

# **Estimation of volatility values from diffusion data observed at random times**

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## OUTLINE

- Motivation: estimating current volatility from high-frequency data
- Methodology: Nonlinear filtering with jumps and random time observations
- Numerical Examples: good performance on simulated and real data

## **CLOSELY RELATED LITERATURE**

- Elliott, R.J., Hunter, W.C. and Jamieson (1998), Frey, R. and Runggaldier (2001)
- ARCH-GARCH literature, Quadratic Variation literature, Maximum Likelihood (Ait-Sahalia et al.)

## **Asset price and volatility model**

- Log-price:

$$X_t = \int_0^t \left( \mu(\theta_s) - \frac{1}{2} v^2(\theta_s) \right) ds + \int_0^t v(\theta_s) dB_s$$

- Random observation times:  $\tau_k$
- Volatility

$$d\theta_t = b(t, \theta_t)dt + \sigma(t, \theta_t)dW_t$$

$$+ \int u(\theta_{t-}, x)(\mu^\theta - \nu^\theta)(dt, dx)$$

## Notation and Assumptions

Conditional distribution of  $X_t - X_s$ :

$$m(s, t) = \int_s^t \left( \mu(\theta_u) - \frac{1}{2} v^2(\theta_u) \right) du$$

$$\sigma^2(s, t) = \int_s^t v^2(\theta_u) du$$

$$\rho_{s,t}(y) := \frac{1}{\sqrt{2\pi}\sigma(s,t)} e^{-\frac{(y-m(s,t))^2}{2\sigma^2(s,t)}}$$

There exist  $\phi_k, \Phi_k$ , such that

$$\mathbb{P}(\tau_{k+1} \leq t, X_{k+1} \leq y | \mathcal{F}^\theta \vee \mathcal{G}_{\tau_k})$$

$$= \int_{\tau_k}^t \int_{-\infty}^y \phi_k(s) \rho_{\tau_k, s}(z - X_k) dz \Phi_k(ds).$$

Example: Doubly Poisson arrivals

$$\Phi_k(dt) = dt, \quad \phi_k(\omega, t) = n(\theta_t) e^{-\int_{\tau_k}^t n(\theta_s) ds}$$

## Main Result

Filter Estimate:

$$\pi_t(f) = E\left(f(\theta_t) | \mathcal{G}_t\right) = \int f(z) \pi_t(dz)$$

Structure Functions:

$$\psi_k(f; t, y, \theta_{\tau_k}) = E\left(f(\theta_t) \rho_{\tau_k, t}(y - X_k) \phi_k(t) \middle| \sigma\{\theta_{\tau_k}\} \vee \mathcal{G}_{\tau_k}\right)$$

$$\bar{\psi}_k(f; t, \theta_{\tau_k}) = E\left(f(\theta_t) \phi_k(t) \middle| \sigma\{\theta_{\tau_k}\} \vee \mathcal{G}_{\tau_k}\right)$$

**Theorem**

$$\pi_{\tau_{k+1}}(f) = \frac{\pi_{\tau_k}(\psi_k(f; t, y))}{\pi_{\tau_k}(\psi_k(1; t, y))} \Big|_{\{t=\tau_{k+1}, y=X_{k+1}\}}$$

$$d\pi_t(f) = \pi_t(\mathcal{L}^\theta f) dt$$

$$-\frac{\pi_{\tau_k}(\bar{\psi}_k(f; t)) - \pi_{t-}(f) \pi_{\tau_k}(\bar{\psi}_k(1; t))}{1 - \int_{\tau_k}^{t-} \pi_{\tau_k}(\bar{\psi}_k(1; s)) \Phi_k(ds)} \Phi_k(dt)$$

## **Comments**

- Structure functions can be computed "off-line"
- Estimates at observation times given by recursion
- Between observation times it's an ODE

## Main Example: Markov Chain Volatility

- Arrivals with intensity  $n(\theta_t)$
- $v(\theta) = \theta$

$$p_i(\tau_k) := P(v_{\tau_k} = a_i | \mathcal{G}_k)$$

$$p_{ji}(t) = P(v_t = a_i | v_0 = a_j)$$

$$r_{ji}(t, z) = E \left( e^{-\int_0^t n(v_u) du} \rho_{0,t}^j(z) | v_t^j = a_i \right)$$

Optimal filter:

$$\begin{aligned} p_i(\tau_k) &= \\ \frac{n(a_i) \sum_j r_{ji}(\Delta \tau_k, \Delta X_k) p_{ji}(\Delta \tau_k) p_j(\tau_{k-1})}{\sum_{i,j} n(a_i) r_{ji}(\Delta \tau_k, \Delta X_k) p_{ji}(\Delta \tau_k) p_j(\tau_{k-1})} \end{aligned}$$

## Sketch of Proof

Counting measure:

$$\mu(dt, dy) = \sum_{k=1}^{\infty} \delta_{\{\tau_k, X_k\}}(t, y) dt dy$$

Compensator  $\nu$ :

$$E \int_0^\infty \int \varphi(t, y) \mu(dt, dy) = E \int_0^\infty \int \varphi(t, y) \nu(dt, dy)$$

$$\nu(dt, dy) I_{[\tau_k, \tau_{k+1}]}(t) = \frac{d\mathbb{P}(\tau_{k+1} \leq t, X_{k+1} \leq y | \mathcal{G}_{\tau_k})}{1 - \mathbb{P}(\tau_{k+1} < t | \mathcal{G}_{\tau_k})}$$

**Lemma:** For  $\tau_k < t \leq \tau_{k+1}$ ,

$$\nu(dt, dy) = \frac{\pi_{\tau_k}(\psi_k(1; t, y))}{1 - \int_{\tau_k}^{t-} \pi_{\tau_k}(\bar{\psi}_k(1; s)) \Phi_k(ds)} \Phi_k(dt) dy.$$

## Theorem:

$$\begin{aligned}\pi_t(f) &= \pi_0(f) + \int_0^t \pi_s(\mathcal{L}^\theta f) ds \\ &+ \int_0^t \int \left[ \sum_{k \geq 0} I_{\llbracket \tau_k, \tau_{k+1} \rrbracket}(s) \frac{\pi_{\tau_k}(\psi_k(f; s, y))}{\pi_{\tau_k}(\psi_k(1; s, y))} - \pi_{s-}(f) \right] \\ &\quad (\mu - \nu)(ds, dy)\end{aligned}$$

- 1. Show  $\pi_t(f) - \int_0^t \pi_s(\mathcal{L}^\theta f) ds$  is a martingale
- 2. Use martingale representation theorem

## Numerical Implementation

If the parameters of Markov Chain  $v$  are known:

- Discrete approximation of  $v$
- Monte Carlo simulation of  $r_{ji}$

## Filter Performance

- Likelihood ratio

$$S_{LR} = \frac{1}{N} \sum_{k=1}^N \log \frac{p_{a(k)}}{\pi_{a(k)}}$$

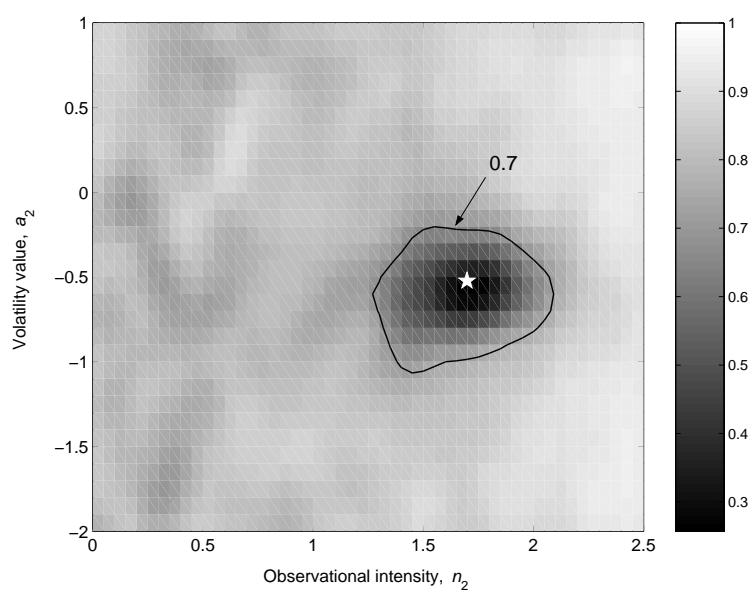
$p_{a(k)}$  is a posteriori probability for the true volatility value  $a(k)$

$S_{LR} = 0$  corresponds to "random guessing". Maximum value for  $p_{a(k)} = 1$ .

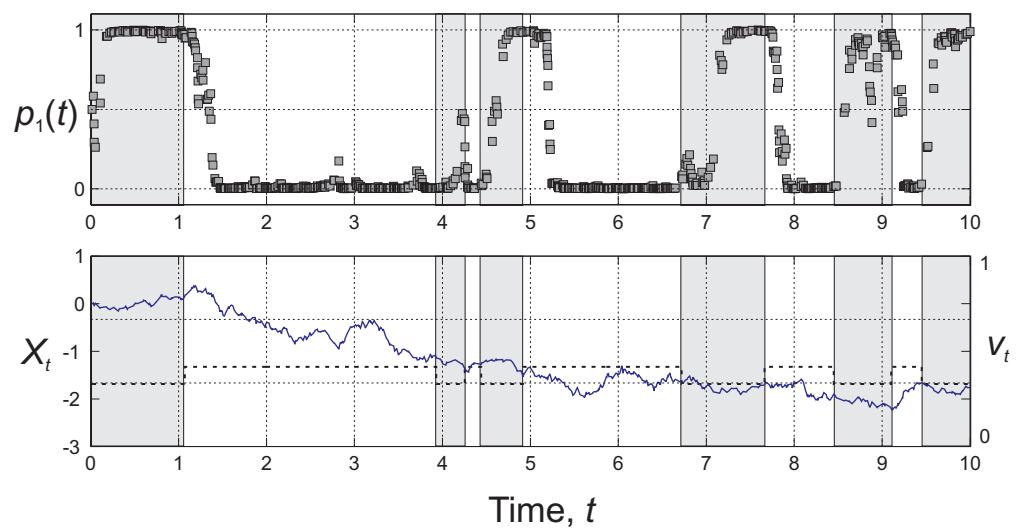
- Conditional frequencies

$$q_{ij} = \text{Prob}\{\hat{v}_t = a_j \mid v_t = a_i\}$$

$$\text{Tr} = \frac{\text{Trace}(q_{ij})}{M}$$



A level  $S_{LR} = 0.7$  is shown by the solid line.



## Dependence on a priori parameters

**Proposition:** The a posteriori probabilities  $p_i(T_k)$  are asymptotically independent of the choice of  $\mu$  as the observational intensity increases,  $n_i \rightarrow \infty$ .

Simulation: a several dozen of observations between volatility jumps are sufficient.

## Change-point problem

$$v_t = a_1 I_{\{0 \leq t \leq t_0\}} + a_2 I_{\{t > t_0\}}$$

$$\widehat{t}_0 = \min(t : p_2(t) \geq p_0)$$

For  $a_2 \geq 2a_1$ , we need less than 10 observations. For  $a_2 = \frac{19}{20}a_1$  we need less than 100 observations.

Errors:

$m$  = expected number of observations to detection

$f$  = fraction of false alarms

## Estimating a priori values

We need to decide initial values for

$$p_i, a_i, n(a_i)$$

Method: Multiscale Trend Analysis (MTA) of Zaliapin, Gabrielov, and Keilis-Borok (2004).

Find process  $P$  so that

$$\Delta P_t \approx av_t \Delta t$$

$$P_t := \sum_{k:T_k < t} |X_{T_k} - X_{T_{k-1}}|$$

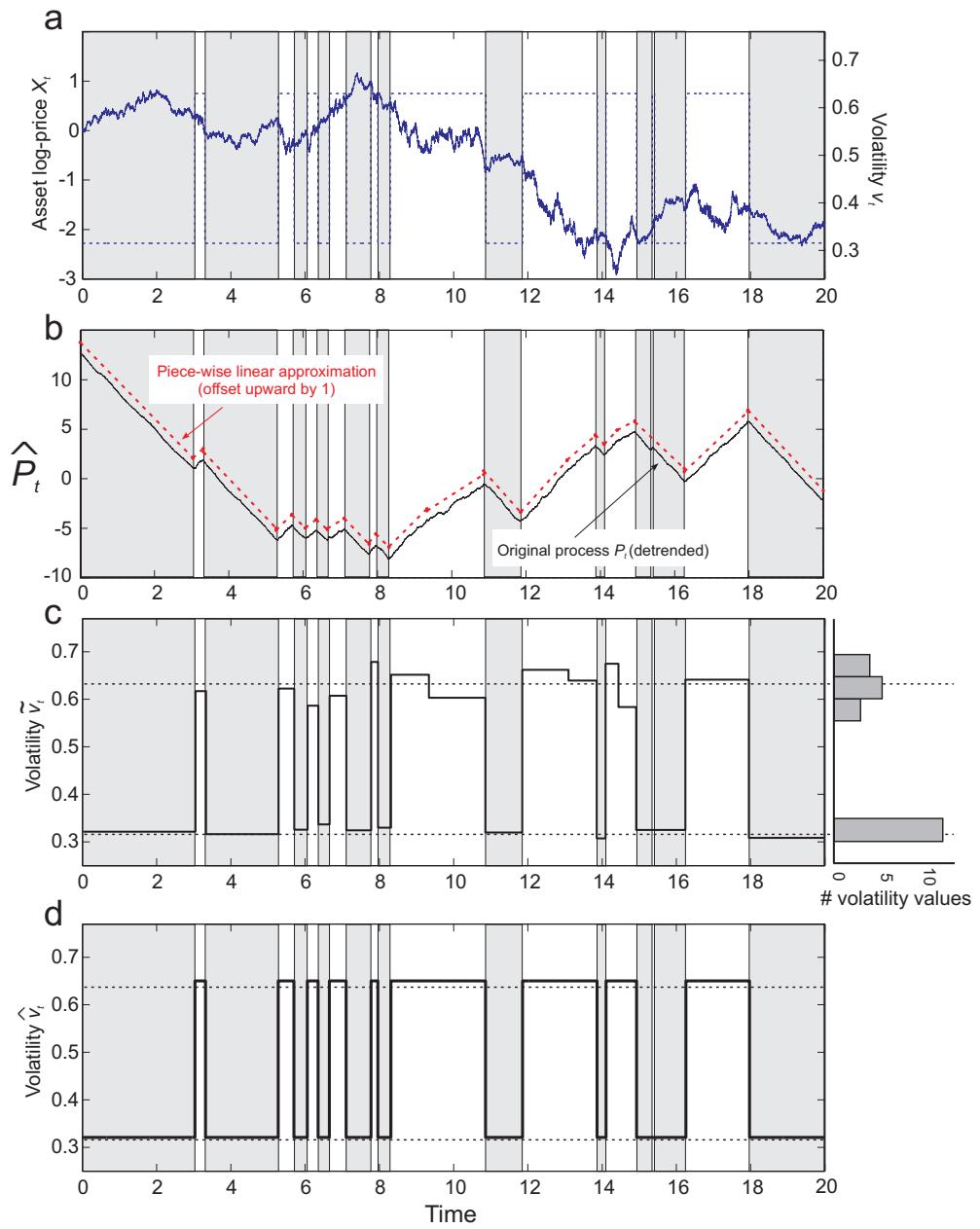
**Proposition:** The following holds assymptotically as  $n \rightarrow \infty$ :

$$P_{t_2} - P_{t_1} = \sum_{k:t_1 \leq T_k \leq t_2} |X_{T_k} - X_{T_{k-1}}| \approx (t_2 - t_1)v\sqrt{n/2}$$

Initial values:

$$\hat{p}_i = \frac{D_i}{T}, \quad \widehat{n(a_i)} = \frac{\#\{k : v_{T_k} = a_i\}}{D_i},$$

$$i, j = 1, \dots, \widehat{M}, \quad i \neq j$$



## MTA procedure

Problem: Decide the number  $\hat{M}$  of distinct values of volatility.

Error of approximating  $X$  with one line  $L_0$ :

$$E_0 = \int_{t_1}^{t_N} (X(t) - L_0(t))^2 dt$$

Second iteration: Optimize, over  $(N, E)$ ,

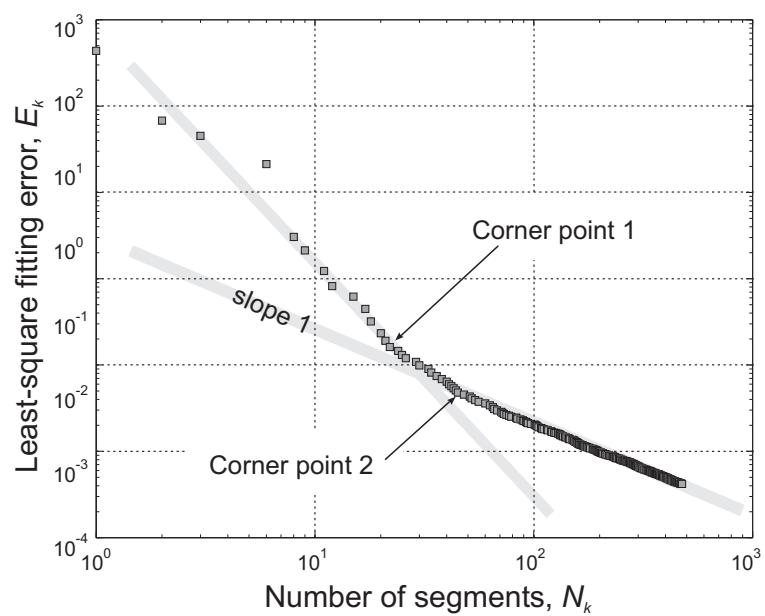
$$\frac{\log(E/E_0)}{N - 1}$$

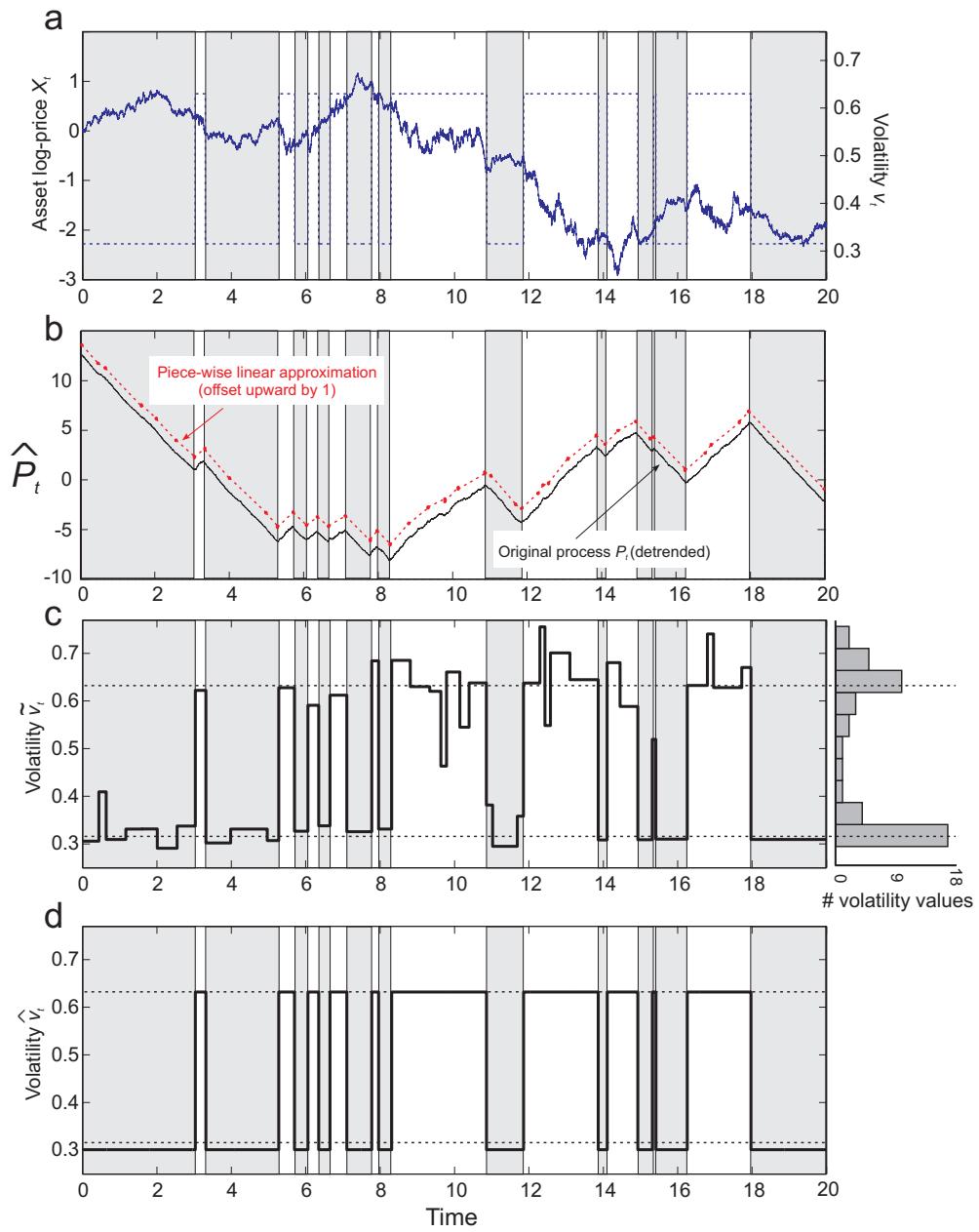
And so on, to get a series of piece-wise linear approximations with  $N_k$  segments.

MTA "spectrum" for self-affine time series:

$$E_k = E_0 N_k^{-H}, \quad \log E_k = \log E_0 - H \log N_k$$

For Brownian Motion  $H = \frac{1}{2}$





## Complete algorithm

**Step 1.** Estimate volatility alphabet.

- 1.1 Construct the process  $P_t$
- 1.2 Construct the MTA decomposition
- 1.3 Calculate preliminary alphabet values  $\{\tilde{a}_i\}$  for piece-wise linear approximation  $L_t$  corresponding to the corner point of MTA spectrum.
- 1.4 Obtain the alphabet estimation  $\{\widehat{a}_i\}_{i=1,\dots,\widehat{M}}$  by grouping the values  $\{\tilde{a}_i\}$  according to their multi-modal distribution.

**Step 2.** Estimate a priori initial probabilities and transitional intensities.

**Step 3.** Estimate time-dependent volatility using the filter.

## Simulated data and General Electric Data

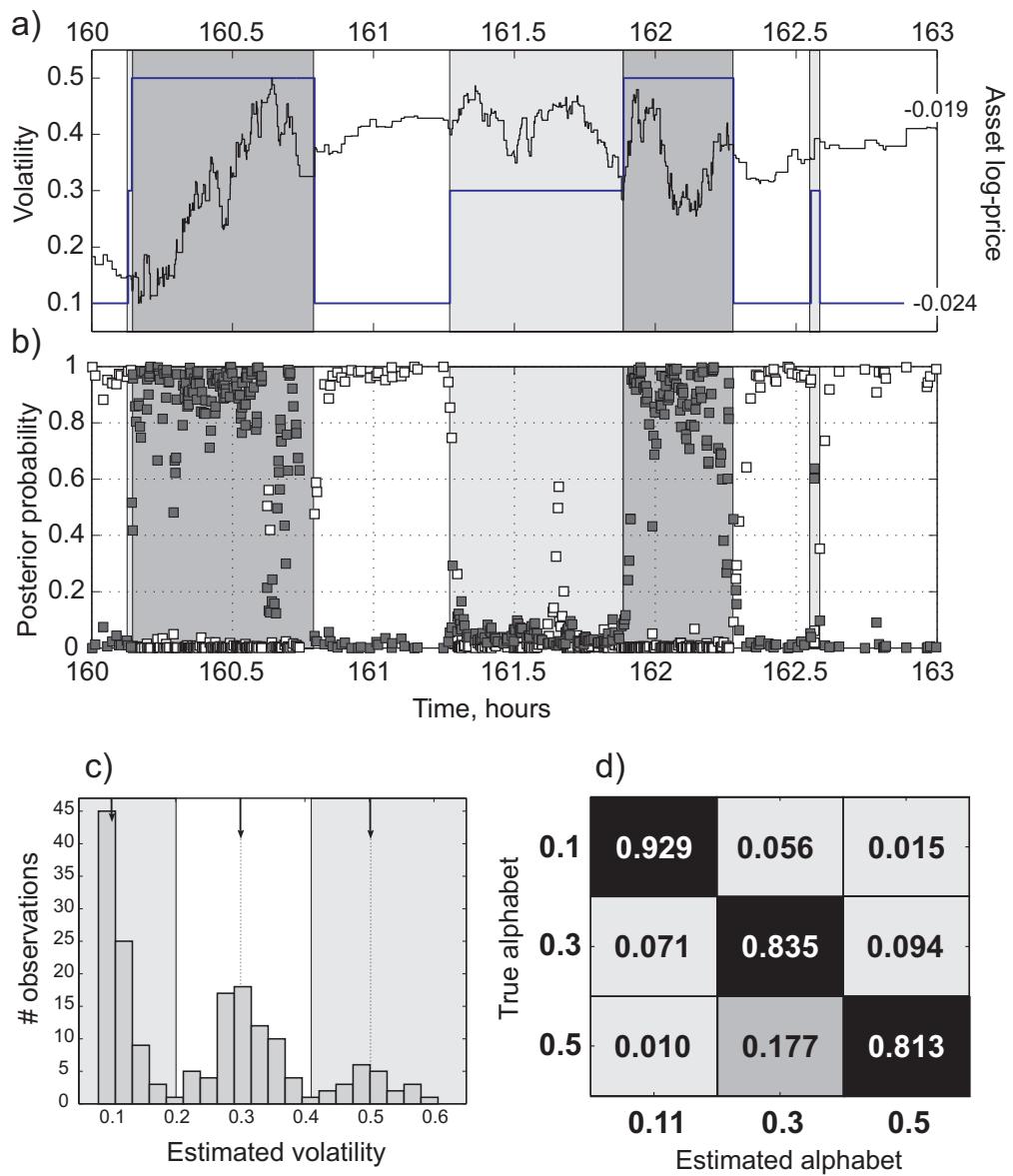
For GE, we get

$$\widehat{M} = 3, \{\widehat{a_i}\} = \{0.06, 0.1, 0.15\}$$

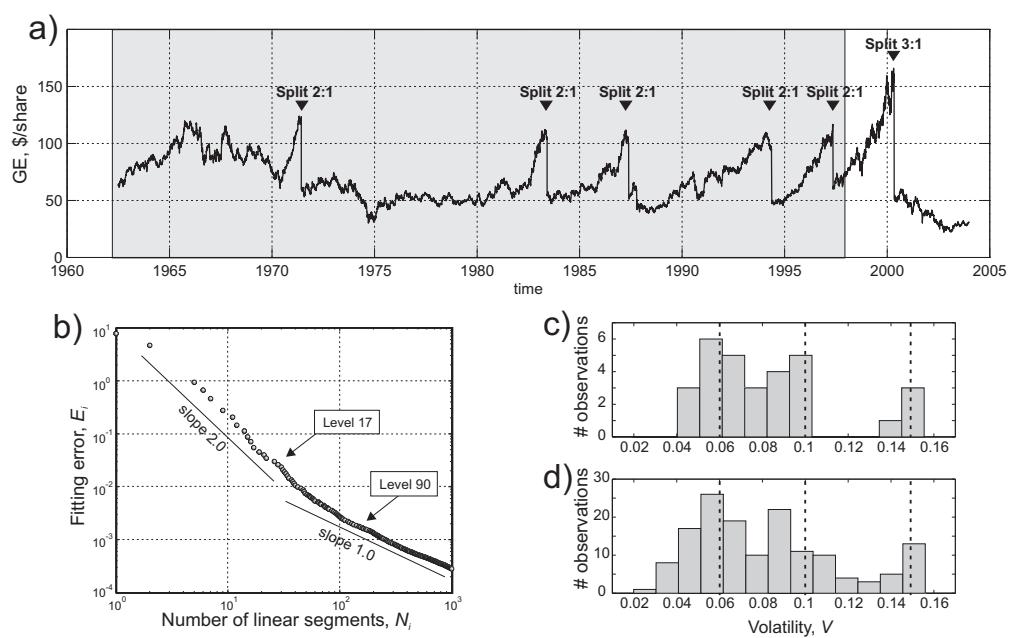
$$\widehat{p}_i = \{0.66, 0.26, 0.07\}, \quad \widehat{\Lambda} = \begin{pmatrix} 0 & 0.16 & 0.5 \\ 1.21 & 0 & 0.6 \\ 3.58 & 2.14 & 0 \end{pmatrix}$$

Smoothing:

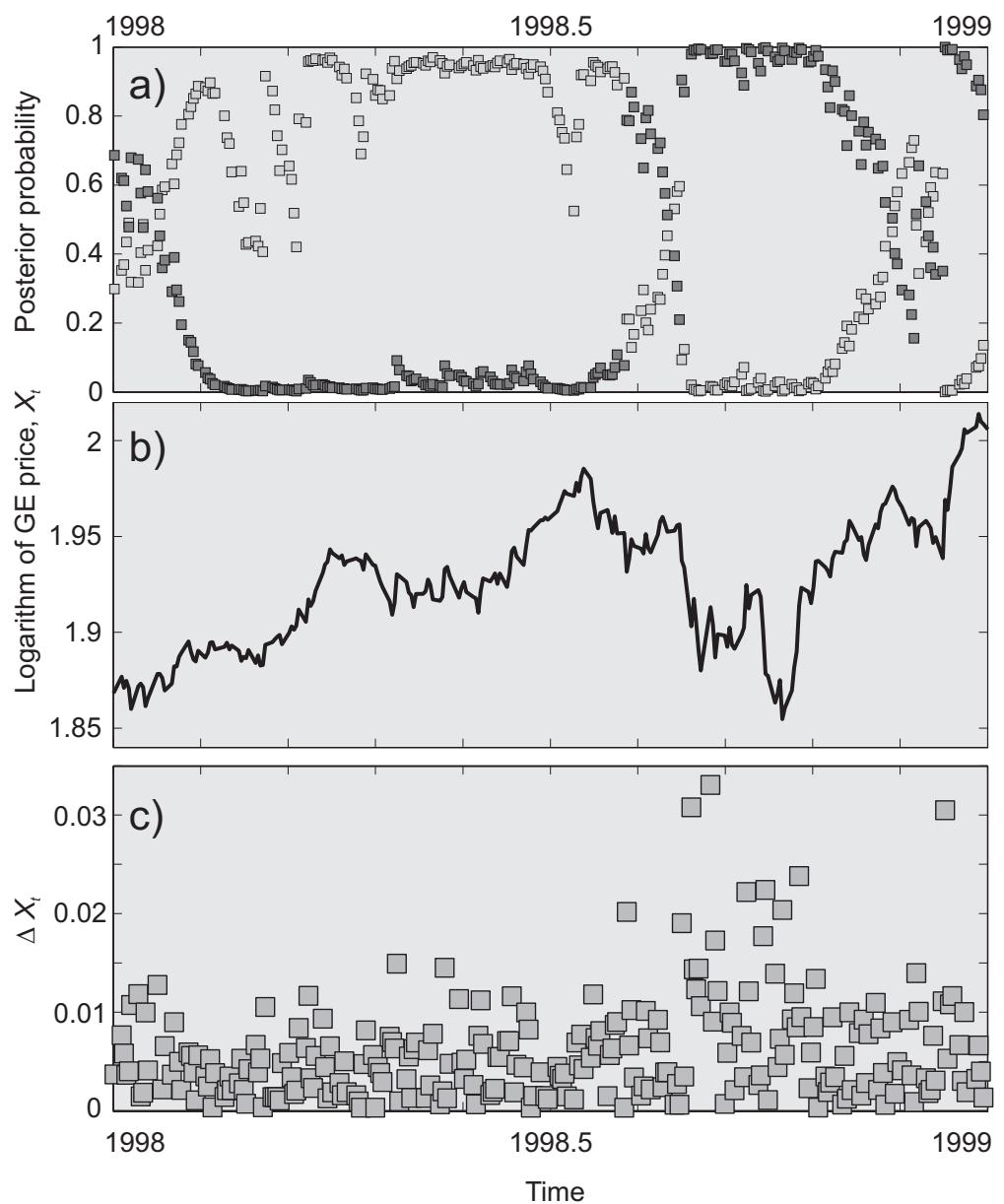
$$\Psi_i(t) := \int K(t-s) p_i(s) ds$$

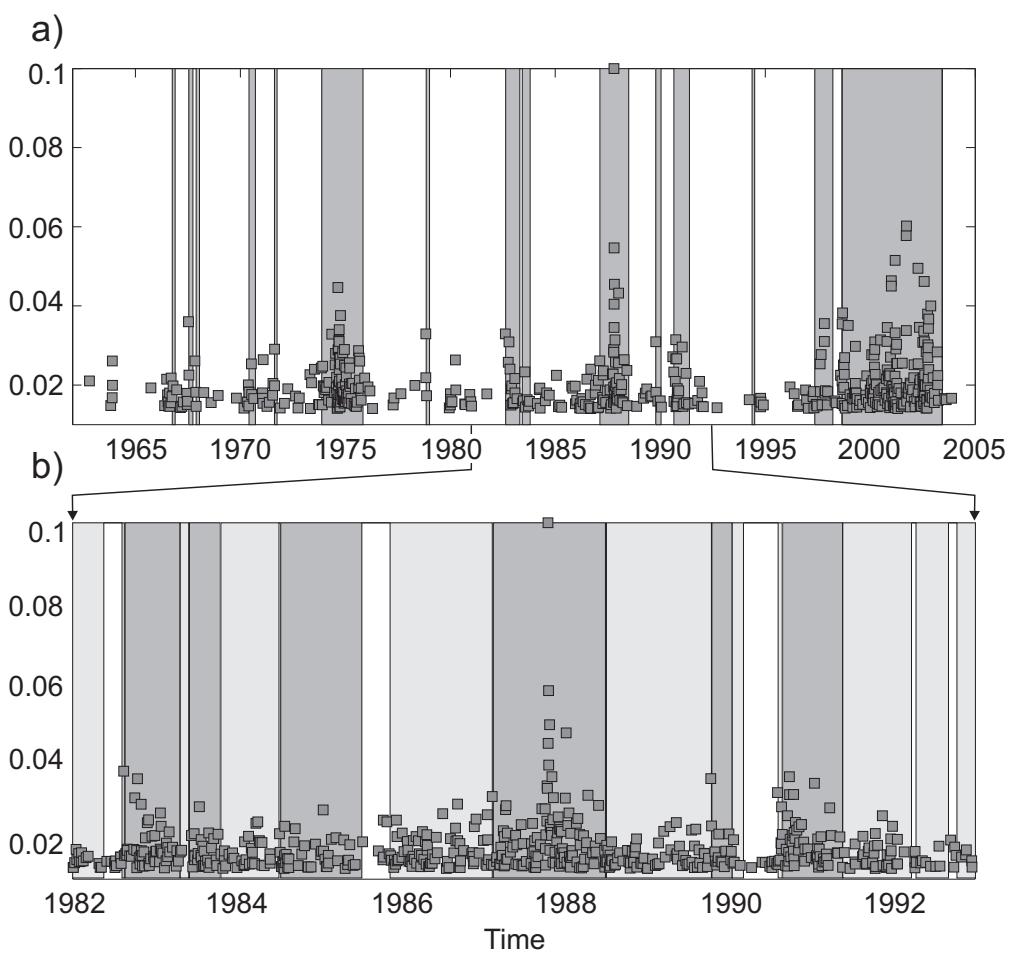


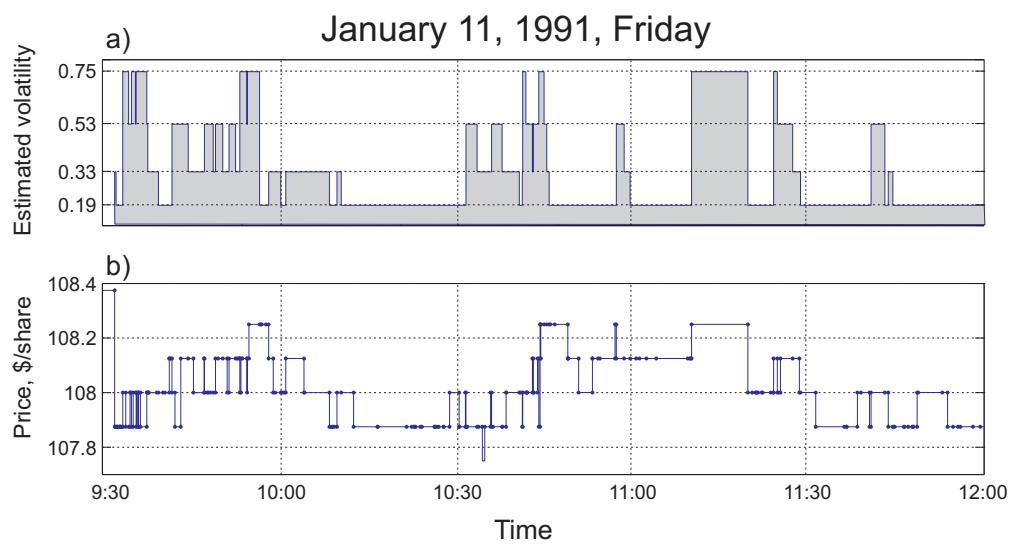
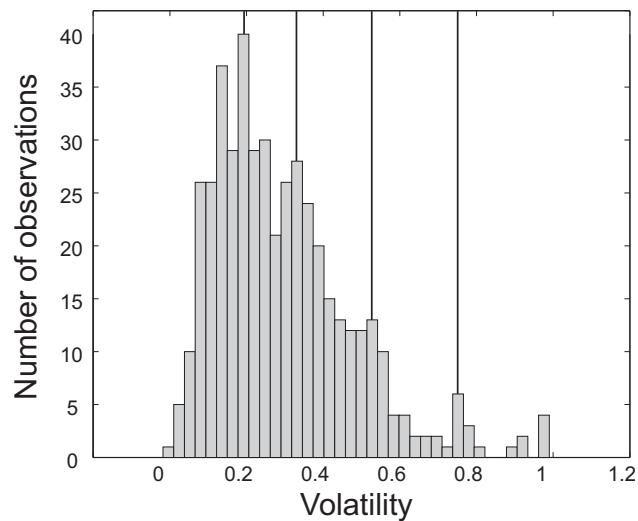
Simulated Data.



GE Data.







IBM Data.

## Conclusions

- Efficient algorithm for estimating current volatility values
- Robust with respect to drift
- Unresolved issues:
  - 1) prediction
  - 2) Volatility depends on asset value
  - 3) jumps in stock price