Robust Replication of Default Contingent Claims

Bjorn Flesaker
Quantitative Financial Research
Bloomberg LP, New York

Based on joint work with Peter Carr, Bloomberg QFR

Presentation at the Fields Institute – April 25, 2007

Overview

- The presentation is about credit derivatives pricing based on traditional arbitrage based replication arguments.
- We treat single name credit default swaps as fundamental hedging instruments.
- We give a motivating introduction with a short detour into the world of credit default swap indices.
- Our analysis applies regardless of the default generating mechanism – i.e. it is equally applicable to a reduced form model as to a structural framework.

Index Replication and Pricing

- In what sense should the pricing of a credit default swap on an index reflect the pricing in the underlying single name CDS market?
- Under what conditions can you replicate a position in an index CDS with positions in single name CDS contracts?
- In the classic derivatives pricing paradigm, the two questions above cannot be separated.
- The credit derivatives world somehow developed along a different path… Pricing without replication was the norm from the start.
- This may be rational on the basis that credit default swaps are truly non-redundant claims, thus completing the market.
- We will explore the extent to which you can indeed create new credit default dependent cash flows with static positions in single name CDS.

Index Arbitrage(?)

- Your friendly banker offers you a deal:
  - you buy protection on the 10Y iTraxx crossover index at a premium of 320 bps running
  - you sell back protection on the corresponding basket of 50 single names, for an average premium of 330 bps running
- You have a perfect hedge (for the default legs) AND the premium legs net to a positive 10 basis points running.
- You want to do it in size? Why not?
- It is by now well known, although poorly understood, that CDS indices should trade with some “basis” to the average of the corresponding single name CDS portfolio, even if there is no mis-match in the default event language in the contracts.
- We will try and shed some light on how to analyze this basis from first principles.

What is a Credit Index anyway?

- Credit default swap indices in their modern form (CDX and iTraxx) are portfolios of single name off-market CDS. Why off-market?
- The balance equation for a single name CDS:

$$S_j(t;T)A_j(t;T) = D_j(t;T)$$

- The corresponding balance equation for the index CDS:

$$S_I(t;T)\sum_j A_j(t;T) = \sum_j D_j(t;T)$$

- Re-arranging slightly:

$$S_I(t;T) = \sum_j w_j(t;T)S_j(t;T) \ with \ w_j(t;T) = \frac{A_j(t;T)}{\sum_j A_j(t;T)}$$

- The risky annuities are the key to generalizing pricing from CDS.

Risky Annuities

- The CDS trade proposed can be broken down into 50 pairs of single name swaps: one on-market and one off-market.
- For each single name CDS pair
  - the default legs will match
  - there will be a piece left over: a net premium leg equal to the single name CDS rate minus the index CDS rate
- Thus, you will be receiving net premium flows until default on the high spread names, and you will be paying net premium until default on the low spread names.
- There is an inherent bias here: your initial positive carry may not last. The bias is greater the more dispersed the portfolio spreads.
- The key to understanding the index basis is the valuation of risky annuities on single name credits.

Static Replication

- We will draw on analogies to the theory of static replication of equity derivatives, as developed by Breeden and Litzenberger (BL) and Ross in the 1970's.
- They demonstrated that all path independent derivatives on a stock could be replicated statically, if you could take positions in European calls of all strikes and maturities.
- BL showed that the second derivative of the European call (or put) price with respect to strike gives the state price density, which is the risk neutral probability density scaled by the discount factor.
- The state price can also be viewed as the Green's function of a particular linear pricing operator, often a second order PDE.
- We will show how, by taking positions in a continuum of CDS for different maturities, we can replicate all claims on default times, and relate our replication to the Green's function of a particular linear operator.

The Model

- The payoff to a European call option occurs on a known date. The only uncertainty is the final stock price.
- In contrast, the pay-offs to a CDS depends on whether default occurs or not, the timing of default, and on the recovery rate.
- There is thus default risk, recovery risk, and (unavoidable) interest rate risk.
- To focus on the default risk, we will assume deterministic interest rates and recoveries.
- We will also make the minor assumption of continuous payment of the CDS premium leg.
- Aside from this, <u>no assumptions</u> will be made on the process generating defaults (structural vs. reduced form models, predictable vs. inaccessible stopping times, continuous vs. jumpy underlying processes, etc.)

The Goal

- Fix a credit risky name, and denote its random default time with τ.

- Consider a target that pays a continuous coupon $c(t)dt$ for $t < min(\tau,T)$ as well as a payment on default, $R(\tau)$, if $\tau<T$.

- Our objective is to replicate the payoffs to the target claim by static positions in par CDS, along with a position in a money market account.

- The only dynamic aspect to the replicating strategy are self-financing flows into and out of the money market account from coupon and CDS premium payments and interest earned, as well as contractual settlements upon default.

Notation and Data

- We assume a fixed recovery rate or, equivalently, that loss given default is some fraction L of the protected notional, where $0 < L \leq 1$.
- Future interest rates earned on the money market account are denoted by $r(t)$, and are assumed to be deterministic.
- Current par CDS rates of different maturities are given by $S_0(t)$, and we assume that we can take continuous positions across maturities.
- We assume that both $r(t)$ and $S_0(t)$ are differentiable as a function of time.
- The assumptions above are not innocuous, but fairly typical in the current generation of credit derivatives models used in the market.
- Future CDS rates can do whatever they feel like; we will not need them. We also do not need liquidity in the CDS market in the future.

Solution Strategy

- We will try a binomial perspective: for each future point in time until maturity, as long as default has not yet happened, be prepared for either of two states: default or no default.

- We have two controls available, namely:

  M(t): the amount of money in the (survival contingent) money market account at time t (if $t \leq \tau$)

  Q(t)dt: the notional amount of CDS protection we write for maturity t

- This looks different from the classic binomial problem, since M and Q are functions on $[0,T]$, but with two controls and two possible states at each time we explore target matching in each state.

Matching if *t* is the Default Time

- We need to be prepared for $t = \tau$, which means:

$$M(t) - L\int_t^T Q(u)du = R(t)$$

- Differentiating with respect to *t* and solving for *Q(t)* gives:

$$Q(t) = \frac{R'(t) - M'(t)}{L}$$

- Since the loss given default *L* and the recovery function of the target claim *R(t)* are both given, the CDS hedge *Q(t)* is determined, once we determine the size of the survival contingent money market account *M(t)*.

Matching if *t* is not the Default Time

- In the case where $\tau > t$, money will flow in and out of the bank account at time *t* as follows:

$$M'(t) = r(t)M(t) + \int_t^T S_0(u)Q(u)du - c(t)$$

- Differentiating again with respect to *t* gives:

$$M''(t) = r(t)M'(t) + r'(t)M(t) - S_0(t)Q(t) - c'(t)$$

Solving for the Controls

- Combining the two, by substituting in the value of *Q(t)* from the default case into the last equation from the no-default case gives the following inhomogeneous linear second order ODE for *M(t)*:

$$M''(t) - \left[r(t) + \frac{S_0(t)}{L} \right] M'(t) - r'(t) M(t) = f(t)$$

where the forcing function *f(t)* is given by:

$$f(t) = -c'(t) - \frac{S_0(t)}{L} R'(t)$$

- A unique solution for *M(t)* on *[0,T]* arises from two terminal conditions:

$$M(T) = 0, \;\; and \;\; \lim_{t \uparrow T} M'(t) = -c(T)$$

The Complete Solution

- So, by solving a terminal value problem, involving a linear second order ordinary differential equation, we have the path of the survival contingent money market account value, *M(t)* on *[0,T].*
- The rate at which we write CDS contracts for maturity *t*, is given by:

$$Q(t) = \frac{R'(t) - M'(t)}{L}$$

- By assumption, all the CDS contracts are initially written at par, and thus at zero up-front cost.
- The no-arbitrage value of the replicated claim is therefore given by the initial value of the money market account, i.e. *M(0)*.
- The ODE must, in general, be solved numerically, but this is very quick and straightforward, e.g. with Runge-Kutta methods.

Replicating a Risky Annuity

- Suppose the target claim is a unit risky annuity to maturity *T* and let $M_a(t;T)$ denote the survival contingent money market account balance at time *t* for its replication strategy.
- Setting $c(t)=1$ and $R(t)=0$ for *t* in $[0,T]$ implies that the forcing term goes to zero on $(0,T)$ and $M_a(t;T)$ solves the homogeneous ODE:

$$\frac{\partial^2}{\partial t^2} M_a(t;T) - \left[r(t) + \frac{S_0(t)}{L} \right] \frac{\partial}{\partial t} M_a(t;T) - r'(t) M_a(t;T) = 0$$

$$s.t. \; M_a(T;T) = 0 \; and \; \lim_{t \uparrow T} \frac{\partial}{\partial t} M_a(t;T) = -1$$

- $M_a(0;T)$ is the initial value of the risky annuity maturing at time T, and we replicate its default risk by writing CDS protection at the rate

$$Q_a(t;T) = -\frac{\partial}{\partial t} \frac{M_a(t;T)}{L}, \; t \in (0,T)$$

Replicating many Risky Annuities

- We demonstrated how to do replication via the backward equation for *M(t)*, and looked at the special case of a unit risky annuity.
- Let us consider the initial value of a unit risky annuity as a function of its maturity:

$$A(u) = M_a(0;u)$$

- This is of direct independent interest, e.g. for pricing off-market CDS and analyzing index replication.
- We could numerically approximate the function *A(u)* for all *u* in *[0,T]* by repeatedly (and tediously) solving the equation described above, but why be backwards?

The Forward Equation - I

- Differentiability of the initial yield curve and the CDS curve ensures a density representation for the risk neutral default time distribution:

$$\Pr\{\tau \le t\} = \int_0^t h(s)ds$$

- The default leg, for a CDS with notional of *1* and maturity *u*, will pay *L* units at the default time, if default takes place before time *u.* Its PV is thus:

$$S_0(u)A(u) = L\int_0^u e^{-\int_0^s r(v)dv} h(s)ds$$

- Differentiating with respect to *u* and rearranging gives:

$$h(u) = e^{\int_0^u r(v)dv}\left[\frac{S_0(u)A'(u) + S_0'(u)A(u)}{L}\right]$$

The Forward Equation - II

- The value of a unit premium leg, of a CDS contract with notional of 1 maturing at time *u* can similarly be expressed as:

$$A(u) = \int_0^u e^{-\int_0^s r(v)dv} \left(1 - \int_0^s h(v)dv \right) ds$$

- Differentiating with respect to *u*, rearranging and repeating gives an alternative expression for the default density:

$$h(u) = -e^{\int_0^u r(s)ds} \left[r(u)A'(u) + A''(u) \right]$$

The Forward Equation - III

- Equating the two expressions for *h(u)* we get the forward ODE for the time zero spot value of the risky annuity as a function of its maturity.

$$A''(u) + \left[r(u) + \frac{S_0(u)}{L} \right] A'(u) + \frac{S_0'(u)}{L} A(u) = 0$$

- Once again, we have a second order ODE, so we need two boundary conditions, for the complete initial value problem:

$$A(0) = 0 \ \textit{and} \ \lim_{u \downarrow 0} A'(u) = 1$$

- The intuition behind the second condition is that the first dollar received is subject to vanishing discounting and risk of default.
- The forward equation can be solved in one single "sweep" to provide the value of the risky annuities for all maturities in *[0,T]* at once.

Green's Function

- From the standard theory of linear operators, the solution to our backward inhomogeneous ODE $\mathcal{L}\, M(t) = f(t)$ can be written as:

$$M(t) = \int_t^T g(t;u) f(u) du$$

where the Green's function *g(t;u)* as a function of *t* satisfies:

$$\mathcal{L}\; g(t;u) = \delta(t-u)$$

$$s.t.\; g(T;u) = 0 \;\; and \;\; \lim_{t \uparrow T} \frac{\partial}{\partial t} g(t;u) = 0$$

- Note that $g(t;u)=M_a(t;u)$ *for* $0 \le t \le u$*, and so* $g(0;u)=A(u)$*.*

- The Green's function *g(t;u)* also satisfies an adjoint ODE in *u*, which generalizes the forward equation we just derived for the initial value of the risky annuity to the square *[0,T] x [0,T]*.

Closed form Solutions

- If the CDS curve is flat, straightforward calculus shows that:

$$M_a(t;T) = \int_t^T \exp\left\{-\left[y(t;u) + \frac{S_0}{L}\right](u-t)\right\} du \qquad y(t;u) = \frac{\int_t^u r(v)dv}{u-t}$$

- In particular, the initial value of the risky annuities are given by:

$$A(u) = \int_0^u e^{-\left[y(0;t)+\frac{S_0}{L}\right]t} dt$$

- Differentiating with respect to u and dividing by the discount factor, we have the implied risk neutral survival probability given as:

$$e^{\int_0^u r(v)dv} \frac{\partial A(u)}{\partial u} = e^{-\frac{S_0}{L}u}$$

Summary and Comments

- Using static replication arguments we derived arbitrage free valuation relationships between credit default swaps and risky annuities in a model free context, subject only to restrictive assumptions on recoveries and interest rates.
- Through financial arguments, we deduced forward and backward equations for the valuation and replicating portfolios of general credit default derivatives.
- We showed examples of closed form solutions.
- Direct applications include credit index pricing and curve stripping, providing a formal justification for (and suggesting some limitations of) current market practices.