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# Information, Inflation, and Interest

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# Modelling framework for inflation

In this presentation we are concerned with the development of models for interest rates and inflation, and the pricing of inflation-linked products.

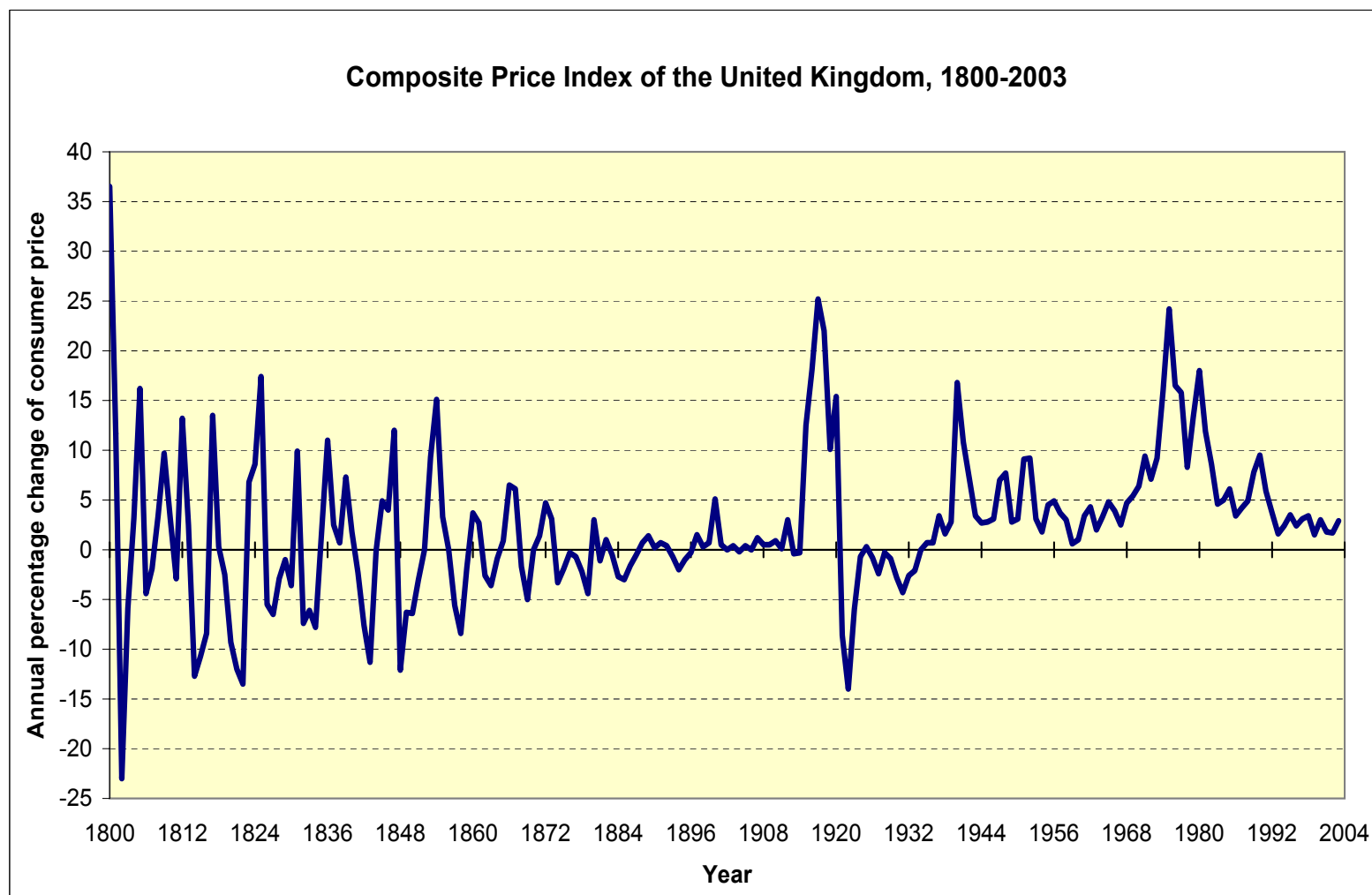
The basic principles of how one most naturally extends the HJM theory to accommodate inflation-linked structures are known (Hughston 1998, Jarrow and Yildirim 2003).

The essential idea here is that one treats the CPI as if it were the price of a foreign currency.

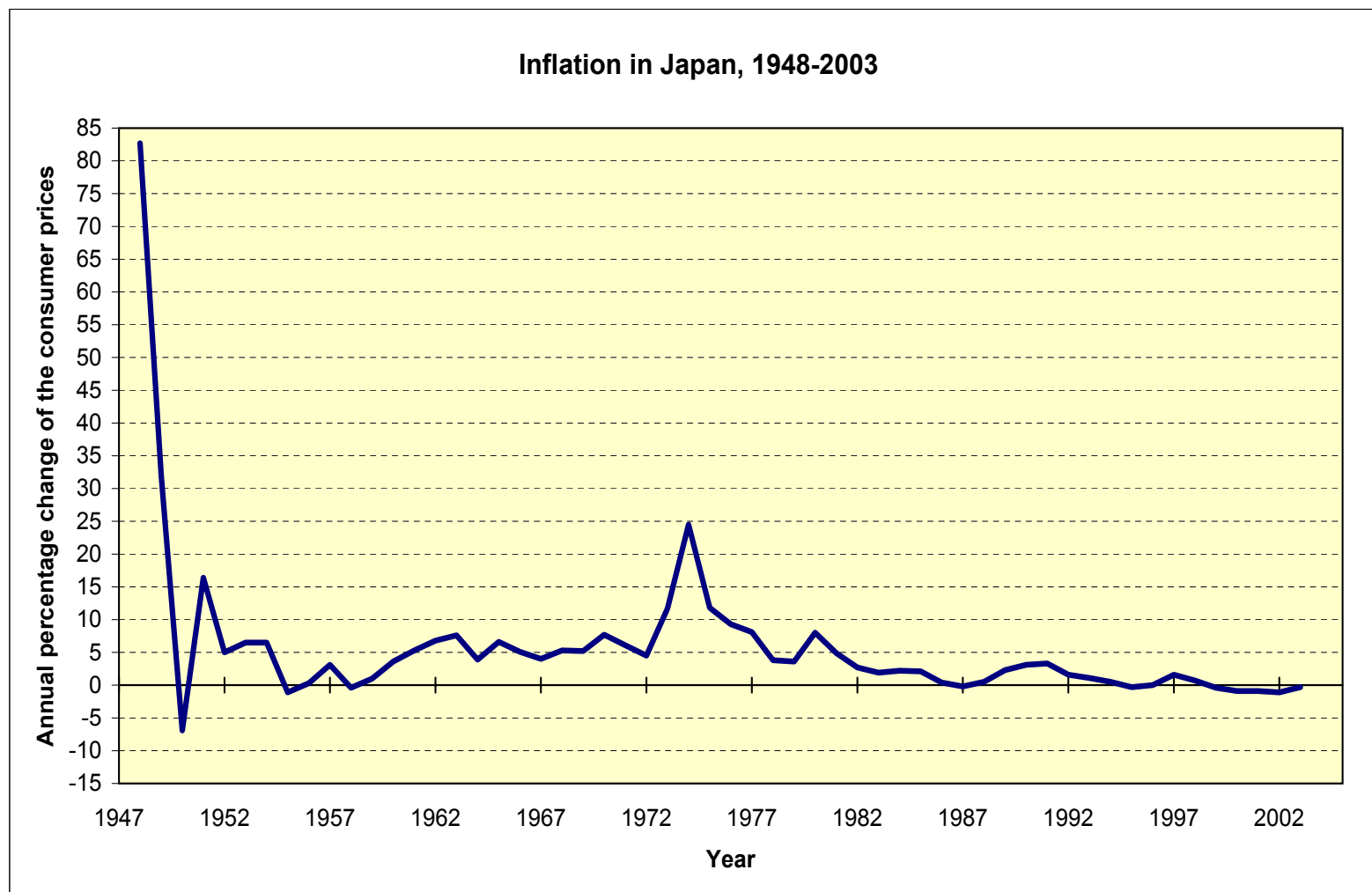
The associated “foreign” interest rate system then has the interpretation of being the system of “real” interest rates.

There is no single preferred measure of inflation. For each different choice of a price index we get a different type of inflation.

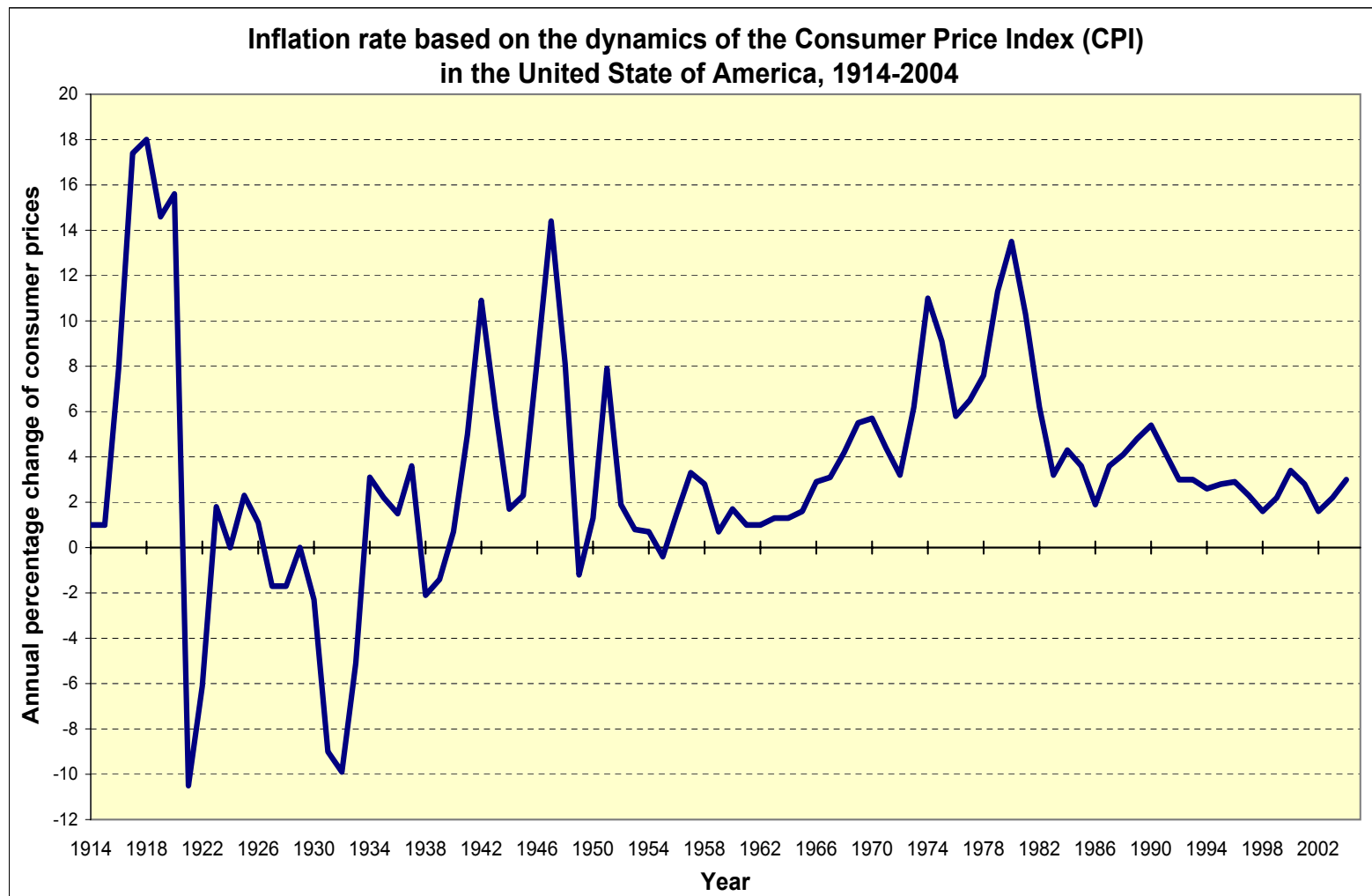
It is also important to recognise that there is no single generally accepted economic model for the “origins” of inflation. It remains very much an open subject.



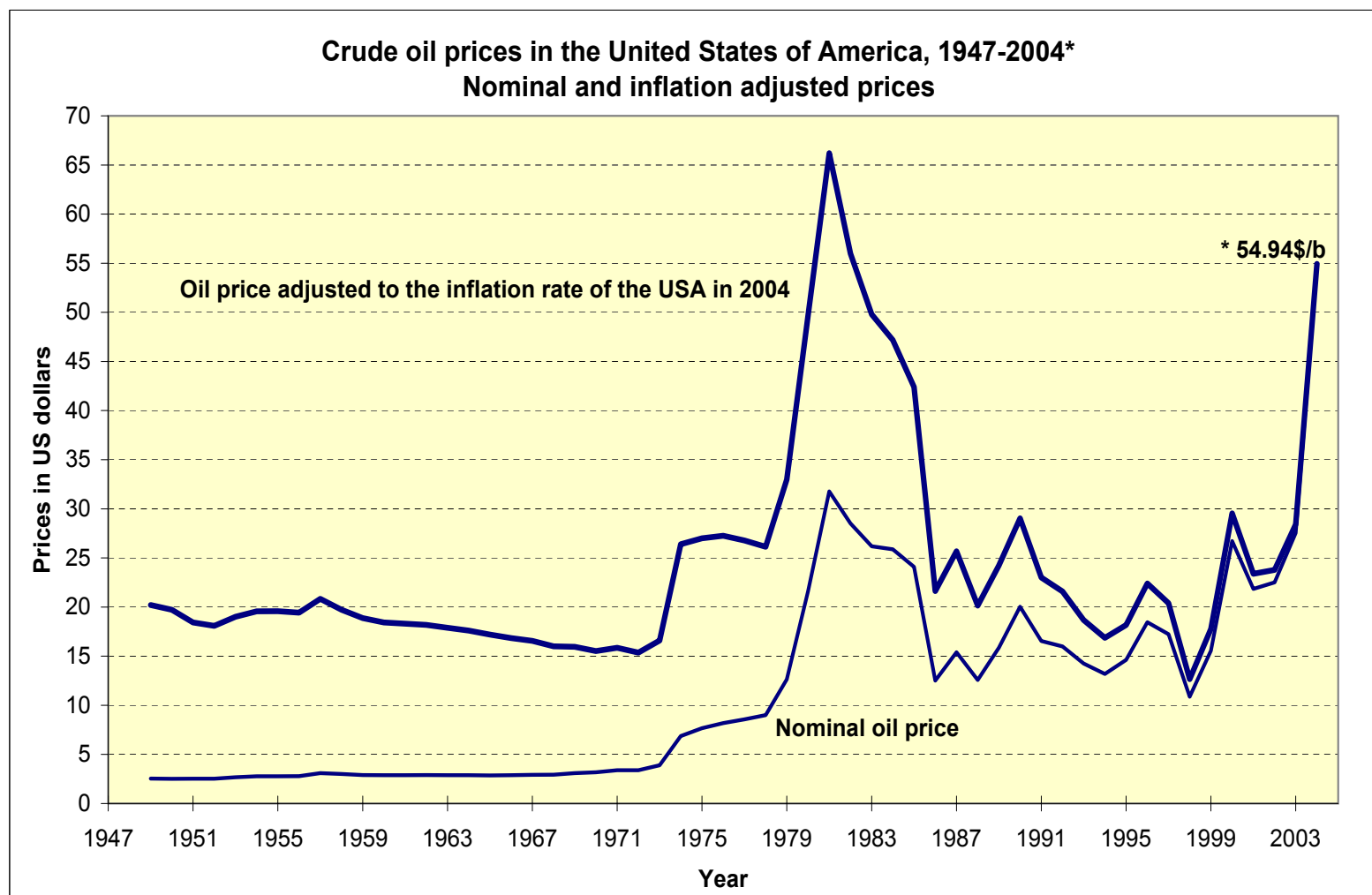
Data source: National Statistics, UK



Data source: Statistics Bureau, Ministry of Internal Affairs and Communications, Japan



Data source: Federal Reserve Bank of Minneapolis, USA



Data source: Financial Trend Forecaster

\* \$54.93/b is the most recent oil price, it is not the annual average in 2004

In the fall of 1972 President Nixon announced that the rate at which inflation was increasing was decreasing.

This was the first time that a sitting US President used a third derivative to advance his case for re-election.

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# Interest rate models and asset pricing

To begin, let's say what we mean by an interest rate model.

An interest rate model consists of the following ingredients:

- (i) A probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with a “market” filtration  $\{\mathcal{F}_t\}_{0 \leq t < \infty}$  to which all asset price processes are adapted.
- (ii) A process  $\{B_t\}_{0 \leq t < \infty}$  which we call the “nominal money market account” process, and is assumed to be of the form

$$B_t = \exp \left( \int_0^t r_s ds \right), \quad (1)$$

where  $\{r_t\}$  is called the “short rate”. We assume  $\{r_t\}$  is strictly positive.

- (iii) A system of nominal discount bond price processes  $\{P_{tT}\}_{0 \leq t \leq T < \infty}$ . For fixed  $T$  we call  $\{P_{tT}\}_{0 \leq t \leq T}$  the price process of the bond with maturity  $T$ . We require that  $P_{TT} = 1$  for all  $T$ , that  $0 < P_{tT} < 1$  for  $0 \leq t < T$ .

- (iv) A pricing kernel  $\{\pi_t\}_{0 \leq t < \infty}$  (also known as a “state price density”, “stochastic discount factor”, etc.) with the property that  $\{\pi_t B_t\}$  and  $\{\pi_t P_{tT}\}$  are martingales with respect to the given measure and filtration.

More generally, for any asset with price process  $\{S_t\}_{0 \leq t \leq \infty}$ , and continuous dividend process  $\{D_t\}$  we require that the process  $\{M_t\}$  defined by

$$M_t = \pi_t S_t + \int_0^t \pi_s D_s ds. \quad (2)$$

is a martingale.

We shall assume that the market filtration is generated by a multi-dimensional Brownian motion.

With these ingredients in place, we say that we have a “model for the nominal interest rate system”.

The pricing of contingent claim with payoff  $H_T$  is given by

$$H_t = \frac{\mathbb{E}_t[\pi_T H_T]}{\pi_t}. \quad (3)$$

# Models for inflation

The modelling of inflation has a number of subtle features associated with it.

The key idea is that one treats the CPI as if it were the price of a foreign currency.

The associated “foreign” interest rate system then has the interpretation of being the system of “real” interest rates.

The basic setup is as follows.

Let us write  $\{\pi_t^N\}$  for the nominal pricing kernel, and  $\{P_{tT}^N\}$  for the nominal discount bond system.

For the nominal discount bond system we have

$$P_{tT}^N = \frac{\mathbb{E}_t[\pi_T^N]}{\pi_t^N}. \quad (4)$$

The associated dynamics are given by:

$$\frac{dP_{tT}^N}{P_{tT}^N} = (r_t^N + \lambda_t^N \Omega_{tT}^N) dt + \Omega_{tT}^N dW_t. \quad (5)$$

Here  $r_t^N$  (nominal short rate) and  $\lambda_t^N$  (nominal market price of risk) are defined by

$$\frac{d\pi_t^N}{\pi_t^N} = -r_t^N dt - \lambda_t^N dW_t, \quad (6)$$

and  $\Omega_{tT}^N$  is the nominal discount-bond vector volatility.

The dynamics of the discount bond system gives rise to the interpretation of  $r_t^N$  and  $\lambda_t^N$ .

It is interesting to note that there are two distinct but equivalent ways of representing any interest rate system:

1. With the specification of  $\{r_t^N\}$  and  $\{\lambda_t^N\}$ .
2. With the specification of  $\{P_{0t}^N\}$ ,  $\{\lambda_t^N\}$ , and  $\{\Omega_{tT}^N\}$ .

Next we introduce a price index  $\{C_t\}$  which we call consumer price index.

The rate of inflation over a given period is defined to be the percentage increase in the price index over the given period, expressed on an annualised basis.

The interesting question is: how do we model  $\{C_t\}$ ?

One is tempted to write down the usual arbitrage-free dynamics of a risky asset.

But since  $\{C_t\}$  is a price index (including the prices of non-durable items such as the price of a meal in a restaurant) one cannot immediately conclude that the usual arbitrage-free dynamics apply.

Instead, we consider a related family of asset prices: the prices of index-linked bonds.

## Index-linked bonds

An index-linked discount bond pays out the amount  $C_T$  at time  $T$ .

Thus we can think of such a bond as a kind of derivative.

The value  $P_{tT}^I$  of an index-linked bond at time  $t$  is therefore

$$P_{tT}^I = \frac{\mathbb{E}_t[\pi_T^N C_T]}{\pi_t^N}. \quad (7)$$

If we divide  $P_{tT}^I$  by  $C_t$  we get the value (in *real* terms) of a bond that pays one unit of goods and services at time  $T$ :

$$P_{tT}^R = \frac{\mathbb{E}_t[\pi_T^N C_T]}{\pi_t^N C_t}. \quad (8)$$

Now suppose we define the real pricing kernel  $\pi_t^R$  by writing

$$\pi_t^R = \pi_t^N C_t. \quad (9)$$

Then clearly for the price index we have

$$C_t = \frac{\pi_t^R}{\pi_t^N}. \quad (10)$$

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In terms of  $\pi_t^R$ , the real discount bond prices take the form

$$P_{tT}^R = \frac{\mathbb{E}_t[\pi_T^R]}{\pi_t^R}. \quad (11)$$

Now suppose we define  $r_t^R$  and  $\lambda_t^R$  by the relation

$$\frac{d\pi_t^R}{\pi_t^R} = -r_t^R dt - \lambda_t^R dW_t. \quad (12)$$

Then it follows that the dynamics of the real discount bond system are of the form:

$$\frac{dP_{tT}^R}{P_{tT}^R} = (r_t^R + \lambda_t^R \Omega_{tT}^R) dt + \Omega_{tT}^R dW_t. \quad (13)$$

Here  $\Omega_{tT}^R$  is the real discount bond volatility.

This shows  $r_t^R$  and  $\lambda_t^R$  have the interpretation of being the real short rate, and the real market price of risk, respectively.

## Dynamics of the consumer price index

Now we can deduce the dynamics of the price index  $C_t$ .

In particular, by an application of the Ito calculus to the relation  $C_t = \pi_t^R / \pi_t^N$  we obtain:

$$\frac{dC_t}{C_t} = [r_t^N - r_t^R + \lambda_t^N (\lambda_t^N - \lambda_t^R)] dt + (\lambda_t^N - \lambda_t^R) dW_t. \quad (14)$$

We note that the consumer price index volatility is the difference between the nominal and real risk premium vectors.

Thus we can write

$$\frac{dC_t}{C_t} = [r_t^N - r_t^R + \lambda_t^N \nu_t] dt + \nu_t dW_t. \quad (15)$$

Here the CPI volatility vector is defined by:

$$\nu_t = \lambda_t^N - \lambda_t^R. \quad (16)$$

The instantaneous rate of inflation  $I_t$  is by definition the drift of the consumer price index. Thus we have:

$$I_t = r_t^N - r_t^R + \lambda_t^N \nu_t. \quad (17)$$



With these relations at hand, we are now in a position to build models for pricing inflation-linked derivatives.

In particular, it should be apparent that the two required ingredients are  $\pi_t^N$  and  $\pi_t^R$ .

# Macroeconomic models for the pricing of inflation derivatives

Inflation is a monetary phenomenon.

To model a monetary economy with inflation we need three more ingredients.

These are: the rate of real consumption  $\{k_t\}$ , the money supply  $\{M_t\}$ , and the liquidity benefit rate  $\{\beta_t\}$  associated with the money supply.

The process  $\{\beta_t\}$  describes the effective rate of benefit, in dollars per unit of time, associated with a given level of the money supply, on an aggregate basis.

If the money supply at time  $t$  is  $M_t$ , the associated benefit rate is  $\beta_t M_t$  dollars per year.

Thus one thinks of the benefit as a kind of “consumable” commodity.

Of course what really “counts” is the *real* rate of benefit derivable from the presence of the money supply.

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This is given by  $l_t = \beta_t M_t / C_t$ .

The consumer price index is thus used as a kind of “exchange rate” to convert the nominal benefit into a real benefit.

Our assumption is that in equilibrium the price level  $\{C_t\}$  and the rate of real consumption  $\{k_t\}$  will be adjusted to each other in such a way as to maximise the total expected utility.

# Optimisation

We assume the existence of a standard bivariate utility function  $U(x, y)$  of the Skidrauski type such that the expected total utility

$$\mathbb{E} \left[ \int_0^T e^{-\gamma t} U(k_t, l_t) dt \right] \quad (18)$$

is maximised over some time horizon.

This is subject to the budget constraint

$$W_0 = \mathbb{E} \left[ \int_0^T \pi_t^N C_t k_t dt + \int_0^T \pi_t^N \beta_t M_t dt \right]. \quad (19)$$

Here  $W_0$  denotes the total wealth of the economy (including, e.g., the present value of any forthcoming income) available for the given period.

A standard argument shows that the maximum is achieved if

$$U_x(k_t, l_t) = \mu e^{\gamma t} \pi_t^N C_t, \quad (20)$$

and

$$U_y(k_t, l_t) = \mu e^{\gamma t} \pi_t^N C_t \quad (21)$$

for some constant  $\mu$  which is fixed by the budget constraint.

These equations can then be solved to give relations between  $k_t$ ,  $l_t$ ,  $C_t$ , and  $\pi_t^N$ .

In particular, if we specify any two of  $k_t$ ,  $l_t$ ,  $C_t$ , and  $\pi_t^N$ , then the other two can be determined.

The idea is that for modelling purposes we regard the consumption (or GDP) and the money supply processes as being specified exogenously.

Then  $C_t$  and  $\pi_t^N$  are determined, and we can work out  $\pi_t^R$ .

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## A simple example

A simple choice for  $U(x, y)$  is given by

$$U(x, y) = A \ln(x) + B \ln(y), \quad (22)$$

where  $A, B$  are constants.

It follows that

$$U_x(k_t, l_t) = \frac{A}{k_t}, \quad U_y(k_t, l_t) = \frac{B}{l_t}. \quad (23)$$

Equating these we get

$$k_t = \frac{A}{B} l_t = \frac{A}{B} \frac{\beta_t M_t}{C_t}. \quad (24)$$

Hence, rearranging, we have

$$C_t = \frac{A}{B} \frac{\beta_t M_t}{k_t}. \quad (25)$$

Thus the consumer price index is determined completely by the nominal money supply benefit and the rate of real consumption.

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But we can also determine the pricing kernels. A calculation shows that

$$\pi_t^N = \frac{B e^{-\gamma t}}{\mu \beta_t M_t}, \quad \pi_t^R = \frac{A e^{-\gamma t}}{\mu k_t}. \quad (26)$$

Thus both the real and the nominal interest rate systems are determined by the model.

Now suppose that  $H_T$  is the payout of a contingent claim (e.g. an inflation derivative).

Then in this model we have

$$H_0 = \beta_0 M_0 e^{-\gamma t} \mathbb{E} \left[ \frac{H_T}{\beta_T M_T} \right]. \quad (27)$$

This example indicates how expectations concerning monetary policy can have a direct effect on the valuation of derivatives.

We see therefore how with a little mathematical reasoning we can model the relations between interest rates, inflation, consumption (or GDP), and the money supply, and draw some interesting conclusions.

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