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Pricing CDO Tranches of Bespoke Portfolios

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Summary

- **Problem**: find robust practical methodology for modelling, pricing & managing risk of bespoke (synthetic) CDOs – transparent, easy to understand and relate to market practices
  - Characterize and model the effect concentration risk in CDO portfolios
  - Flexible calibration – fit to prices and makes use of market, historical and portfolio information effectively
  - General – extend to other products (e.g. CDO^2)
  - Communication tool

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Summary

Methodology

- *Systematic Weighted MC* – implied risk-factor distribution
  - Non-parametric approach
  - General, flexible, & easy to understand
  - Consistently prices bespoke portfolios, multiple indices, CDO^2
  - Pricing of particular portfolios can depend on “prior” multi-factor model
  - Only explore the “static” version – potentially extensible to dynamic setting

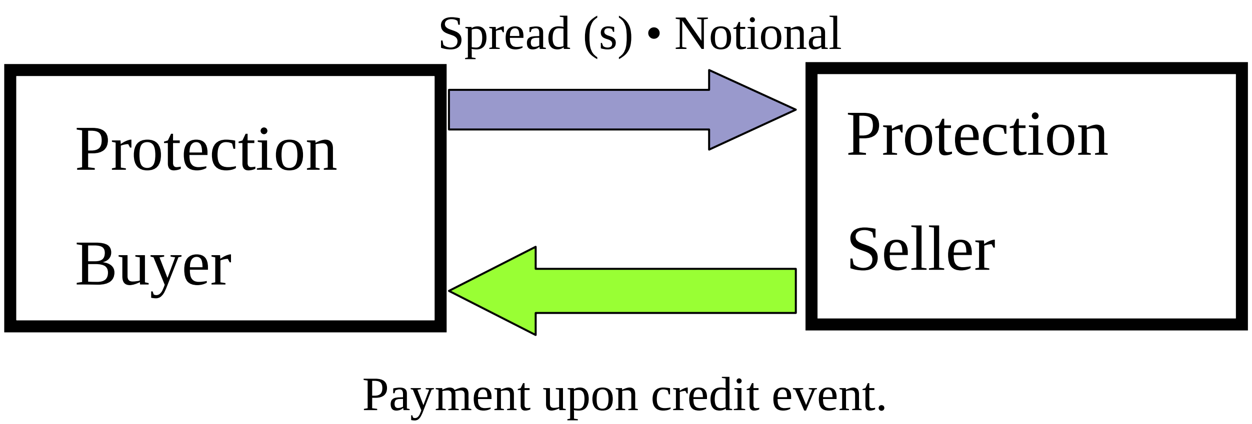
- *Additional “communication tool”: concentration adjusted mappings*
  - Modified base correlations – simple extension to standard *EL (expected loss)* mappings
  - Easy to implement and understand… but same disadvantages of base correlations and mappings… ad-hoc nature, lack of no-arbitrage, and sometimes counterintuitive results

Outline

- Background: credit derivatives, pricing synthetic CDOs and implied correlations
  - Pricing bespoke portfolios and *EL* mappings
  - Concentration risk in credit portfolios
- Methodology
  - Concentration adjusted mapping (base correlations)
  - Systematic Weighted Monte Carlo
- Examples
- Concluding remarks and ongoing work

Credit Derivatives: Credit Default Swaps

- Basically, an insurance contract on the creditworthiness of a given entity.



- D_i : Discount factor for time t_i.
- τ : Time of credit event (default).
- LGD: Payment upon default.
- $q(t) = Q[\tau \geq t]$: (risk-neutral) survival probability.

Pricing Credit Default Swaps

- The value of the payments received by the protection seller is:

$$PV_{Sell} = s \cdot \sum_{i=1}^{N} q(t_i) \cdot D_i \cdot (t_i - t_{i-1})$$

- The value of the payments received by the protection buyer is:

$$PV_{Buy} = LGD \cdot \sum_{i=1}^{N} D_i \cdot Q[\tau = t_i]$$

- The initial market price is zero:

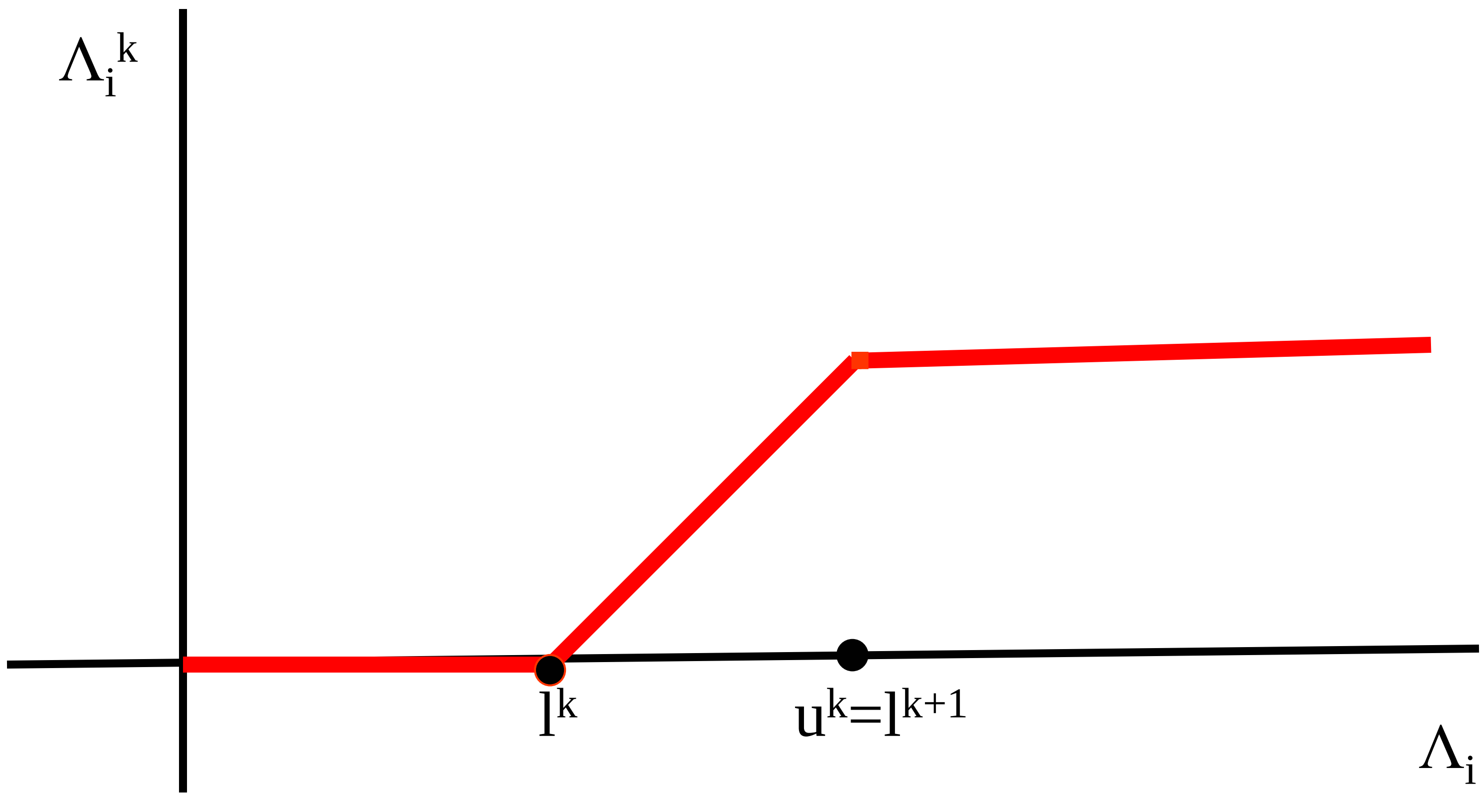
$$s = \frac{LGD \cdot \sum_{i=1}^{N} D_i \cdot Q[\tau = t_i]}{\sum_{i=1}^{N} q(t_i) \cdot D_i \cdot (t_i - t_{i-1})}$$

(Synthetic) CDOs

- Protection is bought/sold on a portfolio of CDS.
- The CDO consist of K tranches. Portfolio losses are absorbed by the tranches according to size and seniority.
- The size of tranche k is $S^k = u^k - l^k$, where:
  - l^k is the tranche's attachment point
  - u^k is the tranche's detachment point
- Λ_{i} : cumulative losses on the portfolio (as a percentage of total portfolio notional) by time t_i.

CDO Tranche Losses

- The losses absorbed by tranche *k* during the time interval (0,t_i) are: $$\Lambda_i^k = \min(S^k, \max(\Lambda_i - l^k, 0))$$



Pricing CDOs

- The value of the payments received by the protection seller (of tranche *k*) is:

$$PV_{Sell}^{k} = E_Q\left(\sum_{i=1}^{N} s_k (t_i - t_{i-1})(S^k - \Lambda_i^k) D_i\right)$$

- The value of the payments received by the protection buyer (of tranche j) is:

$$PV_{Buy}^{k} = E_Q\left(\sum_{i=1}^{N} (\Lambda_i^k - \Lambda_{i-1}^k) D_i\,]\right)$$

- As with CDS, the initial market price is zero:

$$s_k = \frac{\sum_{i=1}^{N} E_Q[(\Lambda_i^k - \Lambda_{i-1}^k)] D_i}{\sum_{i=1}^{N} (t_i - t_{i-1})(S^k - E_Q[\Lambda_i^k]) D_i}$$

Single-Factor Gaussian Copula

- Z systematic factor (Gaussian)
  - Conditional on Z, default times are independent

$$Y_j = \sqrt{\rho_j}\, Z + \sqrt{1-\rho_j}\, \varepsilon_j$$

$$p_j^Z(t) = \Pr\left(\tau_j \le t \,\middle|\, Z\right) \qquad q_j^Z(t) = \Pr\left(\tau_j > t \,\middle|\, Z\right)$$

- Marginal default time distribution functions

$$F_j(\tau_j)$$

- Default times – by mapping to Gaussian

$$\tau_j = F_j^{-1}\left(\Phi\left(Y_j\right)\right)$$

- Simple, explicit formulae for conditional default probabilities

$$p_j^Z(t) = \Phi\left(\frac{\Phi^{-1}\left(F_j(t)\right) - \sqrt{\rho_j}\, Z}{\sqrt{1-\rho_j}}\right)$$

Pricing CDO Tranches

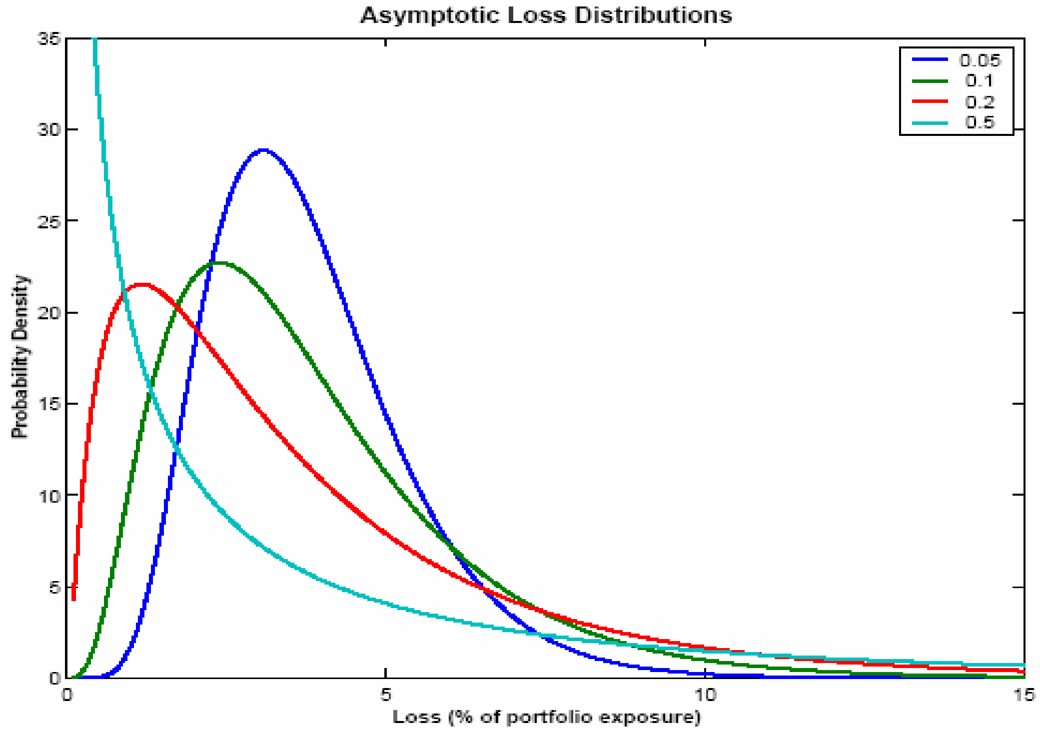
- Cumulative Portfolio Loss:

$$\Lambda_i = \sum_j LGD_j \cdot 1_{\{Y_j \le \Phi^{-1}(p_j(t_i))\}}$$

- Expected Tranche Loss:

$$E[\Lambda_i^k] = \int \underbrace{E[\Lambda_i^k \mid Z = z]}_{\boxed{\text{Many methods exist to calculate this.}}} \varphi(z)\, dz$$

Importance of Correlation: Loss Distributions



Equity tranche is long correlation.

Background – Pricing Synthetic CDOs

- Standard model for pricing synthetic CDOs: single-factor Gaussian copula (Li 2001)
  - Codependence through a one-factor Gaussian copula of *times to default*
  - Single parameter to estimate (correlation for all obligors in portfolio)
- Basic model does not simultaneously match market prices of all traded tranches
  - “Correlation skew” – set of correlations that match the prices of all tranches
- Base correlations – alternative to tranche correlations
  - Implied correlations of equity tranches with different attachment points (mezzanine/senior tranches as difference between two equity tranches)
- Interpolation (or extrapolation) model
  - Calibrated model to observed tranche prices of reference market portfolio (e.g iTraxx or CDX) is useful to price new “bespoke” CDO tranches

Credit Indices and Standard Tranches

- Track CDS spreads and CDO tranche spreads
- Major indices include:
  - Dow Jones CDX NA IG (tracks 125 North American Investment Grade names, 5 and 10 year indices)
  - iTraxx Europe, Asia, Asia ex Japan indices (125 investment grade companies, again 5 and 10 year indices are published).
- Prices are available for a set of standard tranches on each of the main CDS indices. (e.g. CDX.NA.IG 5Y, 5/31/2006)

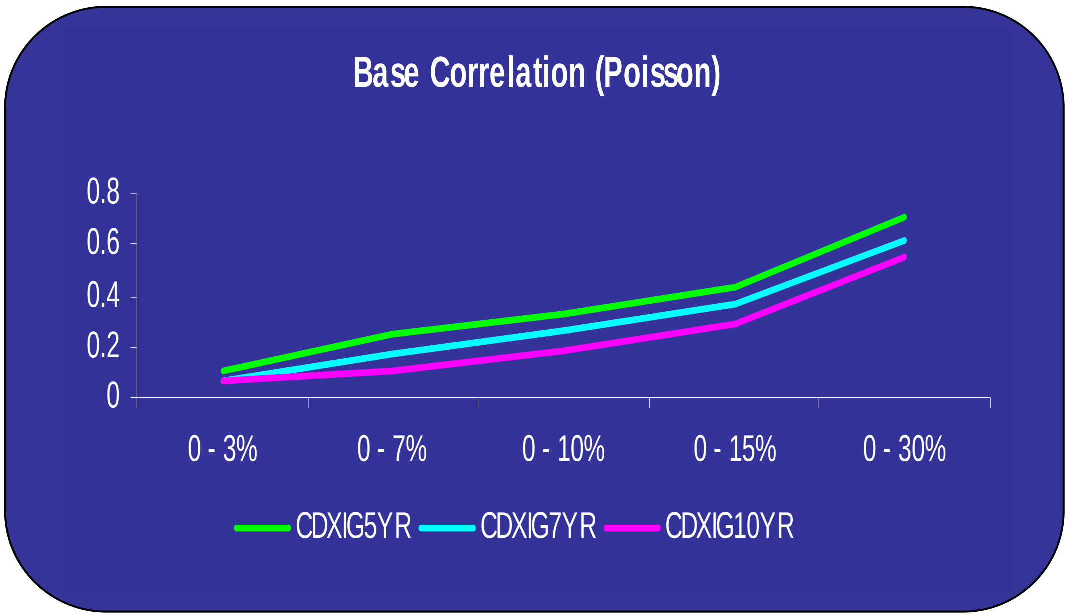
Tranche	Spread (bps)
0-3%	34.81% + 500*
3-7%	99
7-10%	21
10-15%	10.5
15-30%	5.5

* Equity tranches pay an upfront percent of the notional value plus a constant spread of 500 basis points.

Prices are usually quoted simply as upfront percents.

Example – CDX IG Index, May 31st 2006

Market Quotes - CDX							
IndexName	Index Price	Tranch spreads					Up Front
	Spread (%)	0 - 3%	3 - 7%	7 - 10%	10 - 15%	15 - 30%	0 - 3%
CDXIG5YR	42.04	500	99	21	10.5	5.5	31.81
CDXIG7YR	52.00	500	246	46.5	21	7.5	48.69
CDXIG10YR	65.00	500	595	118.5	55.5	16	55.63



Alternative Models – Examples

- Alternative Copula methods:
  - NIG, *t*, double-*t*,, Clayton, Marshall-Olkin copulas (Burtschell et al. 2005a)
  - Stochastic correlations (e.g. Gaussian mixtures, Li and Liang 2005)
  - Local and marginal compound correlations (Burtschell et al. 2005b)
  - Alternative default intensity processes: e.g. Intensity Gamma model (Joshi & Stacy 2006)
- *Implied* loss distribution or hazard rates approaches (non-parametric)
  - Hull-White 2006, Walker 2007, Brigo et al. 2006
- Dynamic models (generally through Monte Carlo methods)
  - Reduced form – dependent default intensities (Duffie & Garleanu 2001)
  - Structural (Merton type) – multi-step default boundary (Hull et al. 2005)
  - Dynamic loss distribution processes (Schönbucher 2005, SPA 2005, Albanese and Vidler 2007, Hull and White 2007)

Bespoke CDO Tranches

- CDO tranche is "bespoke" if it is not among the set of tranches routinely quoted by dealers
  1. *"Bespoke portfolios"* (different that e.g. CDX.NA.IG, iTraxx Europe, etc.)
  2. Attachment and detachment point
  3. Maturity (e.g. 3, 5, 7 and 10 years quoted)

- Bespoke maturity and attach/detach commonly treated through standard interpolation/extrapolation
- Bespoke portfolios… *"it seems reasonable to use the same [calibrated model] when pricing products of a different basket which has similar properties to the index to which one has calibrated"* (Pugachevsky & Reyfman)
  - Base correlation mapping: "mapping" new CDO to reference CDOs with "similar risk"
  - Scale correlation skew to fit the "riskiness" of the new portfolio
  - Most common mapping based on matching expected losses (*EL*) of underlying and reference portfolios

EL Mapping (Base Correlations) – Pugachevsky & Reyfman

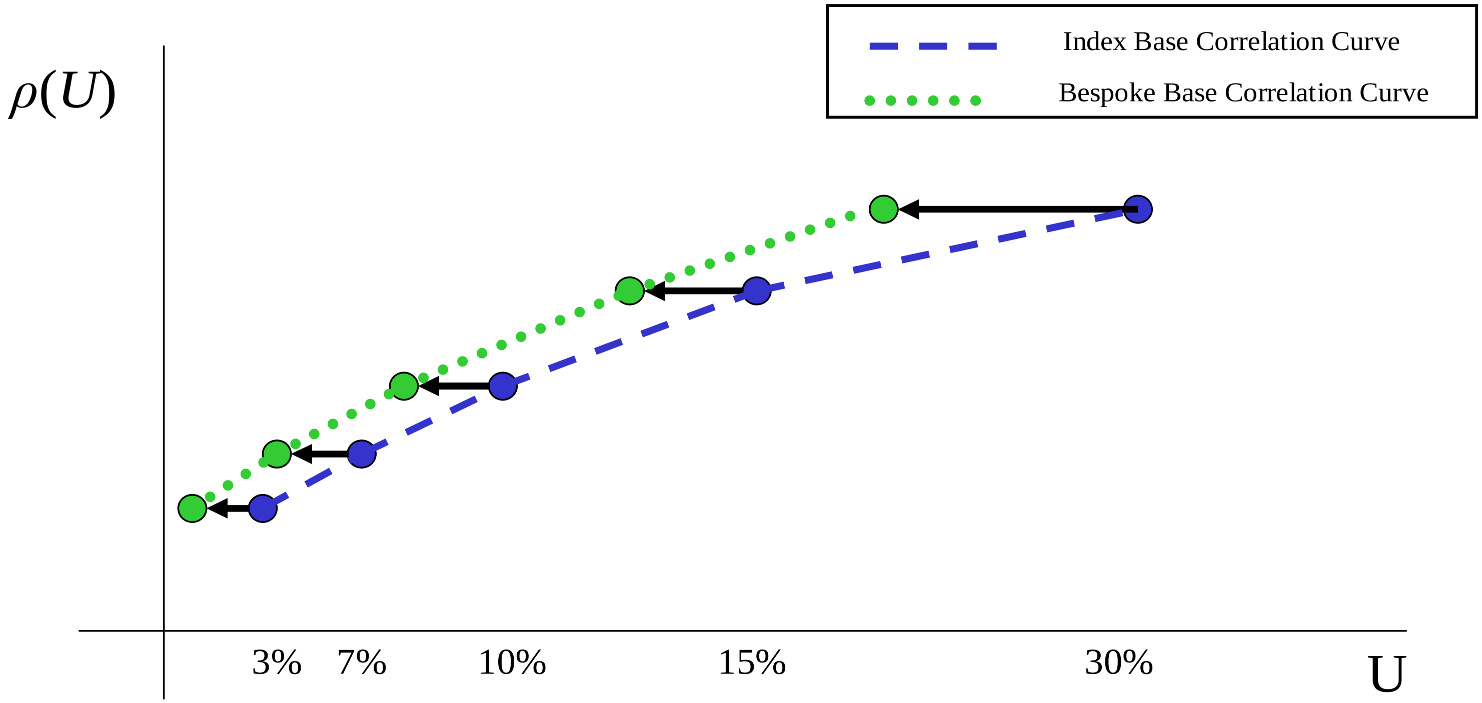
- Main idea: base correlations correspond to different levels of risk in the reference portfolio.
  - By finding the same risk levels on the bespoke portfolio – transfer the base correlation structure from the standard portfolio to the bespoke
- Generally, they solve an equation of the form:

$$S(\hat{P}, \hat{u}, \rho) = S(P, u, \rho)$$

S is a "risk statistic"; *P* the portfolio, *u* the detachment point), and *rho* is the base correlation (the unknown is $\hat{u}$).

  - Note: this is the BASE CORRELATION for the STANDARD PORTFOLIO - but it is used on both sides

EL Mapping (Base Correlations)



EL Mapping (Base Correlations)

- Main idea: base correlations correspond to different levels of risk in the capital structure of the reference portfolio.
  - By finding the same risk levels on the bespoke portfolio – transfer the base correlation structure from the standard portfolio to the bespoke

- *EL* Mapping: solve for the detachment point(s) in the bespoke portfolio which matches the equation:

$$\frac{E[\text{Tranche Loss} \mid u, P, \rho]}{E[\text{Loss} \mid P]} = \frac{E[\text{Tranche Loss} \mid \hat{u}, \hat{P}, \rho]}{E[\text{Loss} \mid \hat{P}]}$$

- We can construct a new base correlation “skew” for the bespoke portfolio (and interpolate from there).

EL Mapping (Base Correlations)

$$\frac{E[\text{Tranche Loss}\,|\,u,P,\rho]}{E[\text{Loss}\,|\,P]}=\frac{E[\text{Tranche Loss}\,|\,\hat{u},\hat{P},\rho]}{E[\text{Loss}\,|\,\hat{P}]}$$

What “risks” should the *EL* mapping account/adjust for?

- Difference in *EL*s of portfolios (~ “shift” of the loss distribution) – implicit in both numerators and denominators of mapping equation
- Difference in portfolio granularity (at least partially) for the of the portfolios (numerators in the mapping equation)

- Does not account for “sector” concentrations (systematic risk)
  - e.g., a portfolio with same number of names, ratings distributions and LGDs as a given index same price/spread, even if it is fully concentrated in one or two sectors
  - Ad-hoc adjustments used in practice
  - Need “concentration adjustment”

Concentration Risk in Credit Portfolios

- Types of concentration: regional, business sector/industry sector, name concentration or *granularity*
- Concentration risk measures
  - Name concentration
    - Herfindahl index (HI) on name exposures, *EL*s
    - *Granularity adjustment* (for capital and tail risk)
  - Sector/geographical concentration
    - HI on sector exposures (ELs or stand-alone capital)
    - Average (weighted) correlation
    - Diversification factor on volatility, capital (e.g. Garcia et al 2006)
- Credit portfolio models generally rely on multi-factor model to model codependence of credit events

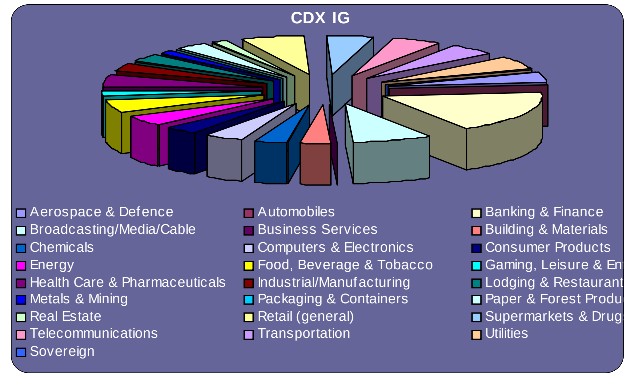
Concentration Reports – CDX & iTraxx

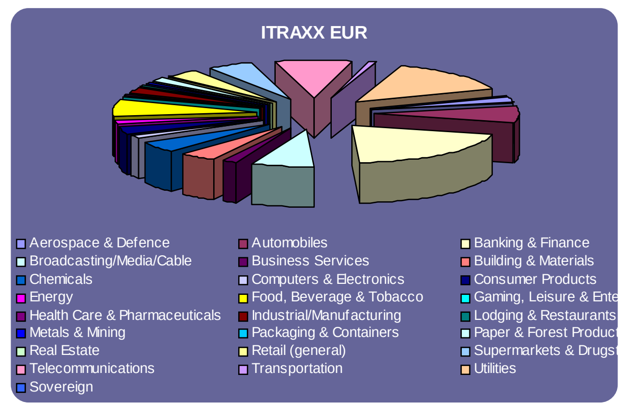
Concentration by Expected Loss					
Rating	**CDX HY**	**CDX IG**	**ITRAXX EUR**	**ITRAXX CJ**	**ITRAXX ASIA**
AAA		1.49%	0.28%	0.38%	1.49%
AA+			0.51%		
AA		1.10%	0.24%		1.10%
AA-		0.39%	5.28%	2.13%	0.39%
A+		4.70%	7.53%		4.70%
A		12.56%	6.70%	9.13%	12.56%
A-		11.16%	11.93%	19.25%	11.16%
BBB+		16.36%	19.54%	5.22%	16.36%
BBB		25.49%	35.00%	13.63%	25.49%
BBB-	0.47%	19.61%	12.98%	14.00%	19.61%
BB+	4.20%	4.40%		10.77%	4.40%
BB	9.14%			10.10%	
BB-	6.85%			15.39%	
B+	12.12%	2.74%			2.74%
B	8.50%				
B-	9.79%				
CCC+	4.22%				
CCC	34.57%				
CCC-	7.34%				
CC	2.81%				

Sector Concentration in Indices

Industry concentration by Notional

	CDX HY	CDX IG	ITRAXX EUR	ITRAXX CJ	ITRAXX ASIA
Exposure Per Name	10MM	8MM	8MM	20MM	1B
Number of Names	100	125	125	50	50
Industry (Fitch)					
Aerospace & Defence	3.00%	4.00%	2.40%		
Automobiles	9.00%		7.20%	4.00%	4.00%
Banking & Finance	3.00%	17.60%	20.00%	20.00%	24.00%
Broadcasting/Media/Cable	7.00%	8.00%	7.20%		
Business Services	4.00%		1.60%	10.00%	
Building & Materials	4.00%	3.20%	4.00%	6.00%	
Chemicals	6.00%	3.20%	4.80%	2.00%	2.00%
Computers & Electronics	10.00%	4.00%	1.60%	8.00%	6.00%
Consumer Products	5.00%	3.20%	3.20%	4.00%	
Energy	10.00%	4.80%	1.60%		10.00%
Food, Beverage & Tobacco	4.00%	5.60%	7.20%	4.00%	
Gaming, Leisure & Entertainment	3.00%	1.60%	0.00%		2.00%
Health Care & Pharmaceuticals	3.00%	5.60%	0.80%	0.00%	
Industrial/Manufacturing	2.00%	3.20%	2.40%	8.00%	6.00%
Lodging & Restaurants	3.00%	3.20%	0.80%		
Metals & Mining	2.00%	1.60%	0.80%	12.00%	4.00%
Packaging & Containers	2.00%				
Paper & Forest Products	5.00%	3.20%	1.60%		
Real Estate		1.60%	0.00%	4.00%	
Retail (general)	3.00%	6.40%	4.00%	4.00%	8.00%
Supermarkets & Drugstores	1.00%	4.80%	4.80%		
Telecommunications	4.00%	4.80%	8.80%	2.00%	12.00%
Transportation	3.00%	4.80%	0.80%	8.00%	4.00%
Utilities	4.00%	5.60%	14.40%	4.00%	4.00%
Sovereign					14.00%
Herfindahl	5.7%	7.0%	9.5%	9.8%	12.2%
Effective number of sectors	17.5	14.2	10.5	10.2	8.2





Example: 40 Names (diversified) Portfolio – CDX

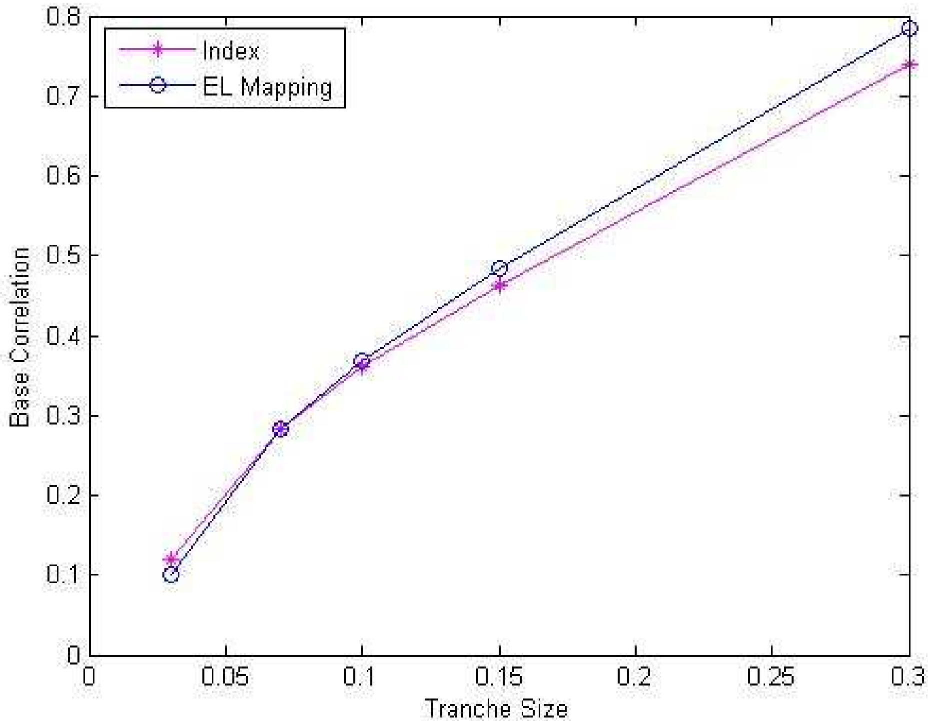
	CDX Index		Bespoke Portfolio	
Sector (aggregate)	**Weight (Notional)**	**Weight (EL)**	**Weight (Notional)**	**Weight (EL)**
TECH	20.8%	21.4%	20%	14.5%
SERVICE	9.6%	10.9%	15%	16.7%
PHARMA	5.6%	3.6%	5%	3.9%
RETAIL	20.0%	29.0%	17.5%	27.2%
FINANCE	19.2%	11.8%	10%	6.6%
INDUSTRIAL	9.6%	9.9%	15%	14.2%
ENERGY	15.2%	13.5%	17.5%	16.9%
HI	0.16	0.19	***0.16***	***0.18***
No. Eff. sectors	6.07	5.40	***6.30***	***5.63***

Simplified sector classification (model) – 7 sectors

23% intra-sector corr
16% cross-sector corr

Average 5yr PD Index = 3.65% 40 Name = 3.92%

EL Mapping 40 Names



CDX Detach Point	EL Mapped Point
3%	3.45%
7%	7.01%
10%	9.75%
15%	14.01%
30%	27.77%

PRICES

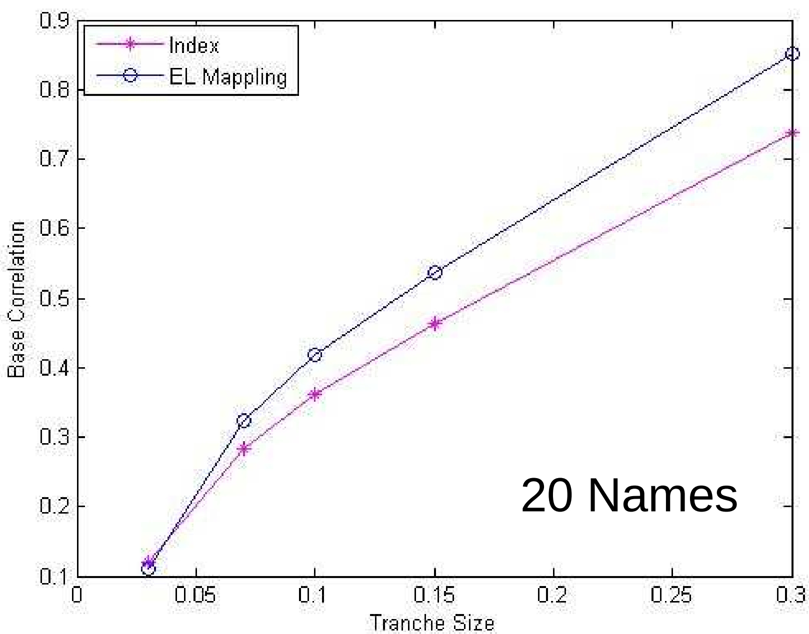
Tranche	Index	Bespoke (EL)
0 - 3%	31.81%	34.04%
3 - 7%	99	139
7-10%	21	43
10-15%	9.9	18
15-30%	5.5	0

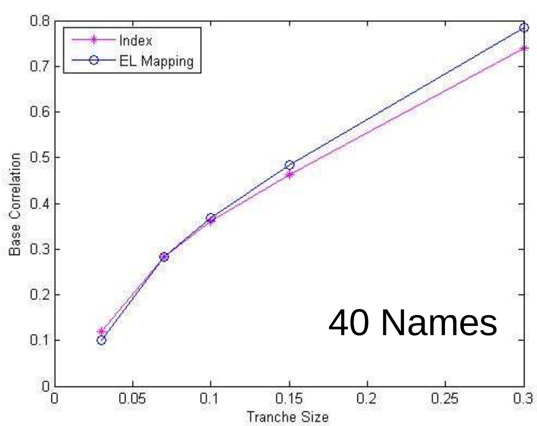
Example: 20 Names Portfolio – CDX

	CDX Index		Bespoke Portfolio (40)		Bespoke Portfolio (20)	
Sector (aggregate)	**Weight (Notional)**	**Weight (EL)**	**Weight (Notional)**	**Weight (EL)**	**Weight (Notional)**	**Weight (EL)**
TECH	20.8%	21.4%	20%	14.5%	**50%**	**72.3%**
SERVICE	9.6%	10.9%	15%	16.7%		
PHARMA	5.6%	3.6%	5%	3.9%		
RETAIL	20.0%	29.0%	17.5%	27.2%		
FINANCE	19.2%	11.8%	10%	6.6%	**50%**	**27.7%**
INDUSTRY	9.6%	9.9%	15%	14.2%		
ENERGY	15.2%	13.5%	17.5%	16.9%		
HI	0.16	0.19	*0.16*	*0.18*	***0.50***	***0.60***
No. Eff. sectors	6.07	5.40	*6.30*	*5.63*	***2.00***	***1.67***

Avg 5yr PD — Index = 3.65% — 40 Name = 3.92% — 20 Name = 3.36%

EL Mapping 20 Names





CDX Point	EL Mapped Point (40)	**EL Mapped Point (20)**
3%	3.45%	**3.17%**
7%	7.01%	**5.89%**
10%	9.75%	**7.98%**
15%	14.01%	**11.55%**
30%	27.77%	**24.65%**

PRICES

Tranche	Index	Bespoke (40)	Bespoke (20)
0 - 3%	31.81%	34.04%	28.26%
3 - 7%	99	139	113
7-10%	21	43	29
10-15%	9.9	18	11
15-30%	5.5	0	0

Concentration-Adjusted Mappings

- Basic Model – “Average correlation adjustment”

$$\rho_{B}^{*Corr}(k) = \left(\frac{\overline{\beta}_B}{\overline{\beta}_I} \right) \cdot \rho_{I}^{*}(k)$$

where the betas are *EL*-weighted “average correlations” of each portfolio. For a given portfolio β solves:

$$w^T \Sigma w = w^T \Sigma_\beta w, \quad w_j = EL_j,$$

$$\Sigma_{ij} = \mathrm{corr}(Y_i, Y_j), \quad \Sigma_\beta = \begin{bmatrix} 1 & \beta & \ddots \\ \beta & 1 & \beta \\ \ddots & \beta & 1 \end{bmatrix}$$

- Justification: roughly assume “equal risk premia” for the *systematic portfolio asset volatility* (under some assumptions and simplifications)

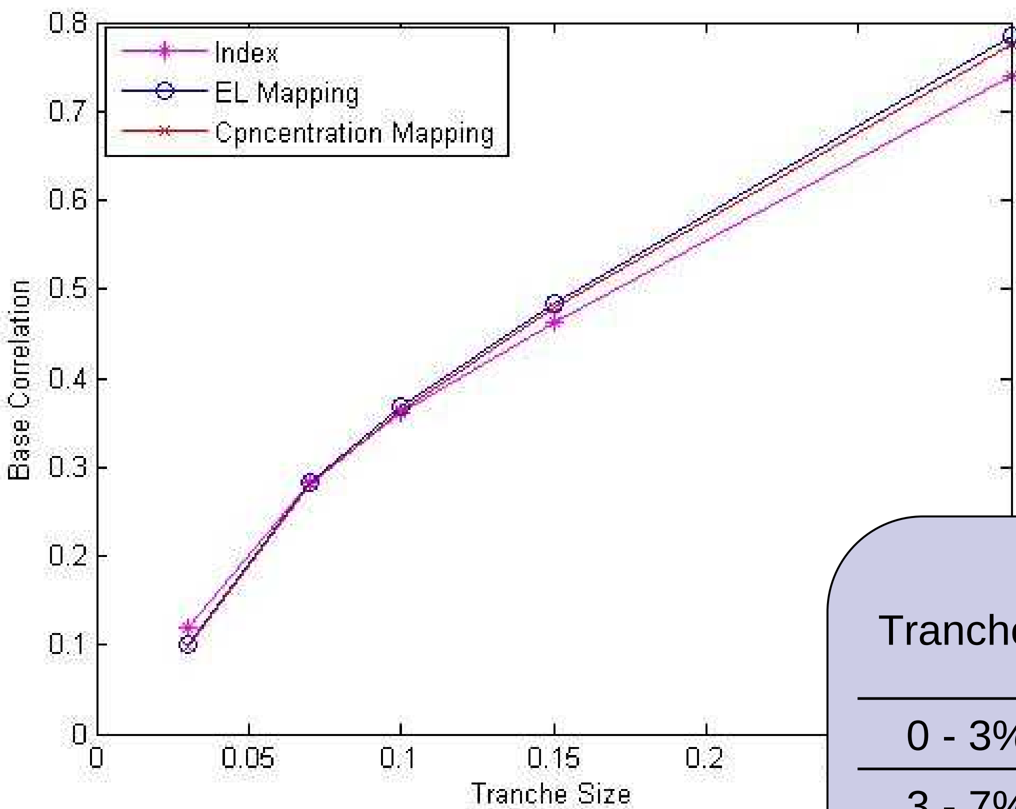
CA Mapping – Comments

- Alternative definitions of average correlation in practice
- Requires “real” multi-factor/correlation model for the portfolio
  - MF model, nested SF Economy-wide systematic, calibrated SF
- Idea extended when portfolio and index do not have same *EL* and number of names (*EL* mapping adjusts for this already)
  - “comparable portfolio” with same EL and name concentration but different sector concentration:

$$\rho_B^{*Corr}(k) = \left(\frac{\overline{\beta}_B}{\overline{\beta}_I}\right) \cdot \rho_B^{EL}(k)$$

Mapping: Index   bespoke (EL, name)   bespoke (EL, name, sector)

CA Mapping – Example (40 Names)



Simple multi-factor (prior) model:

- 23% intra-sector corr
- 16% cross-sector corr

Average correlations

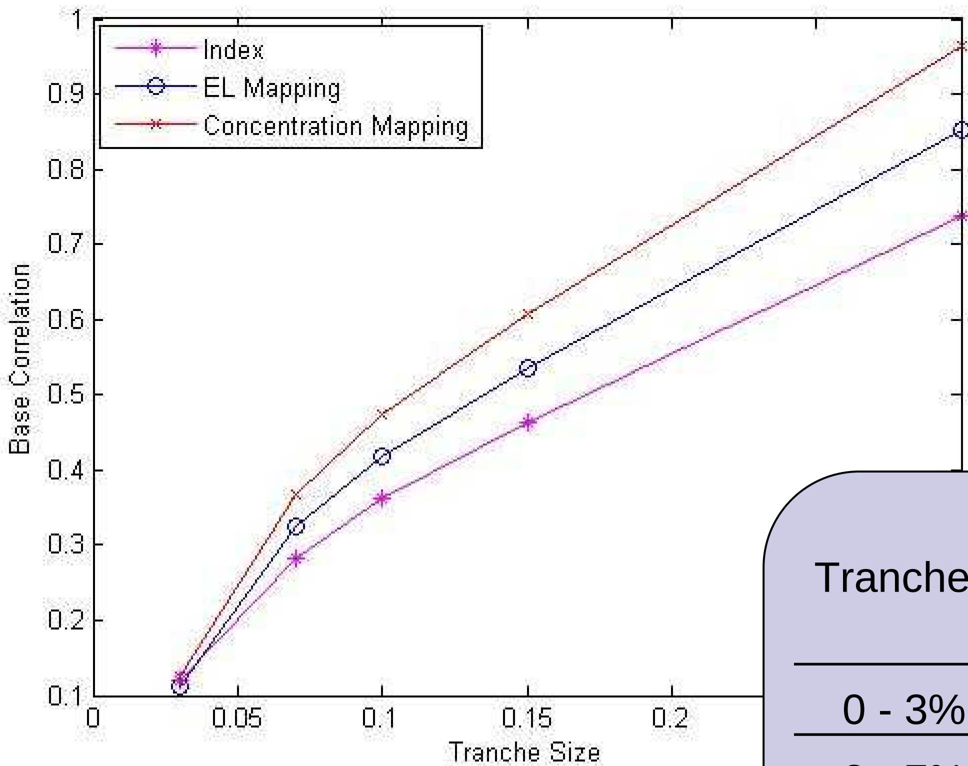
- Index: 0.1805
- Bespoke: 0.1787

Corr. Ratio: 0.9902

PRICES

Tranche	Index	Bespoke (EL)	Bespoke (CA)
0 - 3%	31.81%	34.04%	34.1%
3 - 7%	99	139	141
7-10%	21	43	43
10-15%	9.9	18	18
15-30%	5.5	0	0

CA Mapping – Example (20 Names)



- Simple multi-factor (prior) model:
  - 23% intra-sector corr
  - 16% cross-sector corr
- Corr. Ratio: 1.1312

PRICES

Tranche	Index	Bespoke (EL)	Bespoke (CA)
0 - 3%	31.81%	28.26%	26.64%
3 - 7%	99	113	64
7-10%	21	29	4
10-15%	9.9	11	0
15-30%	5.5	0	0

Systematic Weighted Monte Carlo Method

General Idea

- From observed prices/spreads of indices, tranches, and single-name CDSs, obtain implied discrete distribution (or process in a dynamic setting) of a set of ***systematic factors***
  - Factors drive the systematic credit risk, and hence the joint movement of default probabilities (or hazard rates)
- Multi-factor model links the systematic factors and conditional default probabilities
- Single factor model is generally enough to model an index but a multi-factor model is required to distinguish bespoke portfolio concentrations, and special structures

Systematic Weighted MC Approach

Background

- Weighted MC approach used to price complex options
  - e.g. Avellaneda et al., 2001, Elices and Giménez, 2006
- Similar idea to fitting the implied distribution (or process) of underlying in a (discrete) lattice
- Hull-White “Implied Copula” (2006) is essentially an application of this concept
  - Homogeneous portfolio – cannot be used directly to price bespokes
  - Similar ideas (also for homogeneous portfolios) in Brigo et al (2006).

Hull-White Implied Copula Method

- Assume homogeneous portfolio.
- Specify a set of hazard rate scenarios λ_s.

$$\min_{\pi} \sum \frac{(\pi_{s+1} - 2\pi_s + \pi_{s-1})^2}{(\lambda_{s+1} - \lambda_{s-1})}$$

subject to

$$\sum_s \pi_s = 1, \quad \pi_s \geq 0$$

Match standard tranche prices.

- Extensions for different numbers of names, different maturities, matching CDS term structure,…

Bespoke Portfolios in the Implied Copula Method

- For portfolios that have the same homogeneity as the index, scale hazard rates

$$\lambda_s^* = c\lambda_s$$

by a constant factor so that average CDS spreads are matched.

- For portfolios that have different homogeneity from that of the index:
  - Calculate *y*,*y** the average equity return correlations for companies in the index and bespoke portfolios.
  - Compute:

$$\rho^* = \rho \cdot y \big/ y^*$$

where ρ is the Gaussian copula correlation for the index. Alter hazard rate scenarios to match market CDS spreads and joint default probabilities from the Gaussian copula model.

Bespoke Portfolios in the Implied Copula Method

- "Dealing with portfolios that are less (or more) well diversified than the index requires some judgment and, whether the Gaussian copula/base correlation or implied copula approach is used, is inevitably somewhat *ad hoc*". (Hull-White, 2006).
- SWMC provides a systematic approach to the problem:
  - Can handle bespoke tranches on any portfolio, regardless of concentration/diversification.
  - Implied systematic factor distributions can be used to price other portfolio credit derivatives (e.g. CDO^2).
  - Stronger assumptions are required: relationship between real and risk neutral measure, fully calibrated MF model under *P*.

Implied Risk Factor Distributions – Intuition

Key objective: tractable distribution of joint default times – match marginal distributions and prices of CDSs and quoted CDO tranches

- In a Gaussian copula – conditioning on the systemic factor

$$p_j^Z(t) = \Phi\left(\frac{\Phi^{-1}\left(F_j(t)\right) - \sqrt{\rho}Z}{\sqrt{1-\rho}}\right)$$

- Base correlations the correlation *rho* is a function of the detachment point
- Hull-White “implied copula” model directly conditional PDs through discrete scenarios (on a hazard rate) for homogeneous portfolio
- Implied risk factor distribution model directly the ***distribution of the systematic risk factor*** through discrete scenarios conditional default probabilities through the copula “mapping”
  - Extensible to multi-factor and applied to other portfolios

Systematic Weighted MC Approach – Static Model

- Assumptions
  - MF model joint default behaviour under real world measure *P*
  - Coefficients of MF model for portfolio are known and fixed
  - Difference between real measure *P* and RN-measure *Q* is in the joint distribution of the systematic factors
    - (Marginal) distribution of default times for each name under the risk-neutral measure based on CDS spreads
    - Conditional distribution of default times, as a function of the factor levels under the RN measure still given by the same formula
- Solution
  - Sample discrete "paths" (in this case, single values) for the systematic factors and adjust probabilities of paths to match prices

Systematic Weighted Monte Carlo: Probability Constraints

- For each name *j* in the index and bespoke portfolios, based on default swap spreads, estimate the cumulative default probability $F_j(T)$
- Define the default threshold to be:

$$H_j(T) = \Phi^{-1}(F_j(T))$$

- Constraint that (implied, risk-neutral) probabilities match market implied default probabilities:

$$\sum_{\omega} q_{\omega} \, \Phi\left(\frac{H_j(T) - \sum_{k=1}^{K} \beta_{jk} Z_{\omega}^{k}}{\sqrt{1 - \sum_{k=1}^{K} \beta_{jk}^{2}}} \right) = F_j(T)$$

Systematic Weighted Monte Carlo: Spread Constraints

For each tranche:

- S = tranche size, U = detachment point, L = attachment point
- s = market quoted spread (given)
- Λ_i = the tranche loss random variable at time t_i
- D_i = discount factor for cashflows at time t_i
- Tranche value:

$$V = \sum_{i=1}^{n} s(t_i - t_{i-1})(S - E[\Lambda_i])D_i - \sum_{i=1}^{n} (E[\Lambda_i] - E[\Lambda_{i-1}])$$

- Constraint on matching market spreads is*

$$\sum_{\omega} q_{\omega} E[V \mid Z_{\omega}] = 0$$

* In this exercise, conditional expectations calculated using Normal approximation to portfolio loss

Weighted Monte Carlo: Objective Function

- Optimize a measure of fitness of the probability distribution *G(q),*
  - subject to constraints described above (in practice, use penalties instead)

- Options for *G* include:
  - Maximum Entropy: maximize $-\sum_{\omega} q_{\omega} \ln(q_{\omega})$
  - Maximum Smoothness: minimize quadratic form approximation integral of square of second derivative (same as used in Hull-White)
  - Proximity to a prior distribution *p*. Minimize $\|q - p\|_r$ where:
    - $r = 1$ or ∞ linear programs
    - $r = 2$ quadratic program

Caveat: Interpreting Implied Distributions

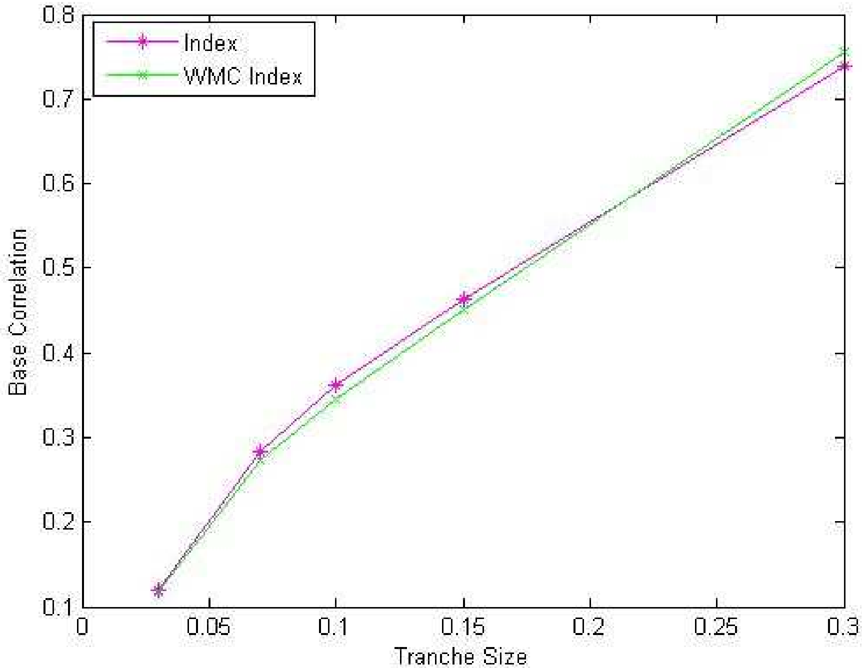
- Consider a homogeneous portfolio, and assume that the "prior" model is the single factor Gaussian copula with correlation ρ_P, and market prices agree with a single factor Gaussian copula with ρ_M.
- Market prices can be reproduced exactly by taking the systematic factor to have a Normal implied distribution with mean and variance:

$$\mu = \Phi^{-1}(PD) \cdot \left(\frac{1}{\sqrt{\rho_P}} - \sqrt{\frac{1-\rho_P}{\rho_P(1-\rho_M)}} \right)$$

$$\sigma^2 = \frac{\rho_M(1-\rho_P)}{\rho_P(1-\rho_M)}$$

- Moral: Implied factor distributions must be viewed in relation to the "prior" factor model. Significant deviations from standard normal factors imply significant changes from the prior model to the implied model.

Systematic Weighted MC – Example



Basic numerical exercise ("raw" calibration)

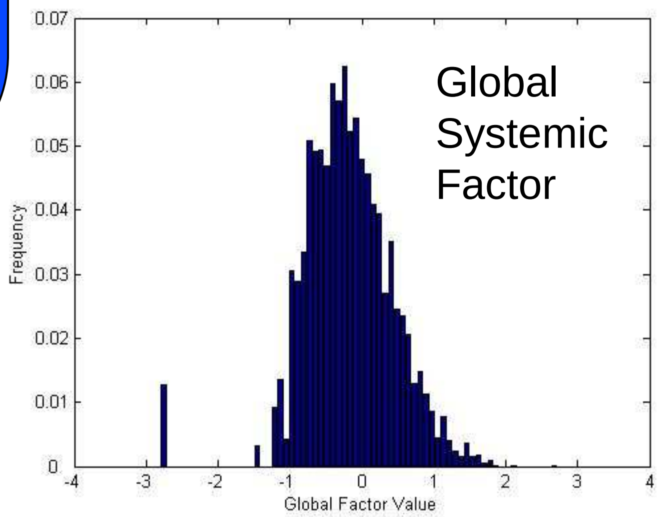
- 200 to 300 Quasi-MC Scenarios (multi-dim. Sobol points)
  - Normal prior scenarios
  - Conditional Normal approx., constant LGD
- Optimization problem
  - min. distance from prior
  - "soft" matching constraints (penalty function)

PRICES

Tranche	Index (Market)	Index (WMC)
0-3%	31.81%	31.93%
3-7%	99	102
7-10%	21	24
10-15%	10.5	9.2
15-30%	5.5	0

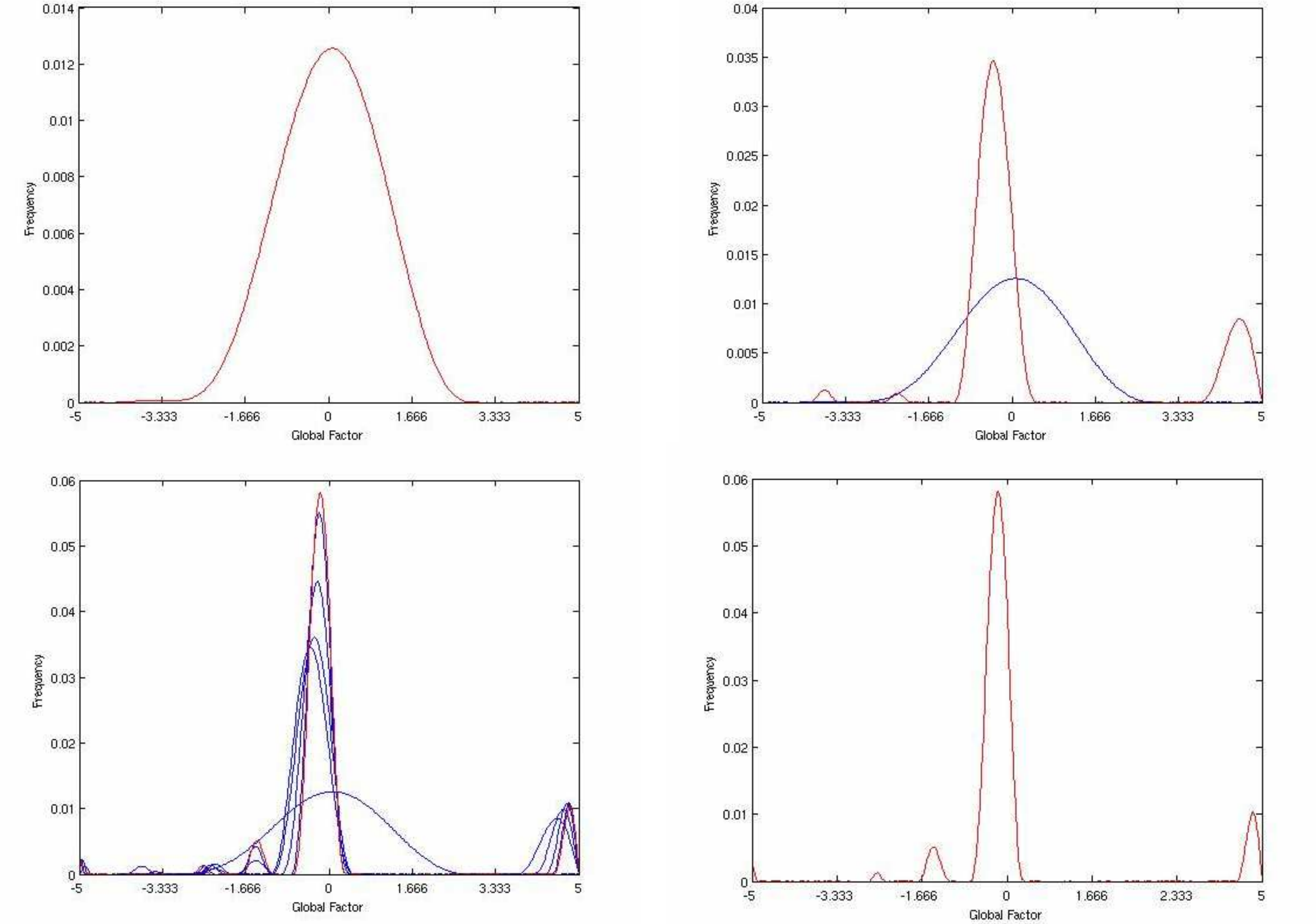
Systematic Weighted MC – Example

Factor	Mean	Std	Skew	Ex. Kurt.
Global	*-0.15*	*0.63*	*-0.48*	*2.76*
TECH	-0.07	1.04	0.27	-0.05
SERVICE	-0.01	1.00	0.04	-0.12
PHARMA	-0.01	0.99	-0.13	-0.20
RETAIL	-0.03	1.00	0.01	-0.39
FINANCE	-0.02	0.97	0.03	-0.23
INDUSTRY	-0.01	1.04	-0.01	-0.04
ENERGY	-0.01	1.00	0.04	-0.22



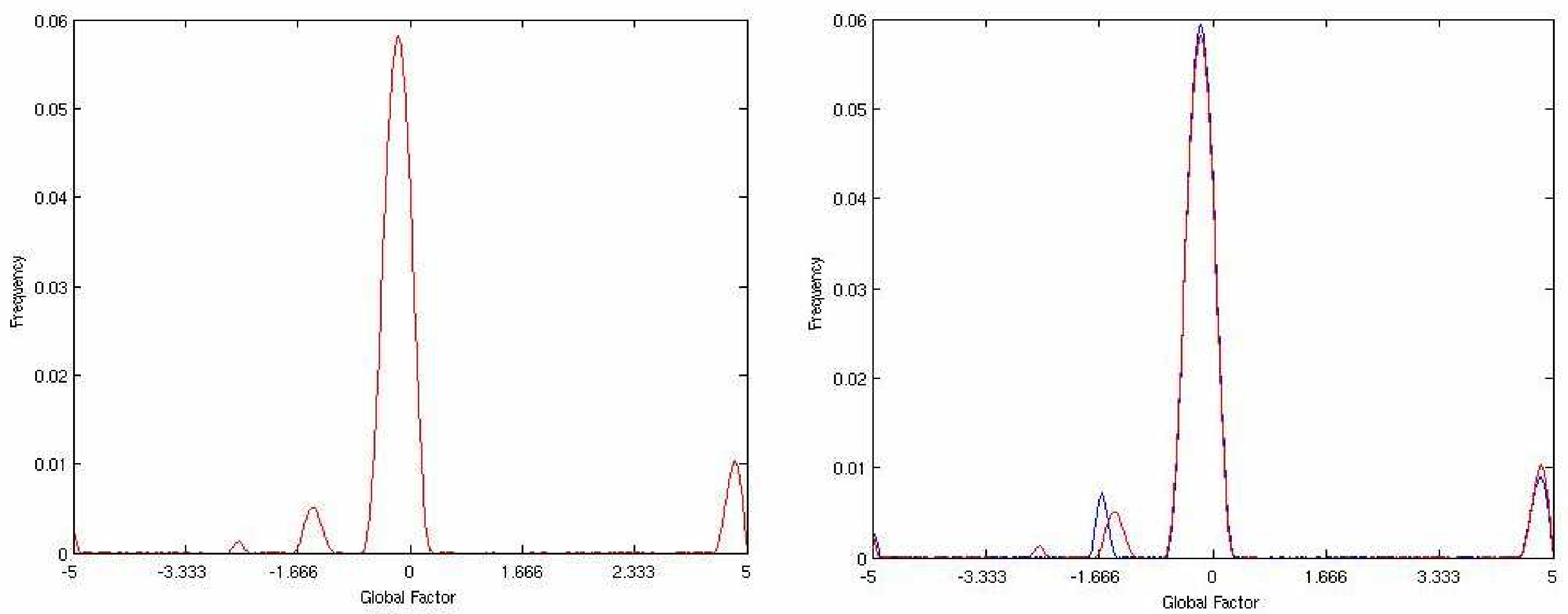
Global Factor Implied Distribution (enhanced calibration)

Evolution of distribution – from maximum smoothness to tight fit of prices



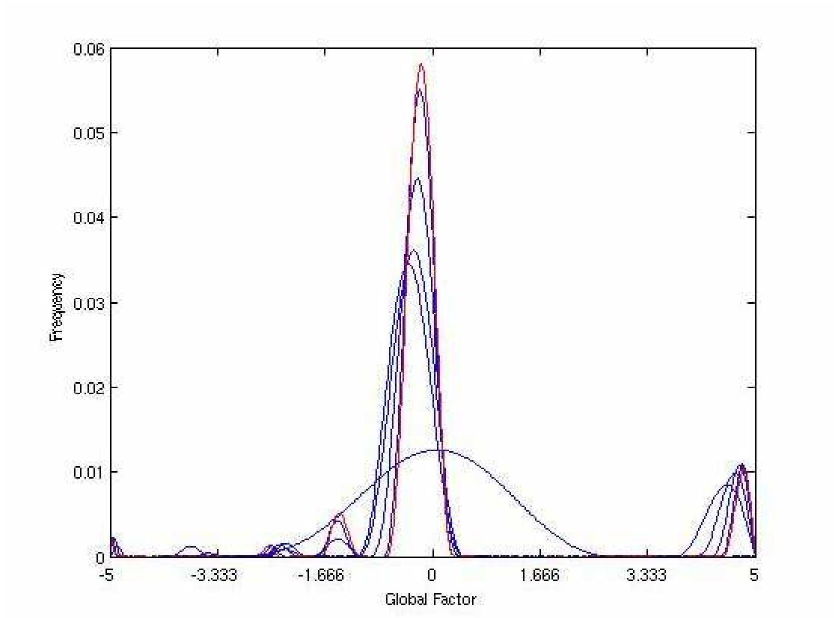
Global Factor Implied Distribution (enhanced calibration)

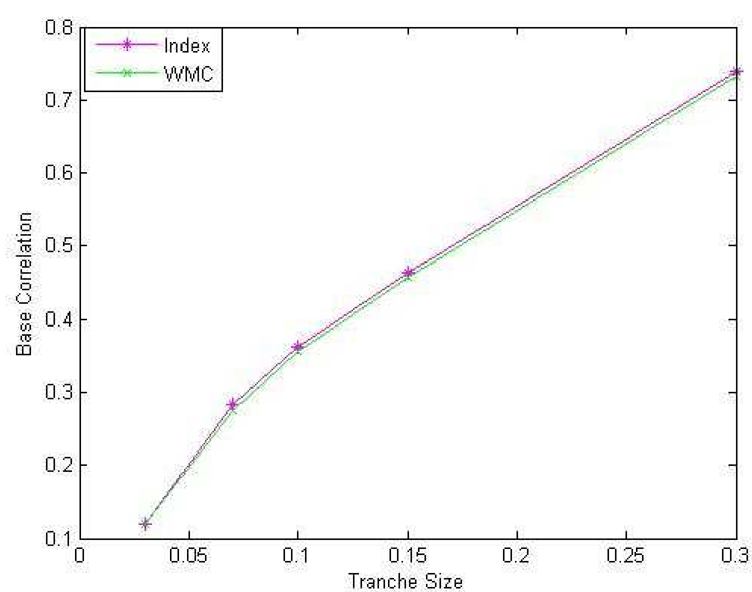
Difference in implied distributions by removing most senior tranche



Systematic Weighted MC – CDO Index Prices

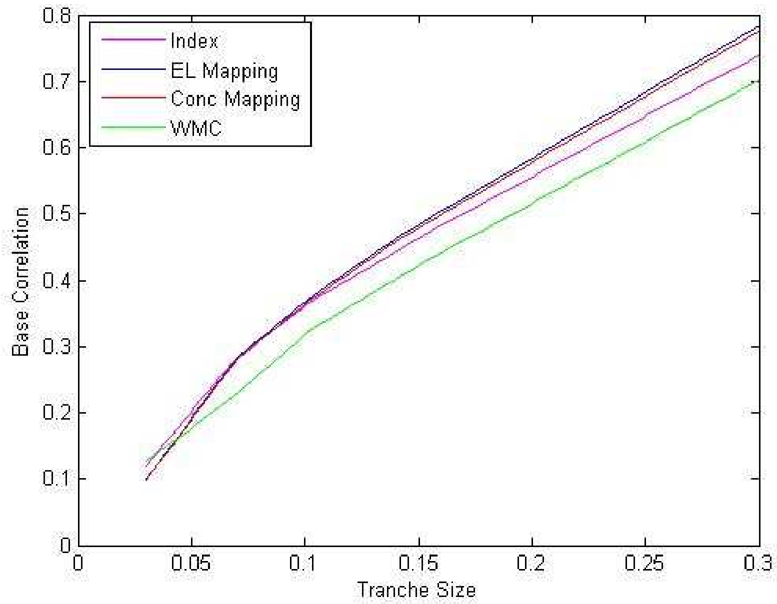
Penalties	Global factor statistics				Tranche Prices				
	MEAN	STD	SKEW	EX.KURT	0-3%	3-7%	7-10%	10-15%	15-30%
None	0.00	0.99	-0.11	-0.27	28.47%	229.06	45.97	9.81	1.12
level 1	0.37	1.90	1.49	1.02	32.31%	118.39	32.85	23.27	9.47
level 2	0.30	1.81	1.68	2.05	32.22%	111.58	34.8	18.67	10.37
level 3	0.22	1.69	1.90	3.33	32.04%	106.54	30.78	16.12	10.77
level 4	0.14	1.54	2.23	5.01	31.89%	101.58	24.46	12.47	7.5
Final	**0.12**	**1.47**	**2.42**	**5.98**	**31.82%**	**99.35**	**21.47**	**10.77**	**5.77**
MARKET					**31.81%**	**99**	**21**	**10.5**	**5.5**





Systematic Weighted MC – Bespoke (40 Names)

Penalties	Tranche Prices				
	0-3%	3-7%	7-10%	10-15%	15-30%
None	29.75%	302.4	77.78	19.53	1.83
level 1	31.58%	244.48	39.2	25.16	10.65
level 2	31.93%	232.1	39.48	22.28	10.85
level 3	32.22%	219.38	38.48	19.67	10.99
level 4	32.48%	207.92	35.2	15.41	7.78
Final	**32.57%**	**203.51**	**33.07**	**13.37**	**6.09**
EL Mapping	**34.04%**	**139**	**43**	**18**	**0**
WMC Index	31.82%	99.35	21.47	10.77	5.77
Market Index	31.81%	99	21	10.5	5.5



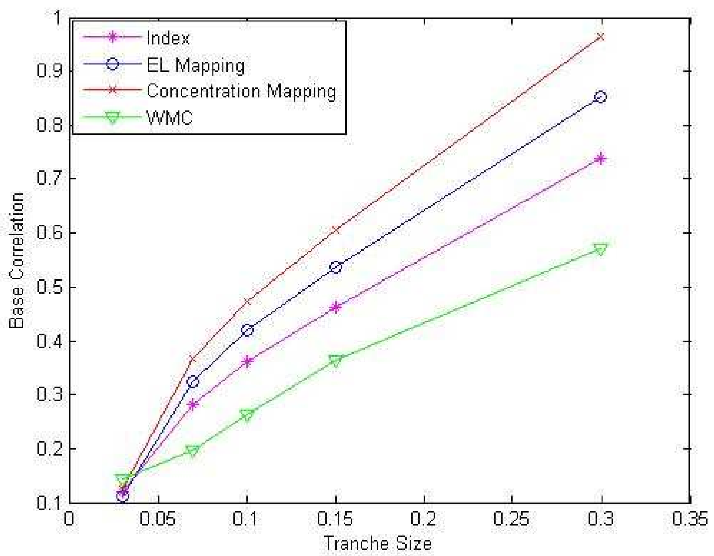
Systematic Weighted Monte Carlo Bespoke (40 Names)

	0-3%	3-7%	7-10%	10-15%	15-30%
MARKET	***31.81***	***99***	***21***	***10.5***	***5.5***
WMC INDEX	***31.81***	***99.53***	***19.96***	***10.86***	***5.48***
BESPOKE (WMC MF)	***33.13***	***214.54***	***24.66***	***11.18***	***5.53***
BESPOKE (WMC SF)	32.57	203.51	33.07	13.37	6.09
BESPOKE EL Mapping	34.04	139	43	18	0

Factor	MEAN	STD	SKEW	KURT
Global	0.66	1.26	1.35	2.89
TECH	0.15	2.91	- 0.14	- 1.13
SERVICE	0.13	2.86	- 0.07	- 1.09
PHARMA	0.31	2.59	- 0.09	- 1.14
RETAIL	- 0.36	2.73	0.13	- 1.01
FINANCE	0.13	2.41	0.06	- 0.72
INDUSTRY	0.39	2.78	- 0.21	- 1.13
ENERGY	0.37	2.83	- 0.16	- 1.10

Systematic Weighted MC – Example (20 Names)

	0-3%	3-7%	7-10%	10-15%	15-30%
MARKET	***31.81***	***99***	***21***	***10.5***	***5.5***
WMC INDEX	*31.81*	*99.53*	*19.96*	*10.86*	*5.48*
BESPOKE (40 name) WMC	*32.57*	*203.51*	*33.07*	*13.37*	*6.09*
BESPOKE *(20 Name) WMC*	**20.98**	**266**	**81**	**30**	**8**
BESPOKE EL Mapping	28.26	113	29	11	0



Intuition: Implied Risk Factor Distributions

- More generally, we can use other “link functions”, satisfying basic matching (no-arbitrage) constraints
  - Logit function, NIG, double-t, etc…
- Generalized linear mixed model (McNeil and Wendin 2006) framework.

$$p_j^Z(t) = G\left(a(t) - \sum_k b_k Z_k\right)$$

  - Example of general multi-factor copula $p_j^Z(t) = G_j\left(\frac{H^{-1}(F_j(t)) - \sum_k \beta_k Z_k}{\sqrt{1-\sum_k \beta_k^2}}\right)$
  - Matching, for each name, the “unconditional” default probability term structure

$$p_j(t) = \int p_j^Z(t)\,df(z)$$

  - … and match quoted CDO prices

Implied Risk Factor Distributions and *GLMMs*

- General formulation:

$$PD_{it}(Z^t) = h\left(a_{it} + \sum_{k=1}^{K} b_{ik}^t Z_k^t\right)$$

- Gaussian copula:

$$a_{it} = \frac{\Phi^{-1}(PD_{it})}{\sqrt{1-\sum_{k=1}^{K}\beta_k^2}} \quad , \quad b_{ik} = \frac{\beta_k}{\sqrt{1-\sum_{k=1}^{K}\beta_k^2}}$$

- Logit function:

$$h(x) = \frac{1}{(1+\exp(-x))}$$

- Poisson mixture (e.g. CreditRisk+)

$$\lambda_i(Z) = E[U_i \mid Z] = c_i \sum_{k=1}^{K} \beta_{ik} Z_k$$

Systematic Weighted MC Approach

- Various possible routes – for a practical and stable solution we have to be pragmatic
  - e.g. calibration problem can be computationally intensive, and with "ugly" underlying distributions
- Issues include:
  - Joint credit evolution model: static (multi-factor Gaussian copula) or dynamic (structural or reduced-form model)
  - Multi-factor model and dimensionality of the problem
  - Discretization (MC method), and number of scenarios
  - Computation of conditional portfolio loss distributions (in each scenario)
  - Optimization problem (objective function, smoothness, constraints, etc.)
  - Solution strategy for optimization
  - Price new instruments via MC or map/calibrate to simple model

Systematic WMC – Improving Performance & Stability

- Systematic Weighted MC is a *modelling framework*
  - Model stability for particular problems must be well understood before using it in practice
- Some available tools to fit and apply the model in practice include
  1. Scenario generation and prior distributions
  2. Non-constant LGDs (as in Altman, Brigo, Moody's)
  3. Effective application of objective function – max. smoothness, max. entropy, min. distance to prior, penalties
  4. Additional "fake constraints" to obtain desired distributions
     - e.g. a solution close to the *EL* mapping can be obtained by including penalties for giving prices different from it
  5. Computational method to compute conditional portfolio loss distributions (conditional Normal, Poisson, recursions, etc.)
  6. Alternative distribution functions for factors and residuals and even time-dependent parameters

Concluding Remarks and Future Directions

- EL mappings and base correlations can be misleading at times
  - Better used as communication and (perhaps) interpolation tools
- Weighted MC methodology is quite general and can price consistently other instrument types (e.g.CDO^2)
  - Sensitivities via MC Greek techniques or also through fitting simplified model
- Key areas of focus include
  - Detailed understanding of sensitivity to model parameters and concentration risk
  - Tail scenarios and non-normal priors
  - Computational “tuning” (scenario sampling, convolution, optimization problem, etc.)
  - Use empirically validated multi-factor model and understanding of its impact
  - Extensions to other instruments – e.g. CDO^2 (calibrating to prices & multiple indices)
- Extensions to dynamic models via hazard rate processes (Cox processes) or multi-step structural models (e.g. Kreinin et al. 1999, HW 2001)
- Measurement and aggregation of risk concentrations across credit portfolios (including CDOs)

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Prior to joining the University of Waterloo, Dr. Saunders taught in the Department of Mathematics at the University of Pittsburgh (2002-2004), where he also served as co-director of the Professional Masters Program in Mathematical Finance, directing numerous collaborative research projects between students and industrial sponsors (including Toronto Dominion Bank, PNC Bank, Mellon Bank and RiskMetrics), and at the Cyprus International Institute of Management, where he was CLR Chair in Corporate Finance and deputy director of RiskLab Cyprus (a financial risk management research laboratory).

Dr. Saunders has served as a consultant for many institutions, including the Bank of Nova Scotia, Ontario Teachers' Pension Plan Board, the Cyprus Development Bank, the Central Bank of Cyprus and R2 Financial Technologies.

While a graduate student at the University of Toronto, Dr. Saunders was an associate member of the Quantitative Research group at Algorithmics Inc., as well as a member of RiskLab Toronto, the University of Toronto's research laboratory on financial mathematics. At Algorithmics, Dr. Saunders worked on a number of projects included mutual fund ratings, applications of optimization duality to pricing derivative securities, stochastic programming for risk management, and the development of software for a general system for solving financial stochastic optimization problems (a joint project with the University of Cambridge).

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