utility Valuation of Credit Derivatives

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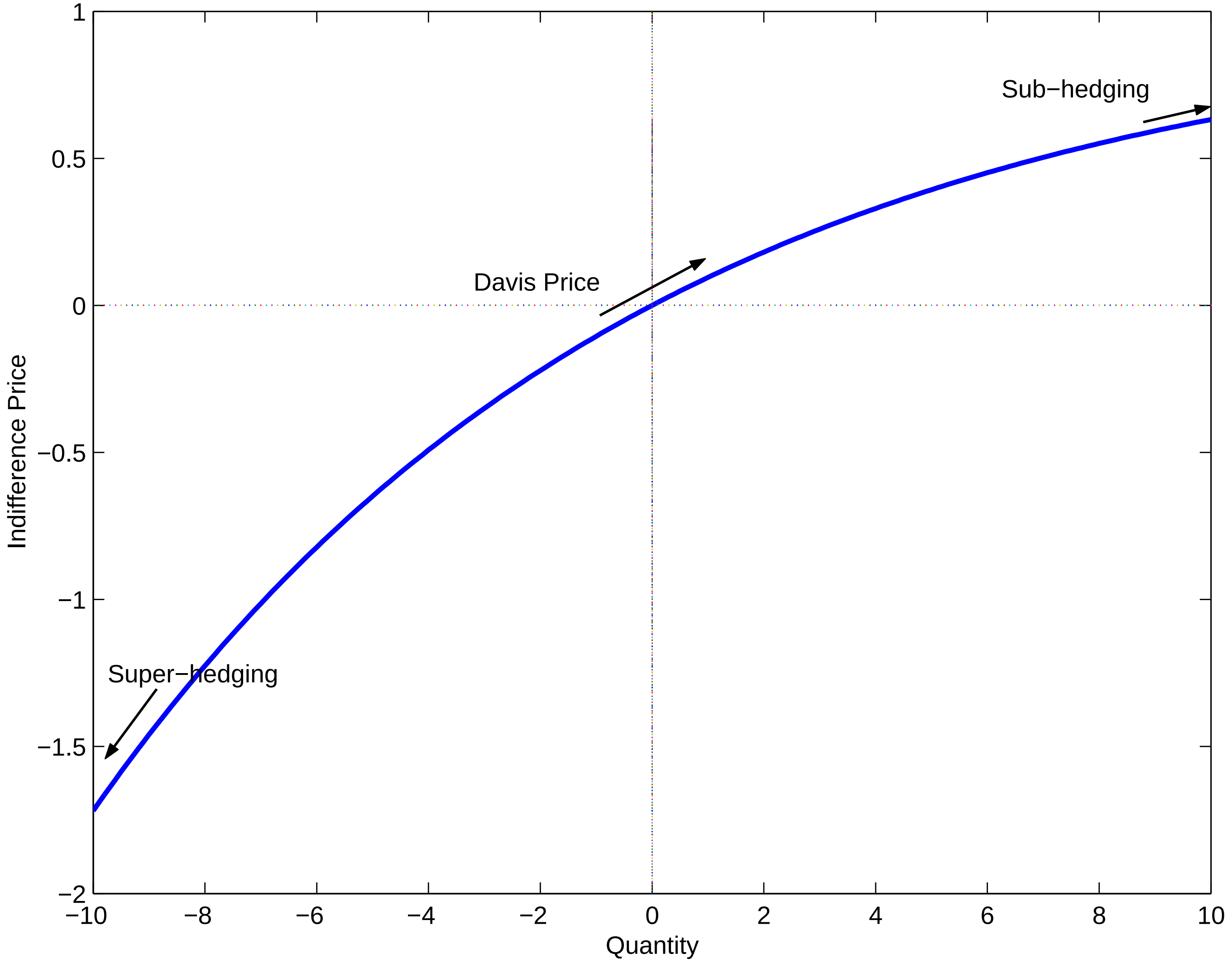
Credit Derivatives

- The market in credit-linked derivative products has grown from \$631.5 billion global volume in the first half of 2001 to above \$12 trillion through the first half of 2005.
- Credit derivatives are increasingly complex, but the quantitative technology for their valuation (and hedging) lags behind.
- Major Problem: high-dimensionality of the basket derivatives, which are typically written on hundreds of underlying names. Computational tractability severely limits model choice.
- A major challenge: to capture and explain high premiums observed for unlikely events.
- Our approach: try to explain such phenomena as a consequence of risk aversion, quantified through the mechanism of utility-indifference valuation.

Utility Indifference Derivative Pricing

- Dynamic generalization of certainty equivalent : $U(p) = \mathbb{E}\{U(X)\}$
- Reasonable preference-based valuation methodology in illiquid/OTC markets.
- *E.g.* options on non-traded assets, weather derivatives; (PUP book on indifference pricing , 2007).
- Computationally tractable (and wealth-independent) under exponential utility : $U(x) = -e^{-\gamma x}$, $\gamma > 0$.
- Nonlinear pricing rule.
- Credit & Indifference Pricing : see also Collin-Dufresne *et al.*, Bielecki-Jeanblanc-Rutkowski, Becherer & Schweizer, Shouda.



Sub−hedging
Davis Price
Super−hedging
Indifference Price
Quantity
1
0.5
0
−0.5
−1
−1.5
−2
−10
−8
−6
−4
−2
0
2
4
6
8
10

Indifference Pricing: Single Name Case

- Stock price S and intensity λ:

$$\begin{aligned} dS &= \mu S\,dt + \sigma S\,dW^{(1)} \\ \lambda_t &= \lambda(Y_t) \\ dY &= b(Y)\,dt + a(Y)\left(\rho\,dW^{(1)} + \sqrt{1-\rho^2}\,dW^{(2)}\right). \end{aligned}$$

- Default time τ is first jump of a time-changed (standard) Poisson process:

$$N\left(\int_0^t \lambda_s\,ds\right),$$

where N and λ are independent.

- Draw $\xi \sim \mathrm{EXP}(1)$, then

$$\tau = \inf\left\{t : \int_0^t \lambda_s\,ds = \xi\right\}.$$

- Given a utility function $U(x) = -e^{-\gamma x}$, at what value is the buyer indifferent in terms of maximum expected utility between holding and not holding the derivative?
- i) Solve plain Merton (optimal investment) problem (with default risk); ii) Solve Merton problem with the credit derivative.
- Wealth process X:

$$\begin{aligned} dX &= \pi \frac{dS}{S} + r(X - \pi)\, dt, \quad \{t < \tau\} \\ &= (rX + \pi(\mu - r))\, dt + \sigma \pi\, dW^{(1)}. \end{aligned}$$

- Switch to discounted variable $X_t \mapsto e^{-rt} X_t$ and $\mu \mapsto \mu - r$. Value function for Merton problem:

$$M(x) = \sup_{\pi} \mathbb{E} \left\{ -e^{-\gamma X_T} \mathbf{1}_{\{\tau > T\}} + (-e^{-\gamma X_\tau}) \mathbf{1}_{\{\tau \leq T\}} \right\}.$$

- Reduce to

$$M(t,x,y) = -e^{-\gamma x} u(t,y)^{1/(1-\rho^2)},$$

where

$$u_t + \widetilde{\mathcal{L}}_y u - (1-\rho^2)\left(\frac{\mu^2}{2\sigma^2} + \lambda(y)\right)u + (1-\rho^2)\lambda(y)u^{-\theta} = 0,$$

with $u(T,y) = 1$ and

$$\theta = \frac{\rho^2}{1-\rho^2}.$$

- Reaction-diffusion equation.

Add claim $\mathbf{1}_{\{\tau>T\}}$

- Define $c = e^{-rT}$. Value function

$$H(x) = \sup_{\pi} \mathbb{E}\left\{-e^{-\gamma(X_T+c)}\mathbf{1}_{\{\tau>T\}} + (-e^{-\gamma X_\tau})\mathbf{1}_{\{\tau\le T\}}\right\}.$$

- Reduce $H(t,x,y) = -e^{-\gamma(x+c)}w(t,y)^{1/(1-\rho^2)}$, to

$$w_t + \widetilde{\mathcal{L}}_y w - (1-\rho^2)\left(\frac{\mu^2}{2\sigma^2} + \lambda(y)\right)w + (1-\rho^2)e^{\gamma c}\lambda(y)w^{-\theta} = 0,$$

with $w(T,y) = 1$. A similar reaction-diffusion equation.

- Indifference price: $M(x) = H(x-p)$ given by

$$p = e^{-rT} - \frac{1}{\gamma(1-\rho^2)}\log(w/u).$$

- Constant Intensity Case: when λ is constant, defaultable bond price is

$$p_0(T) = e^{-rT} - \frac{1}{\gamma} \log \left(\frac{e^{-\alpha T} + \frac{\lambda}{\alpha} e^{\gamma c} \left(1 - e^{-\alpha T}\right)}{e^{-\alpha T} + \frac{\lambda}{\alpha} \left(1 - e^{-\alpha T}\right)} \right),$$

with $\alpha = \frac{\mu^2}{2\sigma^2} + \lambda$.

- Plot of yield spread $Y_0(T) = -\frac{1}{T} \log(p_0(T)/e^{-rT})$.

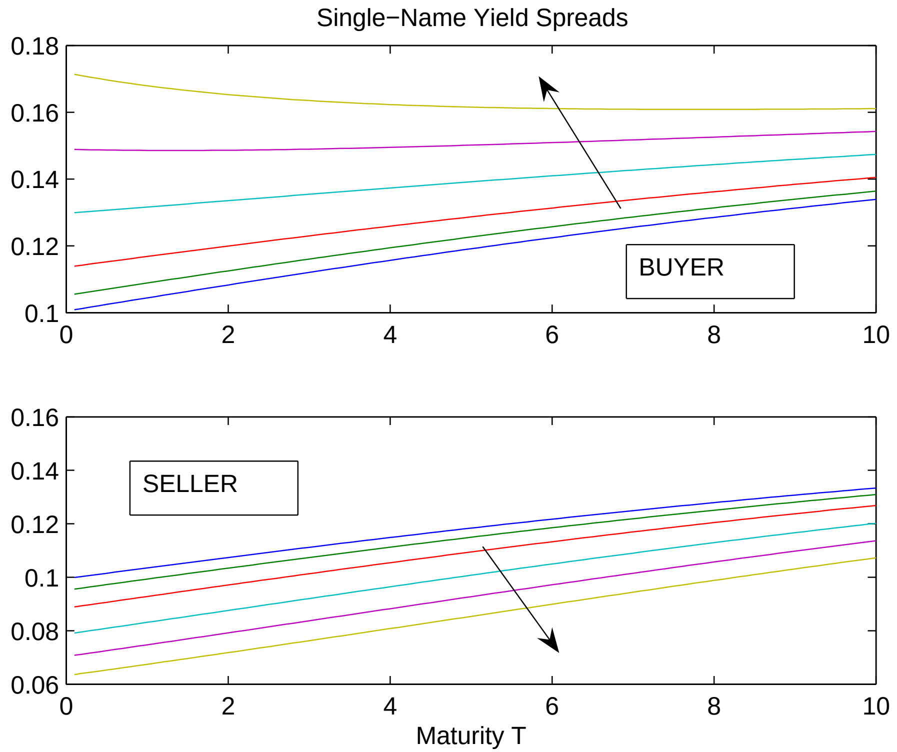


Figure 1: *Single name buyer's and seller's indifference yield spreads. The parameters are* $\lambda = 0.1$, *along with* $\mu = 0.09, r = 0.03$ *and* $\sigma = 0.15$. *The curves correspond to different risk aversion parameters* γ *and the arrows show the direction of increasing* γ *over the values* $(0.01, 0.1, 0.25, 0.5, 0.75, 1)$.

Multi-Name Case: Constant Intensities

- N firms. Stock prices processes

$$\frac{dS_t^{(i)}}{S_t^{(i)}} = (r + \mu_i)\, dt + \sigma_i\, dW_t^{(i)},$$

with $\mathbb{E}\{dW_t^{(i)}\, dW_t^{(j)}\} = \rho_{ij}\, dt$, $i \neq j$.

- Firm i defaults at random time $\tau_i \sim \mathrm{EXP}(\lambda_i)$. Default times are mutually *independent*, and independent of the Brownian motions.

- Discounted wealth process

$$dX_t = \begin{cases} \sum_i \pi_t^{(i)} \mathbf{1}_{\{\tau_i > t\}} \mu_i\, dt + \sum_i \pi_t^{(i)} \mathbf{1}_{\{\tau_i > t\}} \sigma_i\, dW_t^{(i)}, & t < \bar{\tau} \wedge T, \\ 0 & \bar{\tau} \wedge T \leq t \leq T \end{cases},$$

where $\bar{\tau} = \max_i \{\tau_i\}$.

- Merton value function (when all N firms alive)

$$M^{(N)}(t,x) = \sup_{\{\pi^{(i)}\}} I\!\!E\left\{-e^{-\gamma X_T} \mid X_t = x\right\}$$

solves

$$M_t^{(N)} - \frac{1}{2}(\mu^T A^{-1}\mu)\frac{(M_x^{(N)})^2}{M_{xx}^{(N)}} + \sum_{i=1}^{N} \lambda_i \left(M_i^{(N-1)} - M^{(N)}\right) = 0,$$

where $A = \sigma\sigma^T$, and $M_i^{(N-1)}$ is the Merton value function when firm i has dropped out.

- Combinatorial problem: when k firms have defaulted, have to solve the Merton problems for each of $\binom{N}{k}$ combinations of possible firms left.

Symmetric Model

- When there are $n \leq N$ stocks, labelled by the index set

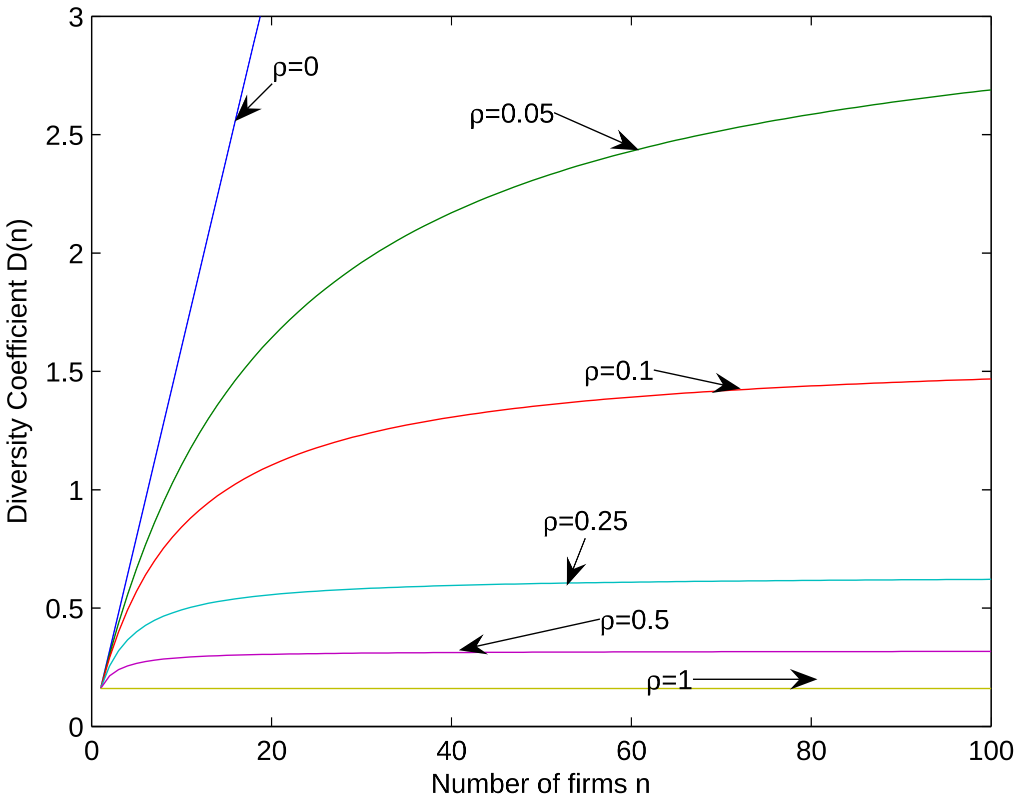
$$I_n = \{i_1, i_2, \cdots, i_n\},$$

$\mu(I_n)$ denotes expected returns; $\sigma(I_n)$ the volatility matrix. Let $A(I_n) = \sigma(I_n)\sigma(I_n)^T$.

- Our assumption is that $D(n) := \mu(I_n)^T A(I_n)^{-1}\mu(I_n)$ is a function only of $n = |I_n|$. The diversity function $D(n)$ is increasing and concave in n.

- *E.g.* $\mu_i \equiv \mu$, $\sigma_i = \sigma$ and correlation structure $\mathbb{E}\{dW^{(i)}dW^{(j)}\} = \rho\, dt, \qquad i \neq j$

$$\Rightarrow D(n) = \frac{\mu^2 n}{\sigma^2(1 + (n-1)\rho)}.$$


ρ=0
ρ=0.05
ρ=0.1
ρ=0.25
ρ=0.5
ρ=1
Diversity Coefficient D(n)
Number of firms n
0
0.5
1
1.5
2
2.5
3
0
20
40
60
80
100

Merton Problem

- Let $M^{(n)}(t,x)$ be the value function when there are $n \in \{0,1,\cdots,N\}$ firms alive. Writing $M^{(n)}(t,x) = -e^{-\gamma x} v_n(t)$,

$$\begin{aligned} v_n' - \alpha_n v_n + n\lambda v_{n-1} &= 0 \\ \alpha_n &:= \frac{1}{2} D(n) + n\lambda. \end{aligned}$$

- It follows that

$$v_n(t) = c_0^{(n)} + \sum_{j=1}^{n} c_j^{(n)} e^{-\alpha_j (T-t)},$$

$$
\begin{aligned}
c_0^{(n)} &= \frac{n\lambda}{\alpha_n}\, c_0^{(n-1)}, \\
c_j^{(n)} &= \frac{n\lambda}{(\alpha_n - \alpha_j)} c_j^{(n-1)}, \qquad j = 1, \cdots, n-1 \\
c_n^{(n)} &= 1 - \left(\frac{n\lambda}{\alpha_n} c_0^{(n-1)} + \sum_{j=1}^{n-1} \frac{n\lambda}{(\alpha_n - \alpha_j)} c_j^{(n-1)} \right),
\end{aligned}
$$

with initial data

$$
c_0^{(1)} = \frac{\lambda}{\alpha_1}.
$$

Special Case:$\rho = 0 \to$ binomial coeffs.

$$
c_j^{(n)} = \binom{n}{j} p^{n-j} (1-p)^j, \quad p := \frac{\lambda}{\lambda + \mu^2/(2\sigma^2)}.
$$

Next: look at the tranche holder's stochastic control problem.

CDO Mechanics

Let N denote the number of firms underlying the CDO and Q the total notional. Attachment points:

Tranche	K_L	K_U
Equity	0%	3%
Mezzanine 1	3%	7%
Mezzanine 2	7%	10%
Senior	10%	15%
Super-Senior	15%	30%

- The tranche holder (protection seller) receives a tranche premium R on his remaining notional, which decreases as the losses start to eat at his tranche.
- The protection buyer receives payments on the losses.

CDO Tranches Spreads

- We want to find the such that he is indifferent between holding the tranche or not.
- Assume fractional recovery $q < 1$, and a coupon payment Re^{rt} paid continuously. Then define

$$F(\ell) = (K_U - \ell)^+ - (K_L - \ell)^+,$$

the tranche holder's percentage notional, and

$$\begin{aligned}
\ell_n &= (1-q)\frac{(N-n)}{N} \\
f_n &= F(\ell_n) - F(\ell_{n-1}) \\
dX_t^{(n)} &= \left(\sum_i \pi_t^{(i)} \mathbf{1}_{\{\tau_i > t\}} \mu_i + RQF(\ell_n)\right) dt + \sum_i \pi_t^{(i)} \mathbf{1}_{\{\tau_i > t\}} \sigma_i \, dW_t^{(i)},
\end{aligned}$$

Tranche Holder's Problem

- Let $H^{(n)}(t,x)$ denote the tranche holder's value function when n firms are left. Writing $H^{(n)}(t,x) = -e^{-\gamma x} w_n(t)$, we have

$$w_n' - \beta_n w_n + n\lambda e^{\gamma f_n} w_{n-1} = 0,$$

with $w_n(T) = 1$ and where

$$\beta_n = \frac{1}{2} D(n) + n\lambda + \gamma R Q F(\ell_n).$$

- Can similarly construct solution as a series of exponentials with coefficients given through recurrence relations.
- The indifference tranche spread value is found by solving for R

$$w_N(0) = v_N(0).$$

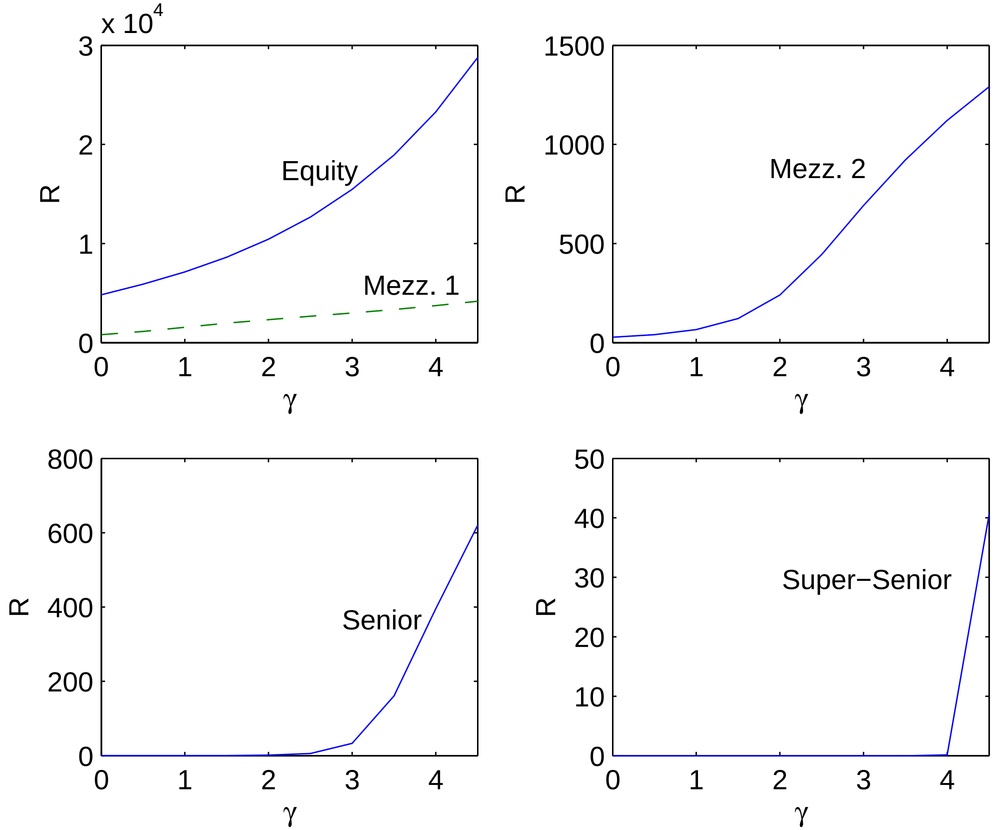


Figure 2: *$N = 25$ (left), $N = 100$ (right); $\lambda = 0.015$, $\mu = 0.07$, $\sigma = 0.15$ and $\rho = 0.3$. The recovery is $q = 40\%$, the interest rate $r = 3\%$ and $T = 5$ years. The notional is normalized to 1 unit per firm, so $Q = N$.*

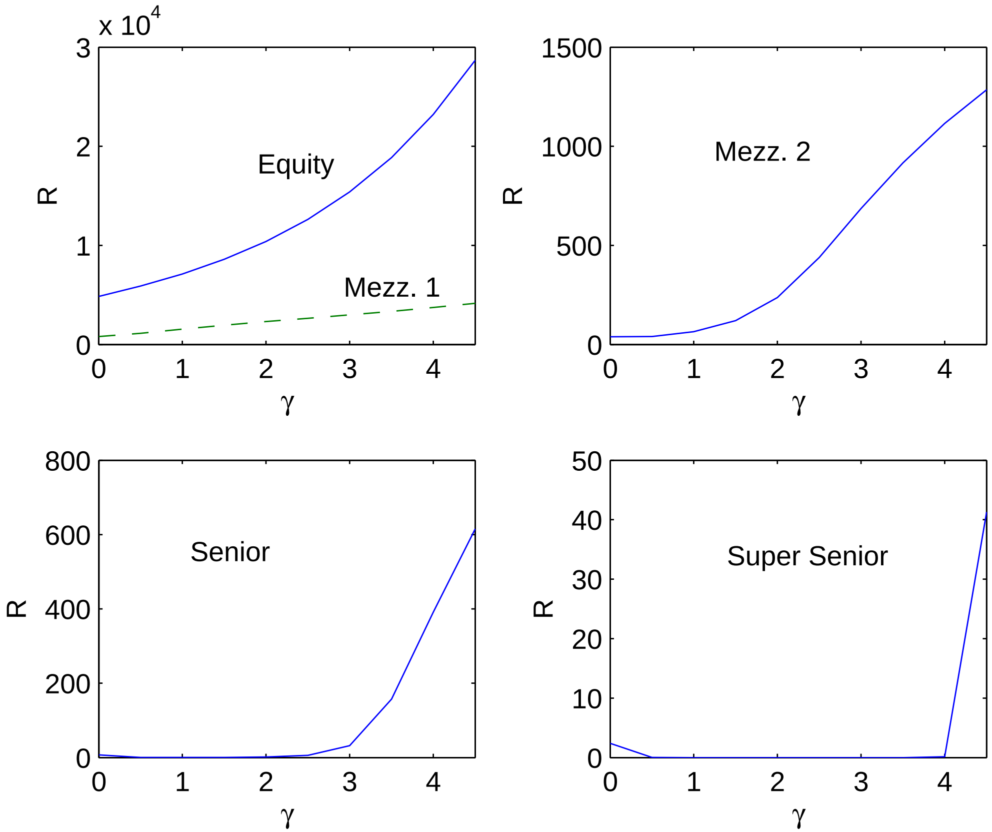


Figure 3: $N = 100$ *firms.*

Stochastic Intensities

- Of course, want to incorporate utility valuation around correlated defaults: Intensities: $\lambda_t^{(i)} = \lambda(Y_t)$

$$\begin{aligned} \frac{dS_t^{(i)}}{S_t^{(i)}} &= (r+\mu)\,dt + \sigma\,dW_t^{(i)} \\ dY_t &= b(Y_t)\,dt + a(Y_t)\,dZ_t \end{aligned}$$

$$\langle dW^{(i)}, dW_t^{(j)} \rangle = \rho\,dt \qquad \langle dW^{(i)}, dZ_t \rangle = m\,dt.$$

  Then in the control problems, have to solve a system of reaction-diffusion PDEs.

- Preliminary computations (both symmetric and heterogeneous cases) suggest utility valuation greatly enhances the real correlation.

- This is the model we are developing, both purely symmetric, and with a small number of heterogeneous groups .

Concluding Remarks

- Non-trivial yield spreads and potential implied correlation smiles even from constant intensities.
- Nonlinearity of indifference pricing rule acts as a correlator of default times via the effect of risk-aversion on portfolios.
- Computational/combinatorial problem remains, but under constant intensities deal with ODEs.
- Symmetric case tractable. Interesting to construct a system in which this is the "homogenized" approximation.
- Related problem: optimal static-dynamic hedging of CDO tranche risk (with CDSs and stocks).