Dynamic Asset Allocation: a Portfolio Decomposition Formula and Applications

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1 Introduction

- Dynamic consumption-portfolio choice:
  - Merton (1971): optimal portfolio includes intertemporal hedging terms in addition to mean-variance component (diffusion)
  - Breeden (1979): hedging performed by holding funds giving best protection agst fluctuations in state variable (diffusion)
  - Ocone and Karatzas (1991): representation of hedging terms using Malliavin derivatives (Ito, complete markets)

    $\rightarrow$ Interest rate hedge

    $\rightarrow$ Market price of risk hedge
  - Detemple, Garcia and Rindisbacher (DGR JF, 2003): practical implementation of model (diffusion, complete markets)

    $\rightarrow$ Based on Monte Carlo Simulation

    $\rightarrow$ Flexible method: arbitrary # assets and state variables, non-linear dynamics, arbitrary utility functions

    $\rightarrow$ Extends to incomplete/frictional markets (DR MF, 2005)

- Contribution:
  - **New decomposition of optimal portfolio (hedging terms):**

    → Formula rests on change of numéraire: use pure discount bonds as units of account

    → Passage to a new probability measure: forward measure (Geman (1989) and Jamshidian (1989))

    → General context: Ito price processes, general utilities

- **New economic insights about structure of hedges:**
  - → Hedge fluctuations in the price of long term bond
    - ∗ pure discount bond with utility of terminal wealth
    - ∗ coupon-paying bond with intermediate utility
    - ∗ this hedge has a static flavor (static hedge)
  - → Hedge fluctuations in future bond return volatilities and market prices of risk
  - → Risk aversion properties:
    - ∗ if risk aversion approaches one both hedges vanish: myopia
    - ∗ if risk aversion becomes large mean-variance term and second hedge vanish: holds just long term bonds
    - ∗ if risk tolerance vanishes all terms are of first order in risk tolerance.
  - → Non-Markovian $N + 2$ fund separation theorem.

- **Technical contribution:**
  - $\longrightarrow$ Exponential version of Clark-Haussmann-Ocone formula
    - $*$ Identifies volatilities of exponential martingale in terms of Malliavin derivatives
  - $\longrightarrow$ Malliavin derivatives of functional SDEs
  - $\longrightarrow$ Explicit solution of a Backward Volterra Integral Equation (BVIE) involving Malliavin derivatives.

- Applications:
  - Preferred habitat
  - Preferences for long term bonds
  - Extreme risk aversion behavior
  - International asset allocation
  - Preferences for I-bonds
  - Integration of risk management and asset allocation
- Road map:
  - Model with utility from terminal wealth
  - The Ocone-Karatzas formula
  - New representation
  - Intermediate consumption
  - Applications
  - Conclusions

2 The Model

- Standard Continuous Time Model:
  - Complete markets and Ito price processes
  - Brownian motion W, d-dimensional
  - Flow of information $\mathcal{F}_t = \sigma(W_s : s \in [0, t])$
  - Finite time period $[0, T]$.
  - Possibly non-Markovian dynamics

▶ Assets: Price Evolution

- **Risky assets (dividend-paying assets):**

$$\boxed{\frac{dS_t^i}{S_t^i} = \left(r_t - \delta_t^i\right) dt + \sigma_t^i \left(\theta_t dt + dW_t\right), \quad S_0^i \text{ given}}$$

→ σ_t^i: volatility coefficients of return process ($1 \times d$ vector)

→ r_t: instantaneous rate of interest

→ δ_t^i: dividend yield

→ θ_t : market prices of risk associated with W ($d \times 1$ vector)

→ $(r, \delta, \sigma, \theta)$: progressively measurable processes; standard integrability conditions

- **Riskless asset:**

  ∗ pays interest at rate r

▶ Investment and Wealth:

- **Portfolio policy π:**

$\rightarrow$ d-dimensional, progressively measurable, integrability conditions

$\rightarrow$ amounts invested in assets: π

$\rightarrow$ amount in money market: $X - \pi'\mathbf{1}$

- **Wealth process:**

$$\boxed{dX_t = r_t X_t dt + \pi_t' \sigma_t \left(\theta_t dt + dW_t \right), \text{ subject to } X_0 = x.}$$

- **Admissibility:**

$\rightarrow$ π admissible ($\pi \in \mathcal{A}$) if and only if wealth non-negative: $X \geq 0$.

- Asset Allocation Problem:
  - **Investor maximizes expected utility of terminal wealth:**

$$\boxed{\max_{\pi\in\mathcal{A}} \mathbf{E}\left[U(X_T)\right]}$$

  - **Utility function:** $U : \mathbb{R}_+ \to \mathbb{R}$

  $\rightarrow$ Strictly increasing, strictly concave and differentiable

  $\rightarrow$ Inada conditions: $\lim_{X\to\infty} U'(X) = 0$ and $\lim_{X\to 0} U'(X) = \infty$

  $\rightarrow$ **Example: CRRA** $U(x) = \frac{1}{1-R} X^{1-R}$ where $R > 0$.

  - **Property:**

  $\rightarrow$ Strictly decreasing marginal utility in $(0, \infty)$

  $\rightarrow$ Inverse marginal utility $I\left(y\right)$ exists and satisfies $U'\left(I\left(y\right)\right) = y$

  $\rightarrow$ Derivative: $I'(y) = 1/U''(I(y))$

  - **Variation:** $U : [A, +\infty) \to \mathbb{R}$

  $\rightarrow$ Strictly increasing, strictly concave and differentiable

  $\rightarrow$ Inada conditions: $\lim_{X\to\infty} U'(X) = 0$ and $\lim_{X\to A} U'(X) = \infty$

  $\rightarrow$ **Example: HARA** $U(x) = \frac{1}{1-R}(X - A)^{1-R}$ where $R > 0$, $A > 0$.

3 The Optimal Portfolio

- Complete Markets:
  - Market price of risk: $\theta_t = (\theta_{1t}, ..., \theta_{dt})'$
  - **State price density:**

$$\boxed{\xi_v \equiv \exp\left(-\int_0^v \left(r_s + \frac{1}{2}\theta_s'\theta_s\right) ds - \int_0^v \theta_s' dW_s\right)}$$

$\rightarrow$ converts state-contingent payoffs into values at date 0

  - **Conditional state price density:**

$$\boxed{\xi_{t,v} \equiv \exp\left(-\int_t^v \left(r_s + \frac{1}{2}\theta_s'\theta_s\right) ds - \int_t^v \theta_s' dW_s\right) = \xi_v / \xi_t}$$

- Optimal Portfolio: Ocone and Karatzas (1991), Detemple, Garcia and Rindisbacher (2003)

$$\boxed{\pi_t^* = \pi_t^m + \pi_t^r + \pi_t^\theta}$$

where

MV: $\boxed{\pi_t^m = \mathbf{E}_t\left[\xi_{t,T}\Gamma_T^*\right](\sigma_t')^{-1}\theta_t}$

IRH: $\boxed{\pi_t^r = -(\sigma_t')^{-1}\mathbf{E}_t\left[\xi_{t,T}\left(X_T^* - \Gamma_T^*\right)\int_t^T \mathcal{D}_t r_s ds\right]'}$

MPRH: $\boxed{\pi_t^\theta = -(\sigma_t')^{-1}\mathbf{E}_t\left[\xi_{t,T}\left(X_T^* - \Gamma_T^*\right)\int_t^T (dW_s + \theta_s ds)'\mathcal{D}_t\theta_s\right]'}$

- Optimal terminal wealth $X_T^* = I(y^*\xi_T)$
- Constant y^* solves $x = E\left[\xi_T I(y^*\xi_T)\right]$ (static budget constraint)
- $\Gamma(X) \equiv -U'(X)/U''(X)$: measure of absolute risk tolerance
- $\Gamma_T^* \equiv \Gamma(X_T^*)$: risk tolerance evaluated at optimal terminal wealth
- $\mathcal{D}_t$ is Malliavin derivative

- Structure of Hedges:

$$\text{IRH:} \quad \boxed{\pi_t^r = -\left(\sigma_t'\right)^{-1} \mathbf{E}_t \left[\xi_{t,T} \left(X_T^* - \Gamma_T^*\right) \int_t^T \mathcal{D}_t r_s ds\right]'}$$

- Driven by sensitivities of future IR and MPR to current innovations in W_t. Sensitivities measured by Malliavin derivatives $\mathcal{D}_t r_s$ and $\mathcal{D}_t \theta_s$
- Sensitivities are adjusted by factor $\xi_{t,T} \left(X_T^* - \Gamma_T^*\right)$: depends on preferences, terminal wealth and conditional state prices.
- Optimal terminal wealth: $I(y^* \xi_T)$
- Date t cost: $\xi_{t,T} I(y^* \xi_T) = \xi_{t,T} I(y^* \xi_t \xi_{t,T})$
- Sensitivity to change in conditional SPD $\xi_{t,T}$

$$\boxed{\frac{\partial\left(\xi_{t,T} I(y^* \xi_t \xi_{t,T})\right)}{\partial \xi_{t,T}} = I(y^* \xi_t \xi_{t,T}) + y^* \xi_t \xi_{t,T} I'(y^* \xi_t \xi_{t,T}) = X_T^* - \Gamma_T^*}$$

- Sensitivity of conditional SPD to fluctuations in IR and MPR

$$\boxed{-\xi_{t,T} \int_t^T \mathcal{D}_t r_s ds \quad \text{and} \quad -\xi_{t,T} \int_t^T \left(dW_s + \theta_s ds\right)' \mathcal{D}_t \theta_s.}$$

- Constant Relative Risk Aversion (CRRA)

$$\boxed{\frac{\pi_t^m}{X_t^*} = \frac{1}{R}\left(\sigma_t'\right)^{-1}\theta_t}$$

$$\boxed{\frac{\pi_t^r}{X_t^*} = -\rho\left(\sigma_t'\right)^{-1}\mathbf{E}_t\left[\frac{\xi_T^\rho}{\mathbf{E}_t\left[\xi_T^\rho\right]}\int_t^T \mathcal{D}_t r_s ds\right]'}$$

$$\boxed{\frac{\pi_t^\theta}{X_t^*} = -\rho\left(\sigma_t'\right)^{-1}\mathbf{E}_t\left[\frac{\xi_T^\rho}{\mathbf{E}_t\left[\xi_T^\rho\right]}\int_t^T (dW_s + \theta_s ds)'\mathcal{D}_t\theta_s\right]'}$$

- $\rho = 1 - 1/R$
- $y^* = \left(\mathbf{E}\left[\xi_T^\rho\right]/x\right)^R$
- $X_t^* = \mathbf{E}_t\left[\xi_{t,T}(y^*\xi_T)^{-1/R}\right]$
- Hedging terms are weighted averages of the sensitivities of future interest rates and market prices of risk to the current Brownian innovations.

4 A New Decomposition of the Optimal Portfolio

4.1 Bond Pricing and Forward Measures

- Pure Discount Bond Price: $B_t^T = E_t\left[\xi_{t,T}\right]$
- Forward T-Measure: (Geman (1989) and Jamshidian (1989))
  - **Random variable:**

$$\boxed{Z_{t,T} \equiv \frac{\xi_{t,T}}{E_t\left[\xi_{t,T}\right]} = \frac{\xi_{t,T}}{B_t^T}}$$

  - **Properties:** $Z_{t,T} > 0$ and $E_t\left[Z_{t,T}\right] = 1$. Use $Z_{t,T}$ as density
  - **Probability measure:** $dQ_t^T = Z_{t,T}dP$

  $\rightarrow$ Equivalent to P

- **Change of Numéraire:** unit of account is T-maturity bond
  - Under Q_t^T price $V(t)$ of a contingent claim with payoff Y_T is

$$\boxed{V(t) = E_t[\xi_{t,T} Y_T] = E_t[\xi_{t,T}]\, E_t\left[\frac{\xi_{t,T}}{E_t[\xi_{t,T}]} Y_T\right] = B_t^T E_t^T[Y_T]}$$

  - $E_t^T[\cdot] \equiv E_t[Z_{t,T} \cdot]$ is expectation under Q_t^T
  - **Martingale property:** $V(t)/B_t^T = E_t^T[Y_T] = E_t[Z_{t,T} Y_T]$.
  - Density $Z_{t,T}$ is **stochastic discount factor:** converts future payoffs into current values measured in bond unit of account.

- Characterization (Theorem 2): The forward T-density is given by

$$\boxed{Z_{t,T} \equiv \exp\left(\int_t^T \sigma^Z(s,T)' \, dW_s - \frac{1}{2}\int_t^T \sigma^Z(s,T)' \, \sigma^Z(s,T) \, ds\right)}$$

  - **volatility at** $s \in [t,T]$**:** $\sigma^Z(s,T) \equiv \sigma^B(s,T) - \theta_s$
  - **bond return volatility:** $\sigma^B(s,T)' \equiv \mathcal{D}_s \log B_s^T$

- Contribution(s):
  - Identify **volatility of forward measure**
  - Application of **Exponential Clark-Haussmann-Ocone formula**
  - **Market price of risk in the numéraire**

4.2 Portfolio allocation and long term bonds

- An Alternative Portfolio Decomposition Formula:

$$\boxed{\pi_t^* = \pi_t^m + \pi_t^b + \pi_t^z}$$

  - **Mean variance demand:**

$$\boxed{\pi_t^m = E_t^T \left[\Gamma_T^*\right] B_t^T \left(\sigma_t'\right)^{-1} \theta_t}$$

  - **Hedge** motivated by fluctuations in price of **pure discount bond with matching maturity**

$$\boxed{\pi_t^b = \left(\sigma_t'\right)^{-1} \sigma^B(t,T)\, E_t^T \left[X_T^* - \Gamma_T^*\right] B_t^T}$$

  - **Hedge** motivated by fluctuations in **density of forward** T**-measure**

$$\boxed{\pi_t^z = \left(\sigma_t'\right)^{-1} E_t^T \left[\left(X_T^* - \Gamma_T^*\right) \mathcal{D}_t \log\left(Z_{t,T}\right)\right]' B_t^T.}$$

- Essence of Formula: change of numéraire

- **SPD representation:** $\xi_{t,T} = B_t^T Z_{t,T}$
- **Optimal terminal wealth:** $X_T^* = I\left(y^* \xi_t B_t^T Z_{t,T}\right)$
- **Cost of optimal terminal wealth:** $B_t^T Z_{t,T} I\left(y^* \xi_t B_t^T Z_{t,T}\right)$
- **Hedging portfolio:** $\mathcal{D}_t\left(B_t^T Z_{t,T} I\left(y^* \xi_t B_t^T Z_{t,T}\right)\right)$
- **Chain rule of Malliavin calculus:**

$$\boxed{\rightarrow \left(Z_{t,T} I\left(y^* \xi_t B_t^T Z_{t,T}\right) + B_t^T Z_{t,T} I'\left(y^* \xi_t B_t^T Z_{t,T}\right) y^* \xi_t Z_{t,T}\right) \mathcal{D}_t B_t^T}$$

$$\boxed{\rightarrow \left(B_t^T I\left(y^* \xi_t B_t^T Z_{t,T}\right) + B_t^T Z_{t,T} I'\left(y^* \xi_t B_t^T Z_{t,T}\right) y^* \xi_t B_t^T\right) \mathcal{D}_t Z_{t,T}}$$

$$\boxed{\rightarrow B_t^T Z_{t,T} I'\left(y^* \xi_t B_t^T Z_{t,T}\right) B_t^T Z_{t,T} \mathcal{D}_t\left(y^* \xi_t\right)}$$

- Long Term Bond Hedge:
  - Immunizes against instantaneous fluctuations in return of long term bond with matching maturity date
  - Corresponds to portfolio that maximizes the correlation with long term bond return
  - This portfolio is synthetic asset or maturity matching bond, if exists
- Forward Density Hedge:
  - Immunizes against fluctuations in forward density $Z_{t,T}$ (instantaneous and delayed)
  - Source of fluctuations are bond return volatilities and MPRs: $\sigma^Z(s,T) \equiv \sigma^B(s,T) - \theta_s$
  - $\mathcal{D}_t \sigma^Z(s,T) = \mathcal{D}_t \sigma^B(s,T) - \mathcal{D}_t \theta_s$.

4.3 Constant Relative Risk Aversion

- Hedging Terms are:
  - Hedge motivated by fluctuations in price of pure discount bond with matching maturity

$$\boxed{\frac{\pi_t^b}{X_t^*} = \rho\left(\sigma_t'\right)^{-1}\sigma^B\left(t,T\right)B_t^T}$$

  - Hedge motivated by fluctuations in density of forward T-measure

$$\boxed{\frac{\pi_t^z}{X_t^*} = \rho\left(\sigma_t'\right)^{-1}E_t^T\left[\frac{Z_{t,T}^{\rho-1}}{E_t^T\left[Z_{t,T}^{\rho-1}\right]}\mathcal{D}_t\log\left(Z_{t,T}\right)\right]' B_t^T}$$

- Highlights knife-edge property of log utility (Breeden (1979))
  - Logarithmic investor displays myopia (hedging demands vanish)
  - More (less) risk averse investors will hold (short) portfolio synthesizing long term bond
  - More (less) risk averse investors will hold (short) portfolio that hedges forward density

  $\rightarrow$ portfolio is individual-specific: depends on risk aversion of utility function

4.4 Application: Demand for long term bonds

- Constant relative risk aversion
  - Market model:

→ T-maturity bond is traded. Two assets: stock and LT bond

→ Volatility matrix:

$$\boxed{\sigma_t = \begin{bmatrix} \sigma_{1t}^S & \sigma_{2t}^S \\ \sigma_{1t}^B & \sigma_{2t}^B \end{bmatrix}}$$

- Optimal portfolio:

$$\boxed{\pi_t^m = \frac{1}{R}\frac{X_t^*}{\sigma_{1t}^S\sigma_{2t}^B\theta_{1t} - \sigma_{2t}^S\sigma_{1t}^B}\begin{bmatrix} \sigma_{2t}^B\theta_{1t} - \sigma_{1t}^B\theta_{2t} \\ -\sigma_{2t}^S\theta_{1t} + \sigma_{1t}^S\theta_{2t} \end{bmatrix}}$$

$$\boxed{\pi_t^b = \rho\frac{X_t^*}{\sigma_{1t}^S\sigma_{2t}^B\theta_{1t} - \sigma_{2t}^S\sigma_{1t}^B}\begin{bmatrix} \sigma_{2t}^B\sigma_{1t}^B - \sigma_{1t}^B\sigma_{2t}^B \\ -\sigma_{2t}^S\sigma_{1t}^B + \sigma_{1t}^S\sigma_{2t}^B \end{bmatrix} = \rho X_t^*\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

$$\boxed{\pi_t^z = \rho \frac{X_t^*}{\sigma_{1t}^S \sigma_{2t}^B \theta_{1t} - \sigma_{2t}^S \sigma_{1t}^B} E_t^T \left[\frac{Z_{t,T}^{\rho-1}}{E_t^T \left[Z_{t,T}^{\rho-1} \right]} \left[\begin{array}{c} \sigma_{2t}^B \mathcal{D}_{1t} \log\left(Z_{t,T}\right) - \sigma_{1t}^B \mathcal{D}_{2t} \log\left(Z_{t,T}\right) \\ -\sigma_{2t}^S \mathcal{D}_{1t} \log\left(Z_{t,T}\right) + \sigma_{1t}^S \mathcal{D}_{2t} \log\left(Z_{t,T}\right) \end{array} \right] \right]}$$

- Remark: Typical models in literature $\pi_t^z = 0$ (Gaussian models)

$\longrightarrow$ Bonds-to-equities ratio

$$\boxed{e_t = \left(\frac{-\sigma_{2t}^S \theta_{1t} + \sigma_{1t}^S \theta_{2t}}{\sigma_{2t}^B \theta_{1t} - \sigma_{1t}^B \theta_{2t}} \right) + (R-1) \left(\frac{-\sigma_{2t}^S \sigma_{1t}^B + \sigma_{1t}^S \sigma_{2t}^B}{\sigma_{2t}^B \theta_{1t} - \sigma_{1t}^B \theta_{2t}} \right)}$$

$*$ Increases with risk aversion if second ratio is positive

$*$ Independent of investment horizon

$*$ Independent of wealth

$\longrightarrow$ Explains Asset Allocation Puzzle (Canner, Mankiw, Weil (1997))

$*$ Typical advice: increase BER for more conservative investors

$*$ Mean-variance model: ratio is independent of risk aversion

$*$ Static bond hedge explains the puzzle (Bajeux-Besnainou, Jordan and Portait (2001))

- Wealth-dependent risk aversion HARA:

$$\boxed{u(x) = \frac{(x-A)^{1-R}}{1-R}\mathbf{1}_{x>A} - \infty\mathbf{1}_{x\le A}}$$

  - Gaussian model: $\pi_t^z = 0$
  - Bonds-to-equities ratio

$$\boxed{e_t = \left(\frac{-\sigma_{2t}^S\theta_{1t}+\sigma_{1t}^S\theta_{2t}}{\sigma_{2t}^B\theta_{1t}-\sigma_{1t}^B\theta_{2t}}\right) + \left(\frac{E_t^T\left[X_T^*\right]}{E_t^T\left[\Gamma_T^*\right]} - 1\right)\left(\frac{-\sigma_{2t}^S\sigma_{1t}^B+\sigma_{1t}^S\sigma_{2t}^B}{\sigma_{2t}^B\theta_{1t}-\sigma_{1t}^B\theta_{2t}}\right)}$$

$$\boxed{\frac{E_t^T\left[X_T^*\right]}{E_t^T\left[\Gamma_T^*\right]} = R\left(1 + \frac{AB_t^T}{X_t^*-AB_t^T}\left(\frac{B_0^T}{B_0^t}\right)^\rho\right)}$$

$$\boxed{\frac{AB_t^T}{X_t^*-AB_t^T} = \left(\frac{AB_0^t}{x-AB_0^T}\right)h\left(t\right)\left(B_t^T Z_t\right)^{1/R}}$$

$$\boxed{h(t) \equiv \exp\left(\frac{\rho}{R}\int_0^t\left(\frac{1}{2}\|\theta_s + \sigma^B(s,T)\|^2 - \|\sigma^B(s,T)\|^2\right)ds\right)}$$

- Changes in risk aversion imply:

→ Direct relative risk aversion effect: outside power R increases BER

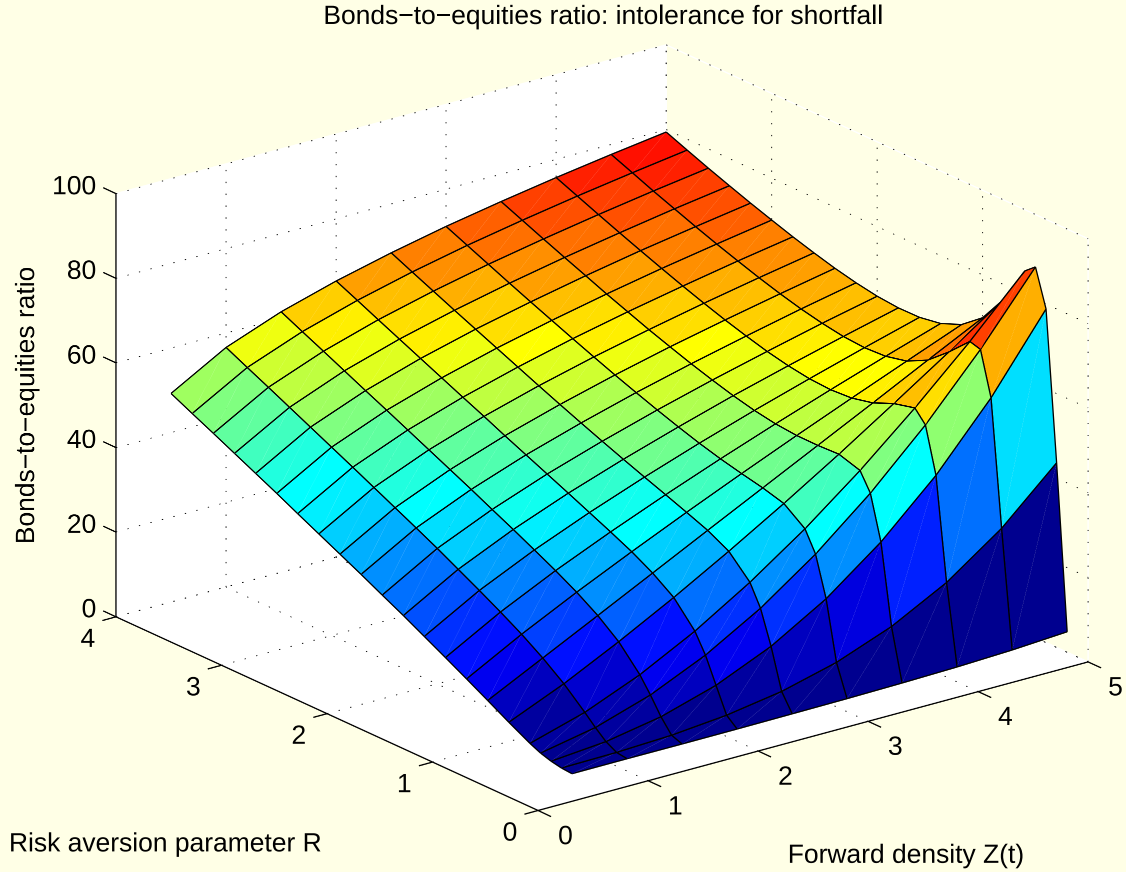
→ Endogenous wealth effect: direction depends on

$$\boxed{h(t)\left(B_t^T Z_t\right)^{1/R}}$$

- nonlinear effects - reduces BER if wealth increases

  * Reduction in dispersion of optimal terminal wealth: consumption smoothing across states
  * Cost of optimal terminal wealth can increase or decrease
  * Budget constraint effect: decreases or increases multiplier $y^{-1/R}$ to satisfy budget (opposite direction)
  * Net effect on wealth at date t can be positive or negative

- Graph illustrates the possibility of a decrease in BER: negative wealth effect dominates in certain regions



$\longrightarrow$ Vasicek interest rate model: $r_0 = \bar{r} = 0.06$, $\kappa_r = 0.05$, $\sigma_{r1} = -0.02$, $\sigma_{r2} = -0.015$ and market prices of risk are constants $\theta_s = 0.3$ and $\theta_b = 0.15$. The interest rate at $t = 5$ is $r_t = 0.02$. Other parameter values are $A = 200,000$, $x = 100,000$ and $T = 10$.

5 Intermediate Consumption

5.1 The Investor's Preferences

▶ Consumption-portfolio Problem:

$$\boxed{\max_{\pi, c \in \mathcal{A}} \mathbf{E}\left[\int_0^T u\left(c_t, t\right) dt + U(X_T)\right]}$$

- **Utility function:** $u\left(\cdot, \cdot\right) : \mathbb{R}_+ \times [0, T] \to \mathbb{R}$ and **bequest function:** $U : \mathbb{R}_+ \to \mathbb{R}$ satisfy standard assumptions
- Maximization over set of admissible portfolio policies $\pi, c \in \mathcal{A}$
- Inverse marginal utility function $J\left(y, t\right)$ exists: $u'\left(J\left(y, t\right), t\right) = y$ for all $t \in [0, T]$
- Inverse marginal bequest function $I\left(y\right)$ exists: $U'\left(I\left(y\right)\right) = y$

5.2 Portfolio Representation and Coupon-paying Bonds

- Decomposition:

$$\boxed{\pi_t^* = \pi_t^m + \pi_t^b + \pi_t^z}$$

  - **Mean variance demand:**

$$\boxed{\pi_t^m = \left(\int_t^T E_t^v\left[\Gamma_v^*\right] B_t^v dv + E_t^T\left[\Gamma_T^*\right] B_t^T\right)\left(\sigma_t'\right)^{-1}\theta_t}$$

  - **Hedge** motivated by fluctuations in price of **coupon-paying bond with matching maturity:**

$$\boxed{\begin{aligned}\pi_t^b \quad = \quad & \left(\sigma_t'\right)^{-1}\int_t^T \sigma^B(t,v)\, B_t^v E_t^v\left[c_v^* - \Gamma_v^*\right] dv \\ & + \left(\sigma_t'\right)^{-1}\sigma^B(t,T)\, B_t^T E_t^T\left[X_T^* - \Gamma_T^*\right]\end{aligned}}$$

  - **Hedge** motivated by fluctuations in **density of forward** T**-measure**:

$$\boxed{\begin{aligned}\pi_t^z \quad = \quad & \left(\sigma_t'\right)^{-1}\left(\int_t^T E_t^v\left[\left(c_v^* - \Gamma_v^*\right)\mathcal{D}_t \log Z_{t,v}\right] B_t^v dv\right)' \\ & + \left(\sigma_t'\right)^{-1}\left(E_t^T\left[\left(X_T^* - \Gamma_T^*\right)\mathcal{D}_t \log Z_{t,T}\right] B_t^T\right)'\end{aligned}}$$

- Static Hedge π_t^b: against fluctuations in value of coupon-paying bond
  - **Coupons** $\boxed{C(v) \equiv E_t^v \left[c_v^* - \Gamma_v^*\right]}$ at intermediate dates $v \in [0, T)$
  - **Bullet payment** $\boxed{F \equiv E_t^T \left[X_T^* - \Gamma_T^*\right]}$ at terminal date T
  - Coupon payments and face value are
    - $\rightarrow$ time-varying
    - $\rightarrow$ tailored to individual's consumption profile and risk tolerance
  - **Bond value**

$$\boxed{B(t, T; C, F) \equiv \int_t^T B_t^v C(v)\, dv + B_t^T F.}$$

  - **Instantaneous volatility**

$$\boxed{\begin{aligned} \sigma(B(t, T; C, F))\, B(t, T; C, F) \quad = \quad & \int_t^T \sigma^B(t, v)\, B_t^v C(v)\, dv \\ & + \sigma^B(t, T)\, B_t^T F \end{aligned}}$$

  - **Hedge:** $\boxed{(\sigma_t')^{-1}\, \sigma(B(t, T; C, F))\ B(t, T; C, F)}$

- Forward Density Hedge π_t^z:
  - Motivation: desire to hedge **fluctuations in forward densities** $Z_{t,v}$
  - Static hedge already neutralizes impact of term structure fluctuations on PV of future consumption
  - Given $\xi_{t,v} = B_t^v Z_{t,v}$ it remains to hedge **fluctuations in discount factor in new numéraire** $Z_{t,v}, v \in [t, T]$.
- Optimal Portfolio Composition:
  - To first approximation portfolio has mean-variance term + long term coupon-bond hedge
  - Under what conditions is this approximation exact (i.e. last term vanishes)?
  - If last term does not vanish what is its size?

5.3 Constant Relative Risk Aversion

- Relative risk aversion parameters R_u, R_U for utility and bequest functions. Portfolio:
  - Mean-variance term

$$\boxed{\pi_t^m = (\sigma_t')^{-1} \left(\int_t^T \frac{1}{R_u} E_t^v \left[c_v^* \right] B_t^v dv + \frac{1}{R_U} E_t^T \left[X_T^* \right] B_t^T \right) \theta_t}$$

  - Hedge motivated by fluctuations in price of coupon-paying bond with matching maturity

$$\boxed{\pi_t^b = (\sigma_t')^{-1} \left(\rho_u \int_t^T \sigma^B (t, v) B_t^v E_t^v \left[c_v^* \right] dv + \rho_U \sigma^B (t, T) B_t^T E_t^T \left[X_T^* \right] \right)}$$

  - Hedge motivated by fluctuations in densities of forward measures

$$\boxed{\begin{aligned} \pi_t^z \quad = \quad & \rho_u (\sigma_t')^{-1} \int_t^T E_t^v \left[c_v^* \mathcal{D}_t \log Z_{t,v} \right]' B_t^v dv \\ & + \rho_U (\sigma_t')^{-1} E_t^T \left[X_T^* \mathcal{D}_t \log Z_{t,T} \right]' B_t^T \end{aligned}}$$

- Static Hedge has two parts:
  - Pure coupon bond (annuity) with coupon given by optimal consumption
  - Bullet payment given by optimal terminal wealth
  - Two parts are weighted by risk aversion factors ρ_u and ρ_U
  - Knife edge property traditionally associated with power utility.
  - Possibility of positive annuity hedge ($R_u > 1$) combined with negative bequest hedge ($R_U < 1$).
- Literature: special cases of this result analyzed by
  - Munk and Sörensen (2004)
    - → CRRA with homogeneous risk aversion coefficients $R_u = R_U \equiv R$.
    - → Portfolio decomposition $\pi_t^m + \pi_t^Q$
      - $\pi_t^Q = (\sigma_t')^{-1}\sigma_t^Q$
      - Hedge against fluctuations in wealth-to-consumption ratio
      - σ_t^Q in terms of unknown volatility function (invoke MRT)

6 Applications

6.1 Preferred Habitats and Portfolio Choice

- Preferred Habitat Theory Modigliani and Sutch (1966):
  - Individuals exhibit preference for securities with maturities matching their investment horizon
  - Investor who cares about terminal wealth should invest in bonds with matching maturity
  - Existence of group of investors with common investment horizon might lead to increase in demand for bonds in this maturity range
  - Implies increase in bond prices and decrease in yields. Explains hump-shaped yield curves with decreasing profile at long maturities.

- Formula shows that optimal behavior naturally induces a demand for certain types of bonds in specific maturity ranges

$$\boxed{\pi_t^* = w_t^m \left(X_t^* - B\left(t, T; C, F\right)\right) + w_t^b B\left(t, T; C, F\right) + \pi_t^z}$$

$$\boxed{w_t^m = \arg\max_w \left\{w'\sigma_t\theta_t : w'\sigma_t\sigma_t'w = k\right\}.}$$

$$\boxed{w_t^b = \arg\max_w \left\{w'\sigma_t\sigma\left(B\left(t, T; C\right)\right) : w'\sigma_t\sigma_t'w' = k\right\}}$$

$$\boxed{\pi_t^z = \arg\max_\pi \left\{\pi'\sigma_t\widehat{\sigma}\left(t, T\right) : \pi'\sigma_t\sigma_t'\pi' = k\right\}}$$

where

$$\boxed{\begin{aligned} \widehat{\sigma}_{t,T}' \quad \equiv \quad & \textstyle\int_t^T E_t^v\left[\left(c_v^* - \Gamma_v^*\right)\mathcal{D}_t \log Z_{t,v}\right] B_t^v dv \\ & + E_t^T\left[\left(X_T^* - \Gamma_T^*\right)\mathcal{D}_t \log Z_{t,T}\right] B_t^T \end{aligned}}$$

- **Any individual has preferred bond habitat:**
  - → Optimal portfolio includes long term bond with maturity date matching the investor's horizon
  - → Preferred instrument is coupon-paying bond with payments tailored to consumption profile of investor
- **Complemented by mean-variance efficient portfolio to constitute static component of allocation**
- Under general market conditions static policy is fine-tuned by dynamic hedge
  - → When bond return volatilities and market prices of risk are deterministic, dynamic hedge vanishes

- Motivation for preferred habitat here is different from Riedel (2001)
  - In his model habitat preferences are driven by structure of subjective discount rates placing emphasis on specific future dates
  - In our setting preference for long term bonds emerges from the structure of the hedging terms
  - Optimal hedging combines static hedge (long term bond) with dynamic hedge motivated by fluctuations in forward measure volatilities

6.2 Universal Fund Separation

- Non-Markovian fund separation
  - Assumptions:

→ N State variables with path-dependent evolution $(N < d)$

$$\boxed{dY_t = \mu(Y_{(\cdot)})_t dt + \sigma(Y_{(\cdot)})_t dW_t}$$

→ $\boxed{B_t^v = B\left(t, v, Y_{(\cdot)}\right)}$

→ $\boxed{\sigma^Z(t, v) = \sigma^Z\left(t, v, Y_{(\cdot)}\right)}$

$\ast$ Path-dependent functionals.

$\ast$ Fréchet differentiable.

- Universal $N+2$-fund separation holds: portfolio demands can be synthesized by investing in $N+2$ *(preference free) mutual funds*:

→ Riskless asset

→ Mean-variance efficient portfolio

→ N mutual funds $(\sigma_t')^{-1} \sigma_t^Y \left(Y_{(\cdot)}\right)'$ to synthesize the static bond hedge and the forward density hedge.

6.3 Extreme Behavior

- Assume risk tolerances go to zero:
  - **Intermediate utility and bequest functions:**

$$\boxed{(\Gamma_u(z,v),\Gamma_U(z)) \to (0,0) \text{ for all } z \in [0,+\infty) \text{ and all } v \in [0,T]}$$

  - **Relative behaviors:** for some constant $k \in [0,+\infty)$:

$$\boxed{\frac{\Gamma_u(z_1,v)}{\Gamma_U(z_2)} \to k \quad \text{for all } z_1, z_2 \in [0,\infty) \text{ and all } v \in [0,T]}$$

$$\boxed{\frac{\Gamma_u(z_1,v_1)}{\Gamma_u(z_2,v_2)} \to 1 \quad \text{for all } z_1, z_2 \in [0,\infty) \text{ and all } v_1, v_2 \in [0,T]}$$

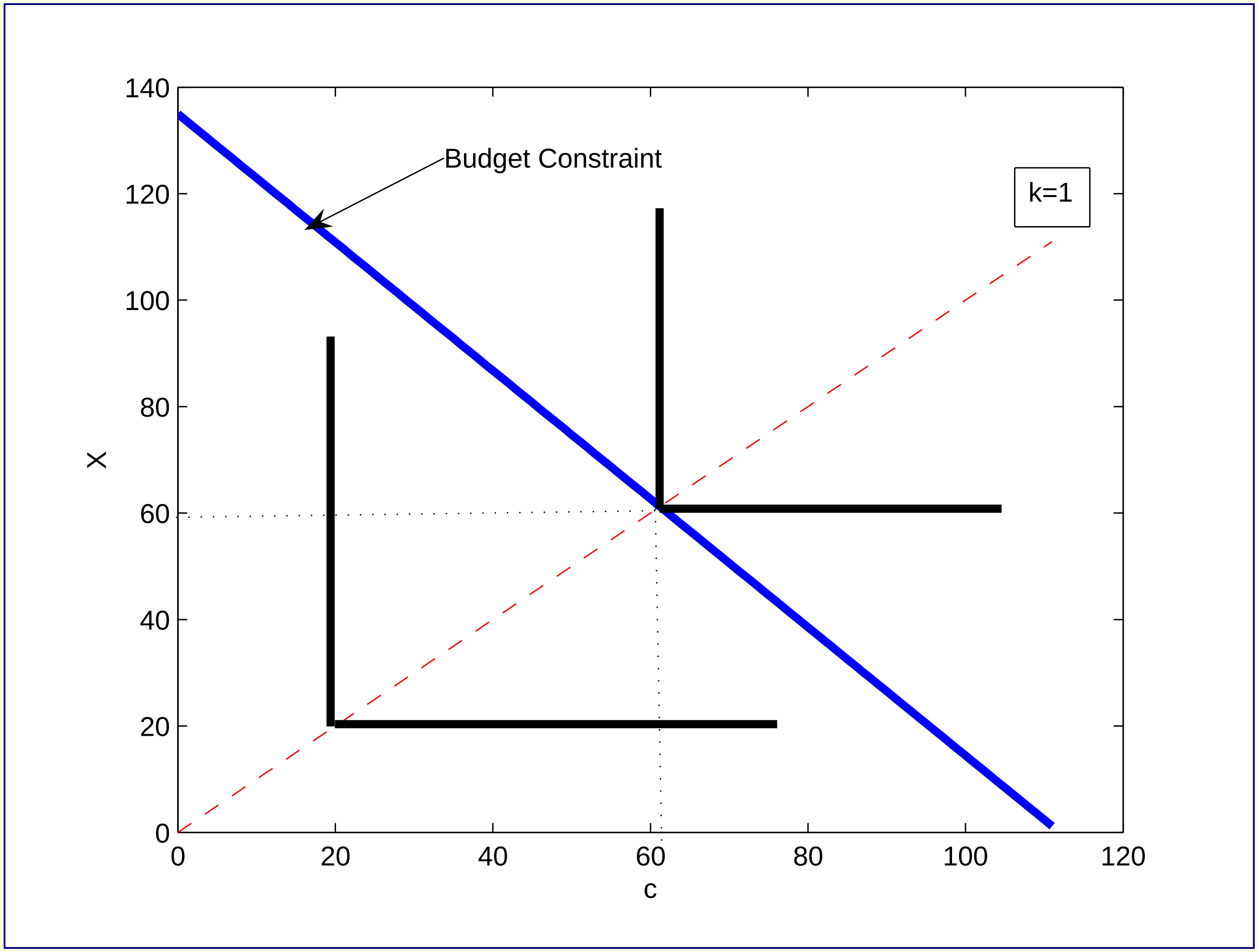
- Limit Allocations: coupon-paying bond with constant coupon C and face value F given by

$$\boxed{C = \frac{x}{\int_0^T B_0^v dv + B_0^T/k} \qquad \text{and} \qquad F = \frac{x}{\int_0^T B_0^v dv k + B_0^T}.}$$

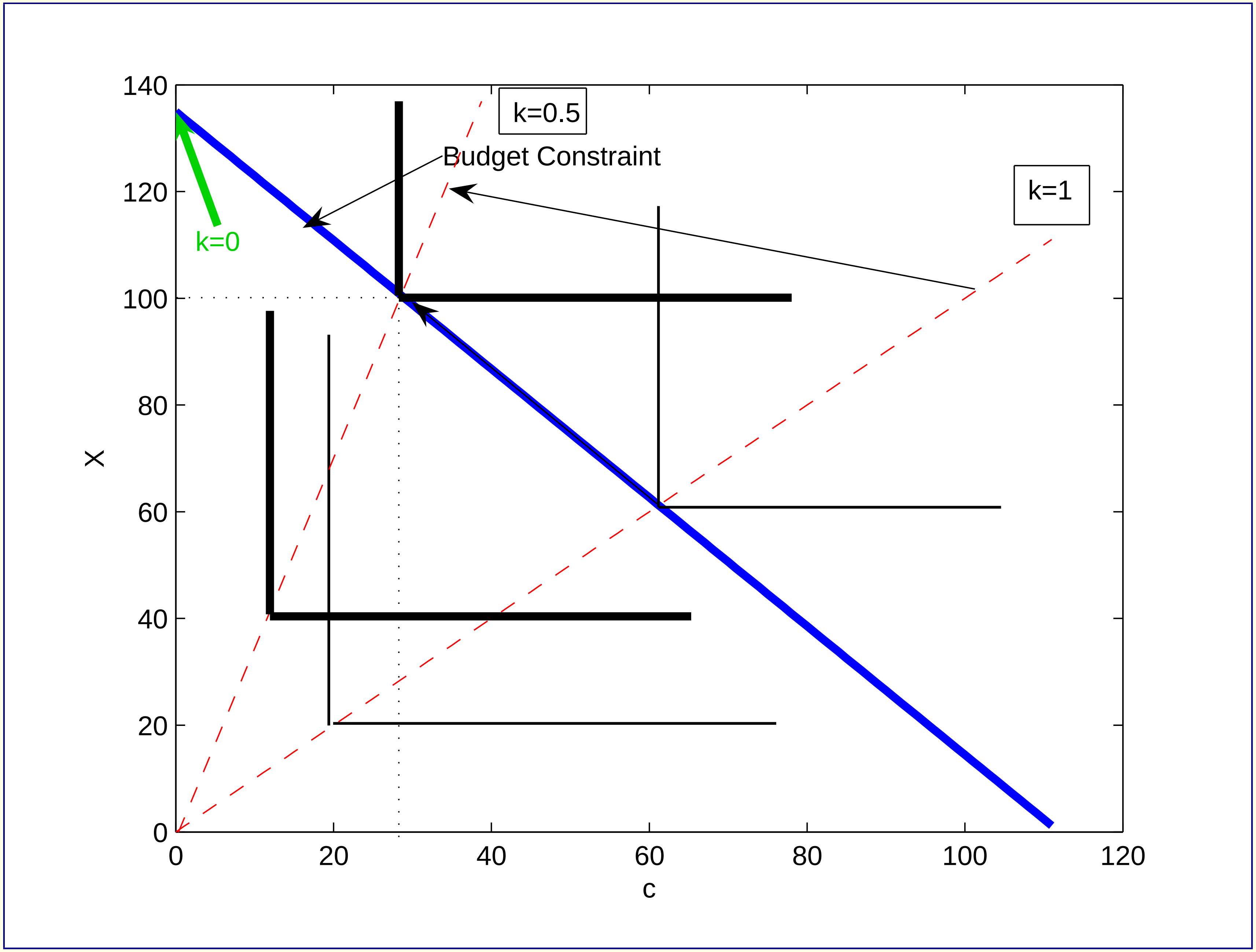
  - If $k = 0$ exclusive preference for pure discount bond, $(C, F) = \left(0, x/B_0^T\right)$
  - If $k \to \infty$ preference is for a pure coupon bond, $(C, F) = \left(x/\int_0^T B_0^v dv, 0\right)$

- Limit Behavior:
  - Governed by relation between utility functions at different dates
  - As risk tolerances vanish, preference for certainty: coupon-paying bond with bullet payment
  - Least extreme of the extreme behaviors drives the habitat:
    - → Given a preference for riskless instruments: individuals puts more weight on maturities where risk tolerance is greater
    - → Exhibits a time preference in the limit.

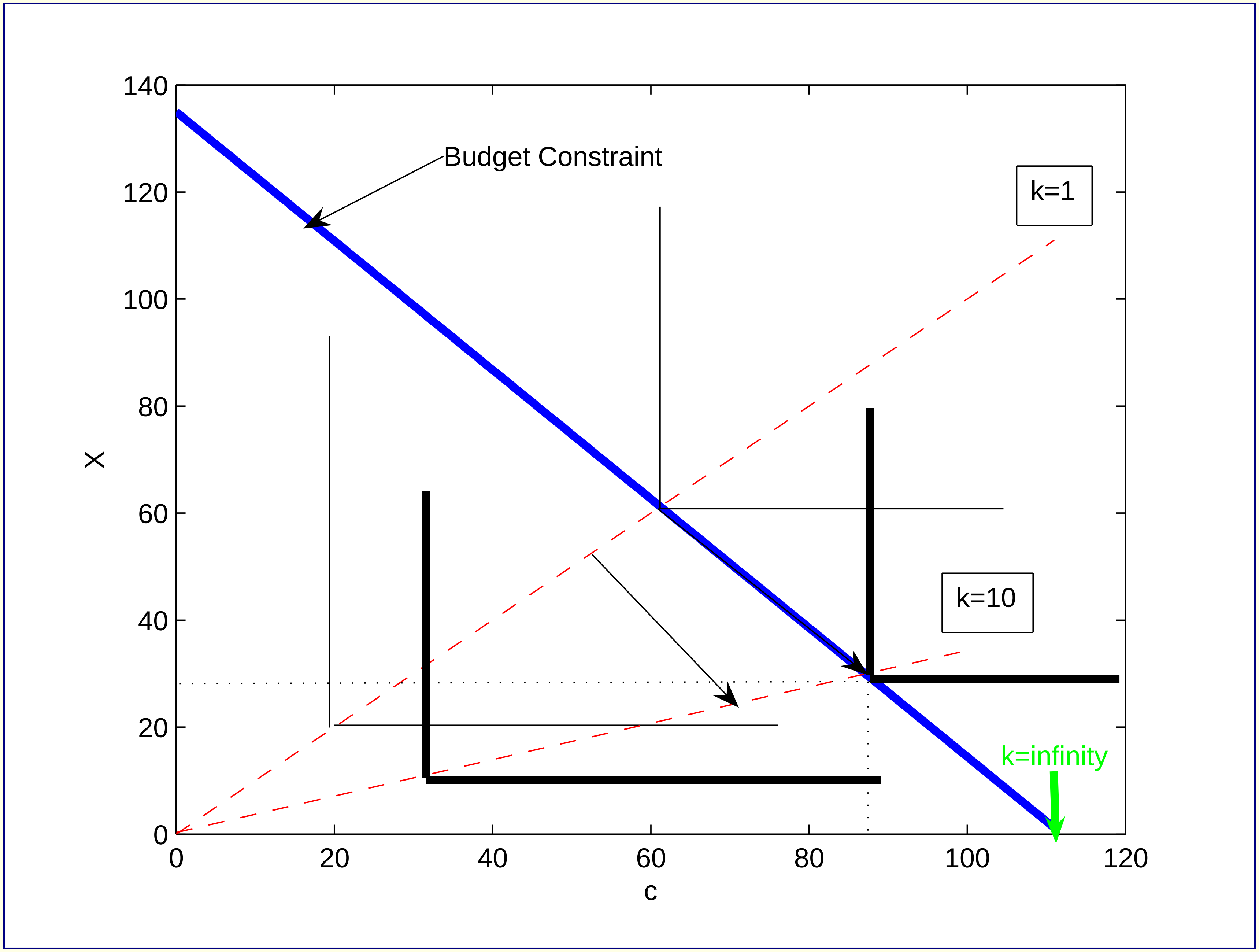
- **Illustration:** CARA preferences Γ_u and Γ_U constant, $k \equiv \Gamma_u / \Gamma_U$.
  - Slope of indifference curves: $\boxed{-\frac{dX}{dc} = \frac{1}{k}\left(e^{X-c/k}\right)^{\frac{1}{\Gamma_U}}}$



- $k \to 0$



- $k \longrightarrow \infty$



- Special case examined by Wachter (2002)
  - Arbitrary utility functions over terminal wealth and markets with general coefficients
  - Documents emergence of preferred habitat when relative risk aversion goes to infinity

    → Pure discount bond with unit face value and matching maturity
  - Our analysis shows that preferred habitat for an extreme consumer may take different forms depending on nature of behavior

    → Pure discount bonds, pure annuities or coupon-paying bonds with bullet payments at maturity can emerge in limit.

▶ Order of Convergence

- As $(\Gamma_u(z,v), \Gamma_U(z)) \to (0,0)$, the limit portfolios

$\to$ $\overline{\pi}_t^m = \overline{\pi}_t^z = 0$

$\to$ $\overline{\pi}_t^b = (\sigma_t') \int_0^T \sigma^B(t,v) B_t^v dv C + \sigma^B(t,T) B_t^T F$

- have scaled asymptotic errors:

$\to$ $\epsilon_t^\alpha(\nu) = (\Gamma_\nu(\cdot))^{-1} (\pi_t^\alpha - \overline{\pi}_t^\alpha)$ with $\alpha \in \{m, b, z\}$ and $\nu \in \{u, U\}$,

$$\boxed{\begin{array}{lcr} \left[\epsilon_t^m(U), \epsilon_t^m(u)\right] & \to & (\sigma_t')^{-1} \theta_t \left[\int_t^T B_t^v dv \; B_t^T\right] \mathcal{K} \\ \left[\epsilon_t^b(U), \epsilon_t^b(u)\right] & \to & -(\sigma_t')^{-1} \left[\int_t^T \sigma^B(t,v) B_t^v dv \; \sigma^B(t,T) B_t^T\right] \mathcal{K} \\ \left[\epsilon_t^z(U), \epsilon_t^z(u)\right] & \to & -(\sigma_t')^{-1} \left[\int_t^T N_{t,v} B_t^v dv \; N_{t,T} B_t^T\right] \mathcal{K} \end{array}}$$

- where

$\to$ $\boxed{N_{t,\tau} \equiv E_t^\tau \left[\left(\int_t^\tau \sigma^Z(r,\tau)' dW_r - \frac{1}{2} \int_t^\tau \|\sigma^Z(r,\tau)\|^2 dr \right) (\mathcal{D}_t \log Z_{t,\tau})' \right]}$

$\to$ $\boxed{\mathcal{K} \equiv \begin{bmatrix} k & 1 \\ 1 & \frac{1}{k} \end{bmatrix}}$

6.4 Term structure models and asset allocation

► Integration of term structure models and asset allocation models:

- Forward rate representation of bonds

$$\boxed{B_t^v = \exp\left(-\int_t^v f_t^s ds\right)}$$

→ Continuously compounded forward rate: $f_t^s \equiv -\frac{\partial}{\partial v} \log\left(B_t^v\right)$

→ Bond price volatility:

$$\boxed{\sigma^B(t,v)' = \mathcal{D}_t \log B_t^v = -\int_t^v \mathcal{D}_t f_t^s ds = -\int_t^v \sigma^f(t,s) ds}$$

→ Volatility of forward rate: $\sigma^f(t,s)$

- Forward rate dynamics:

→ No arbitrage condition (HJM (1992)):

$$\boxed{df_t^v = \sigma^f(t,v)\left(dW_t + \left(\theta_t - \sigma^B(t,v)\right)dt\right), \quad f_0^v \text{ given}}$$

→ Dynamics completely determined by forward rate volatility function and initial forward rate curve

▶ Optimal Portfolio: previous formula with

$$\boxed{\mathcal{D}_t \log Z_{t,v} = \int_t^v \left(dW_s + \left(\theta_s + \int_s^v \sigma^f(s,u)du\right) ds\right)' \left(\mathcal{D}_t\theta_s + \int_s^v \mathcal{D}_t\sigma^f(s,u)du\right)}$$

- Forward density hedge in terms of forward rate volatilities
- Useful for financial institution using a specific HJM model to price/hedge fixed income instruments and their derivatives
- Implied forward rates inferred from term structure model and observed prices

  $\longrightarrow$ estimate volatility function $\sigma^f(s,u)$

  $\longrightarrow$ feed into asset allocation formula
- Simple integration of fixed income management and asset allocation.

- Forward Density Hedge:
  - Immunization demand due to fluctuations in future market prices of risk and forward rate volatilities
  - Vanishes if deterministic forward rate volatilities $\sigma^f(s,u)$ and market prices of risk θ_s
  - Pure expectation hypothesis holds under forward measure: $f(t,v) = E_t^v[r_v]$

  $\rightarrow$ Standard version of PEH ($f(t,v) = E_t[r_v]$) fails when $Z_{t,v} \neq 1$

  $\rightarrow$ Density process $Z_{t,v}$ measures deviation from PEH

  $\rightarrow$ Malliavin derivative $\mathcal{D}_t \log Z_{t,v}$ captures sensitivity of deviation with respect to shocks

  $\rightarrow$ Dynamic hedge = hedge against deviations from PEH

  $\rightarrow$ If $Z_{t,v} = 1$ PEH holds under the original beliefs and hedging becomes irrelevant

  $\rightarrow$ If σ^Z deterministic, deviations from PEH are non-predictable and do not need to be hedged

- Literature:
  - Gaussian models: Merton (1974), Vasicek (1977), Hull and White (1990), Brace, Gatarek and Musiela (1997)
  - Extensively employed in practice
  - Forward rate volatilities σ^f are insensitive to shocks. If MPR also deterministic no need to hedge
  - Bajeux-Besnainou, Jordan and Portait (2001) also falls in this category (one factor Vasicek)

▶ Numerical Results: Forward measure hedges in one factor CIR model

- CIR interest rates:

$$\boxed{dr_t = \kappa_r(\bar{r} - r_t))dt + \sigma_r\sqrt{r}dW_t; \quad r_0 = r}$$

$\longrightarrow$ Parameter values (Durham (JFE, 2003)):

- $\kappa_r = 0.002$
- $\bar{r} = 0.0497$
- $\sigma_r = -0.0062$
- $r = 0.06$

- Market price of risk:

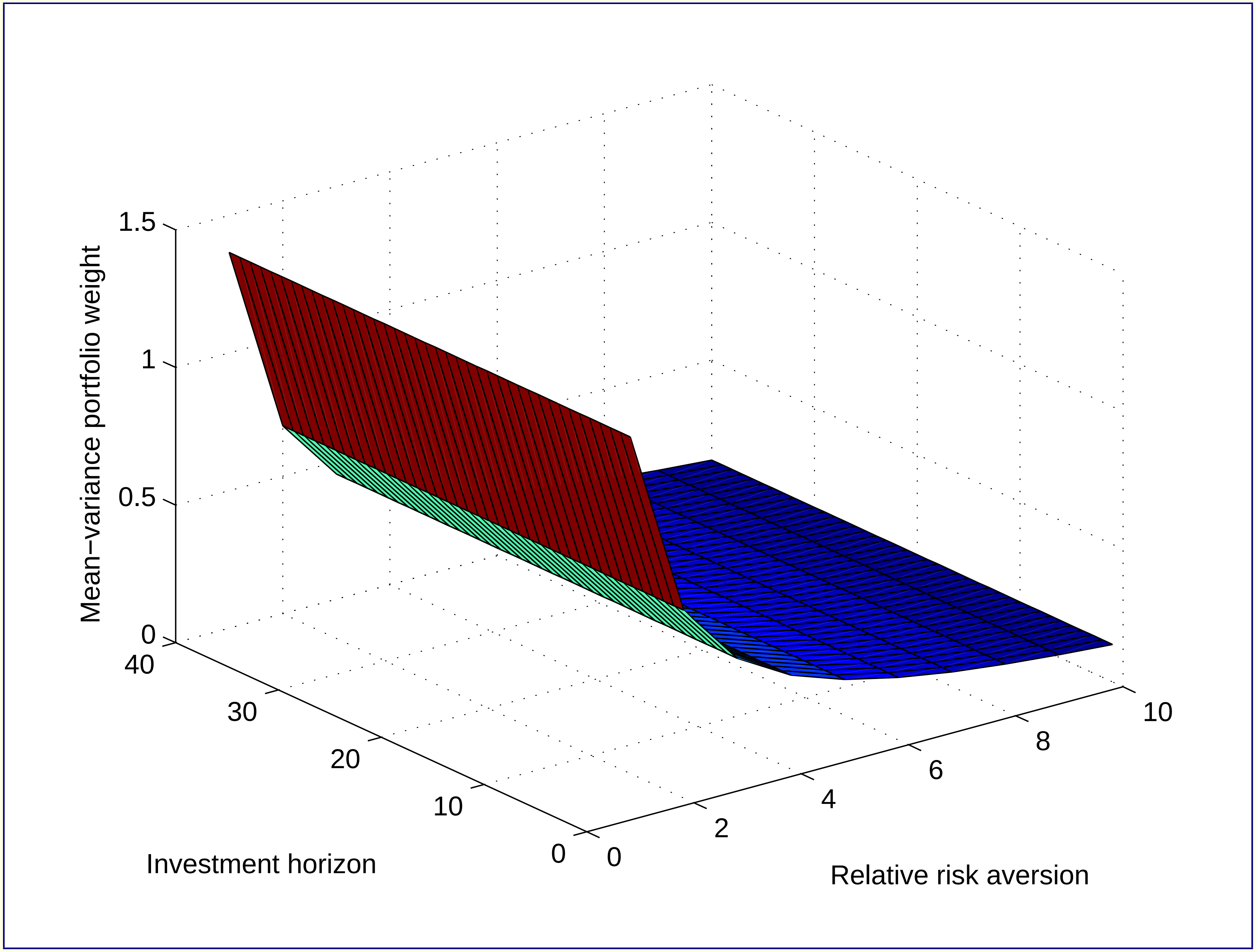
$$\boxed{\theta_t = \gamma_r\sqrt{r_t}}$$

$\longrightarrow$ Parameter values:

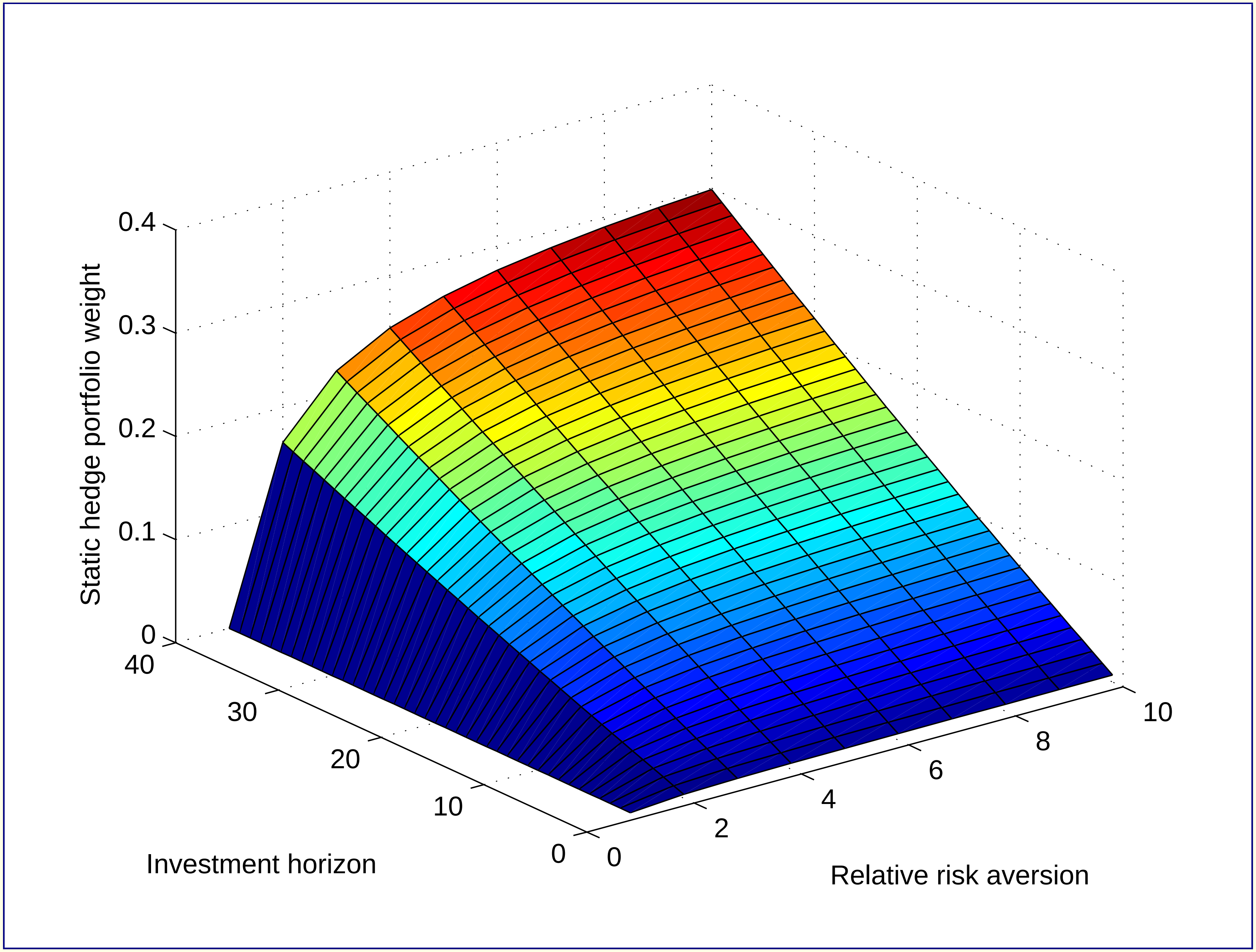
- $\gamma_r = 0.3/\sqrt{\bar{r}}$ such that $\bar{\theta} = \gamma_r\sqrt{\bar{r}} = 0.3$

- CRRA preferences for terminal wealth

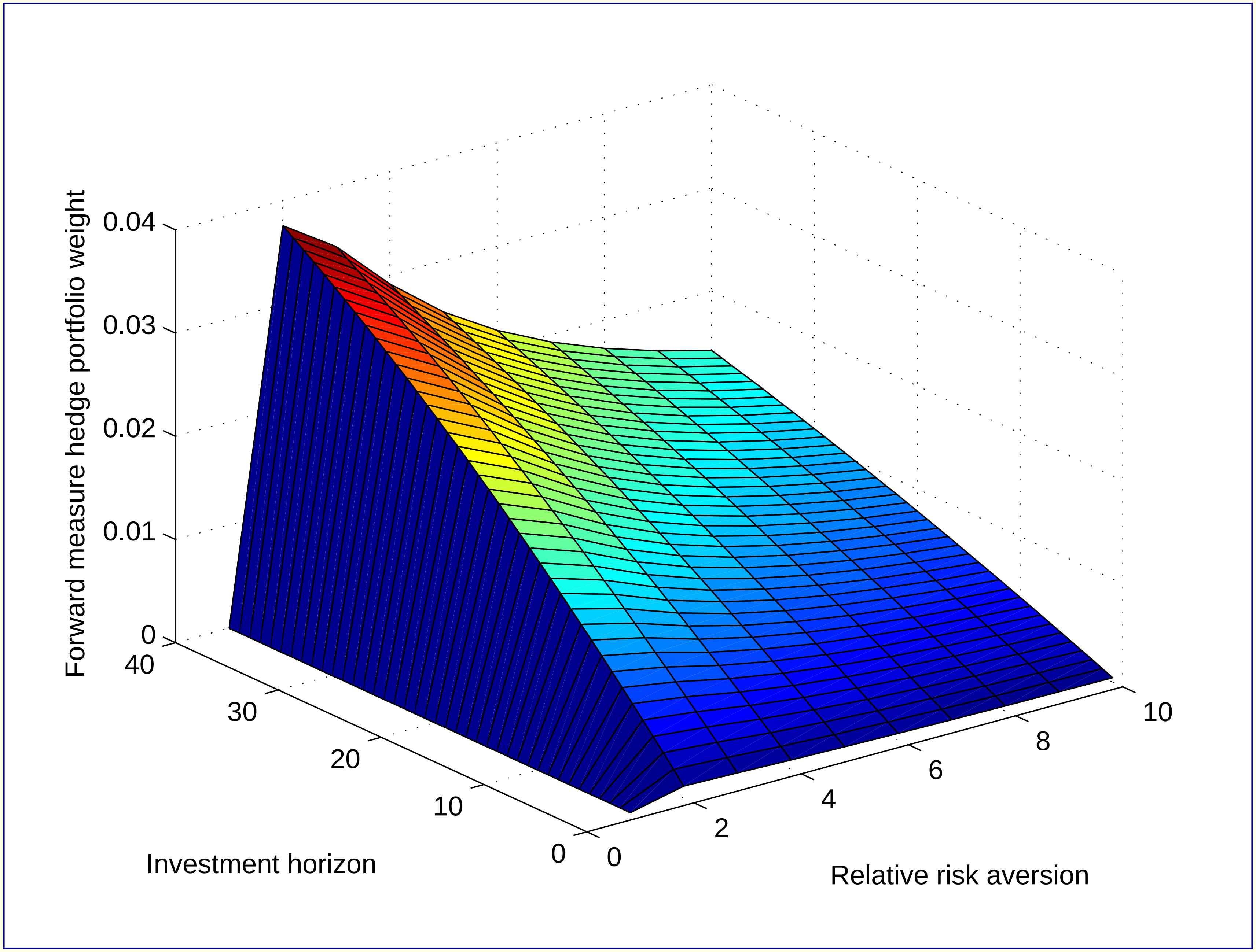
- Mean-variance demand: $\boxed{\pi_t^{mv}/X_t^* = \frac{1}{R}(\sigma_t')^{-1}\theta_t}$



- Static term structure hedge: $\boxed{\pi_t^b / X_t^* = \rho(\sigma_t')^{-1}\sigma^B(t,T)}$

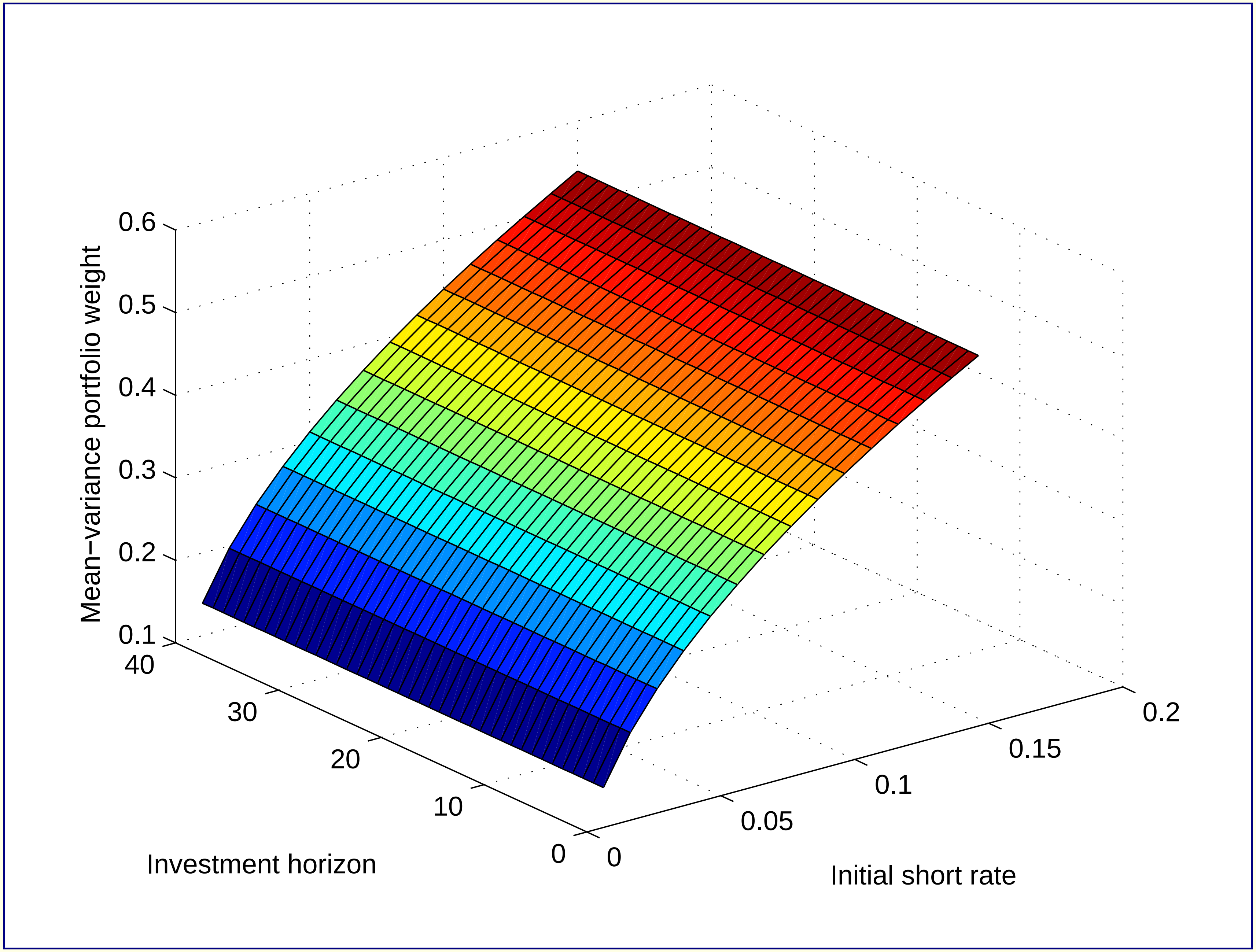


- Dynamic forward measure hedge: $\pi_t^z / X_t^* = \rho\,(\sigma_t')^{-1}\,\mathbf{E}_t^T\left[\frac{Z_{t,T}^{\rho-1}}{\mathbf{E}_t^T\left[Z_{t,T}^{\rho-1}\right]}\,(\mathcal{D}_t \log Z_{t,T})'\right]$

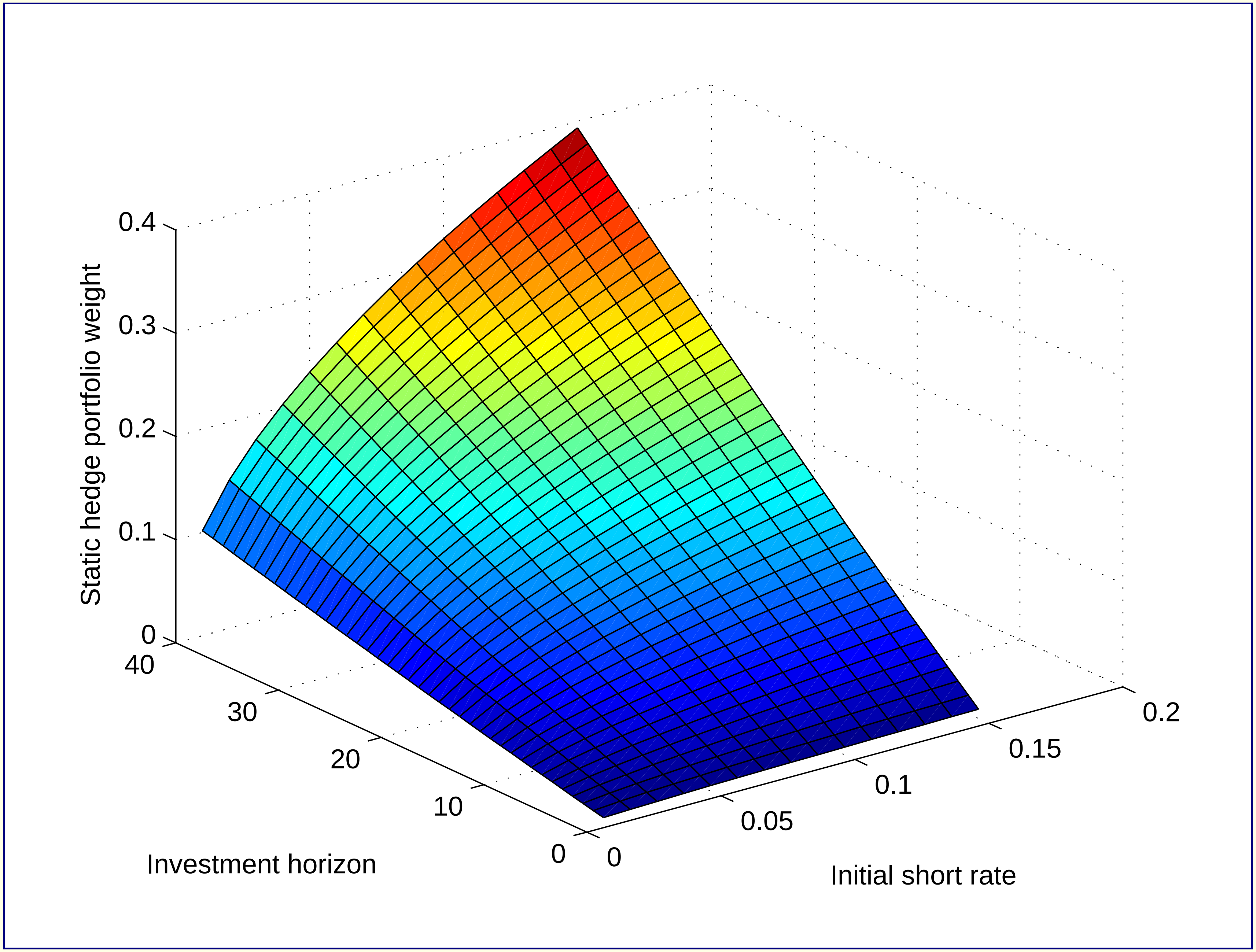


- Changing initial interest rate: Relative risk aversion fixed at $R = 4$

$\rightarrow$ Mean-variance demand: $\boxed{\pi_t^{mv}/X_t^* = \frac{1}{R}(\sigma_t')^{-1}\theta_t}$

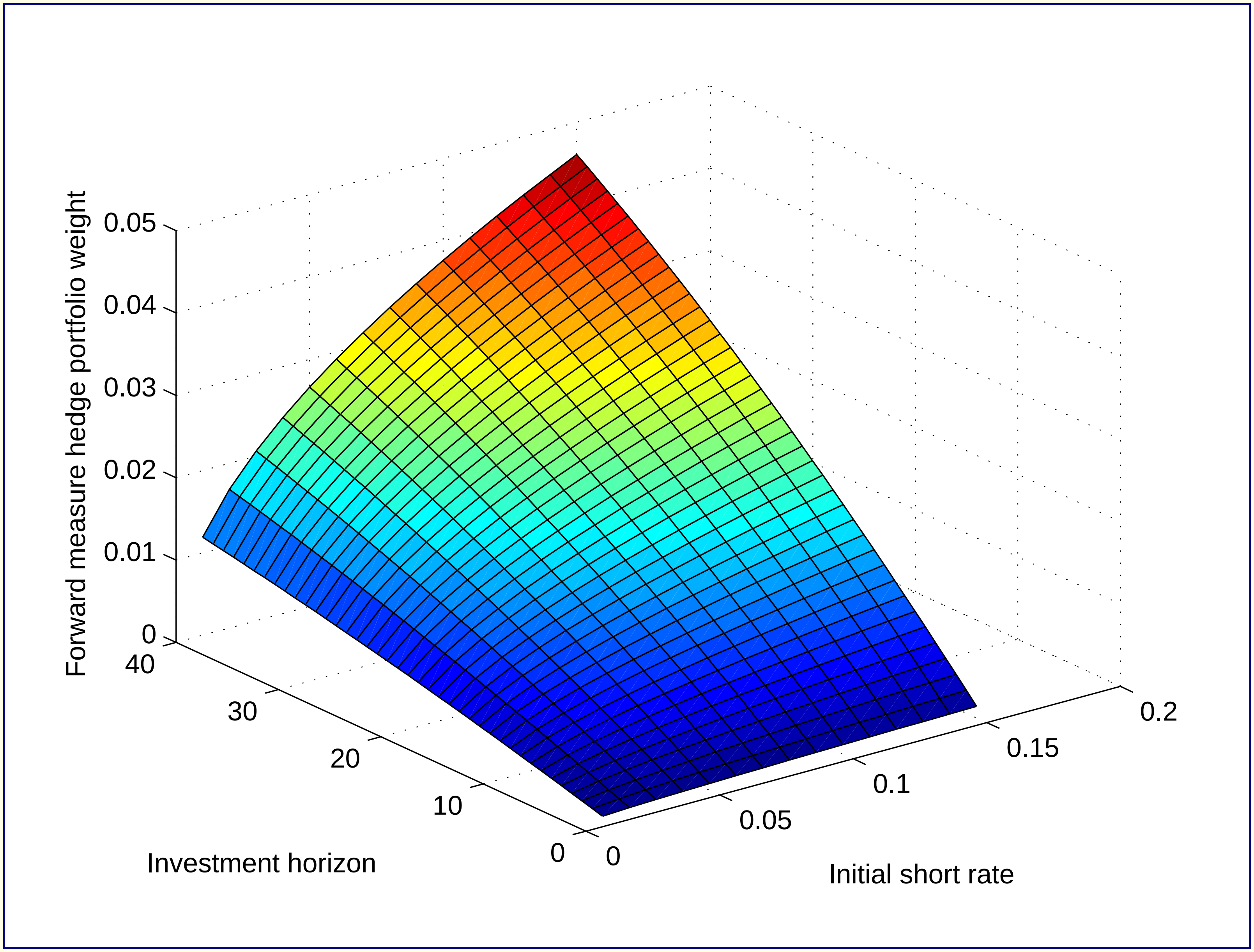


→ Static term structure hedge: $\boxed{\pi_t^b / X_t^* = \rho(\sigma_t')^{-1} \sigma^B(t, T)}$



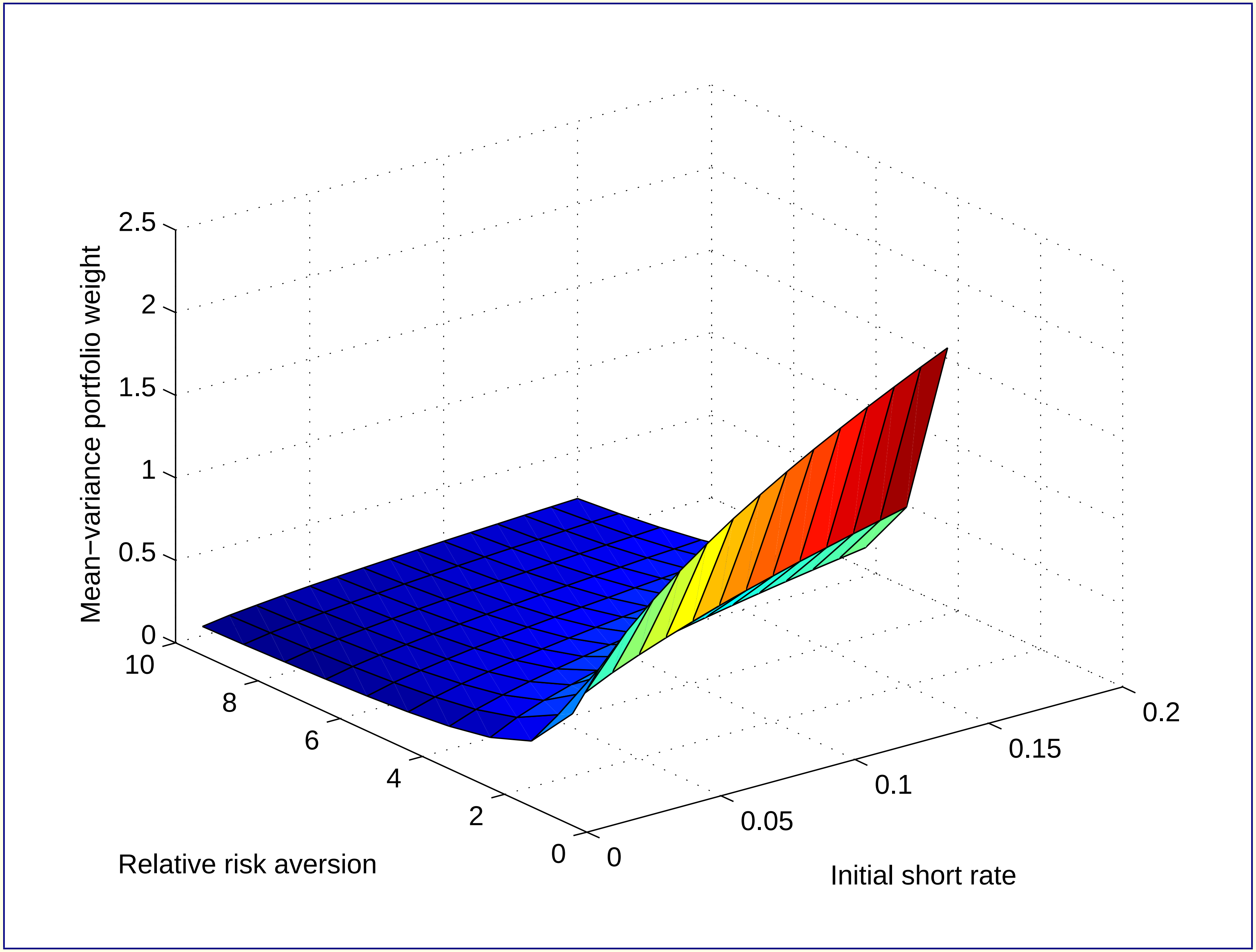
- Dynamic forward measure

hedge: $\boxed{\pi_t^z / X_t^* = \rho\,(\sigma_t')^{-1}\,\mathbf{E}_t^T\left[\frac{Z_{t,T}^{\rho-1}}{\mathbf{E}_t^T\left[Z_{t,T}^{\rho-1}\right]}\,(\mathcal{D}_t \log Z_{t,T})'\right]}$

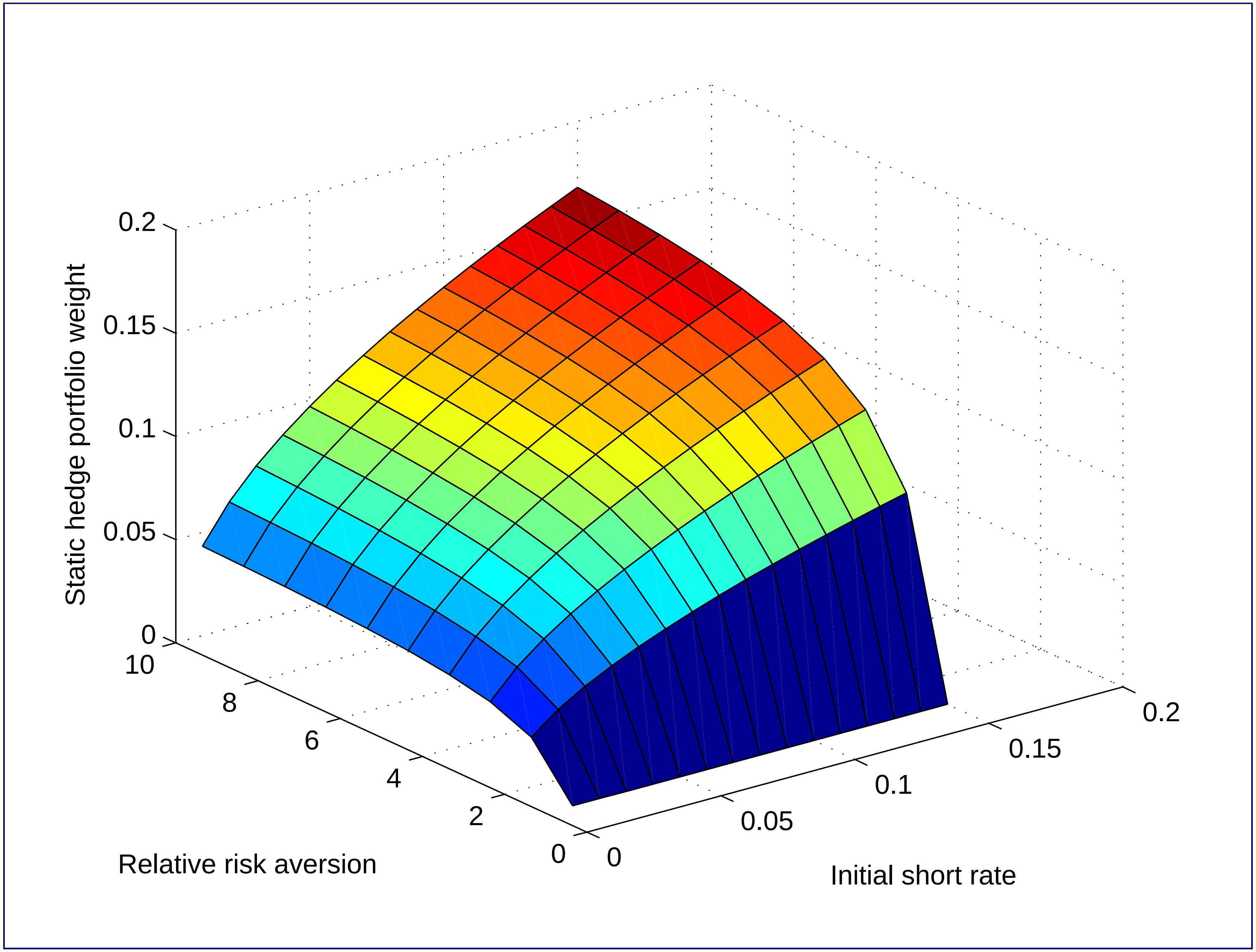


- Changing initial interest rate: Investment horizon fixed at $T = 15$

$\rightarrow$ Mean-variance demand: $\boxed{\pi_t^{mv}/X_t^* = \frac{1}{R}(\sigma_t')^{-1}\theta_t}$

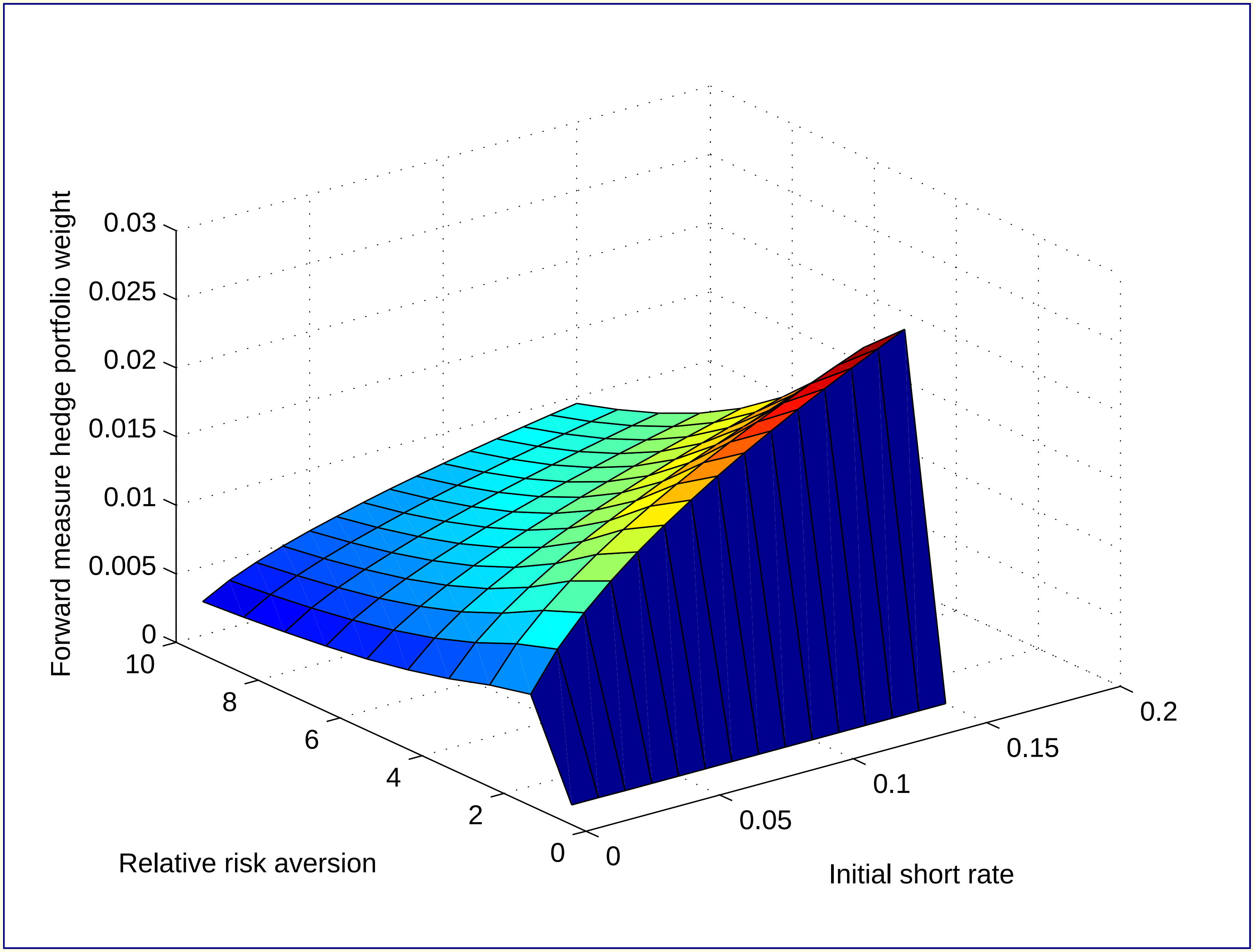


$\rightarrow$ Static term structure hedge: $\boxed{\pi_t^b / X_t^* = \rho(\sigma_t')^{-1} \sigma^B(t,T)}$



- Dynamic forward measure

hedge: $\boxed{\pi_t^z / X_t^* = \rho \left(\sigma_t'\right)^{-1} \mathbf{E}_t^T \left[\frac{Z_{t,T}^{\rho-1}}{\mathbf{E}_t^T \left[Z_{t,T}^{\rho-1} \right]} \left(\mathcal{D}_t \log Z_{t,T} \right)' \right]}$



7 Conclusion

- Contributions:
  - Asset allocation formula based on change of numéraire
  - Highlights role of consumption-specific coupon bonds as instruments to hedge fluctuations in opportunity set
  - Formula has multiple applications: preferred habitat, demand for long term bonds, fund separation, extreme behavior, international asset allocation, demand for I-bonds
  - Technical contributions: exponential Clark-Haussmann-Ocone formula, Malliavin derivatives of functional SDEs, Solution of linear BVIE
- Integration of portfolio management and term structure models
  - Asset allocation in HJM framework
  - Other applications