Adiabaticity in Open Quantum Systems: Geometric Phases & Adiabatic Quantum Computing

Adiabatic Approximation in Open Quantum Systems — *Phys. Rev. A* **71**, 012331 (2005)

Adiabatic Quantum Computation in Open Systems — *Phys. Rev. Lett.* **95**, 250503 (2005)

Abelian and Non-Abelian Geometric Phases in Adiabatic Open Quantum Systems
Phys. Rev. A **73**, 062101 (2006)

Joint work with Dr. Marcelo Sarandy

USC Center for Quantum Information Science and Technology CQIST

$:

The Adiabatic Approximation

Adiabatic Theorem

If Hamiltonian $H(t)$ "changes slowly" then "no transitions", i.e.:
System remains in its instantaneous eigenstate $H(t)\psi_n(t)=E_n(t)\psi_n(t)$ (or degenerate eigenspace) throughout the (adiabatic) evolution, i.e.,
energies $E_n(t)$ evolve continuously and do not cross.

Applications abound… E.g.: Molecular energy level surfaces (Born-Oppenheimer approximation), Berry phase, adiabatic QC, holonomic QC

Standard formulation of adiabatic theorem: applies to *closed* quantum systems only (Born & Fock (1928); Kato (1950); Messiah (1962)).

This talk:

- A generalization of the adiabatic approximation to the case of ***open*** quantum systems.

- Applications to adiabatic quantum computing & geometric phases.

- We'll show (main result): adiabatic approximation generically breaks down after long enough evolution.

Experimental Evidence for Finite-Time Adiabaticity in an Open System

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Experimental Implementation of an Adiabatic Quantum Optimization Algorithm

Matthias Steffen,[1,2,*] Wim van Dam,[3,4] Tad Hogg,[3] Greg Breyta,[5] and Isaac Chuang[1]
[1]*Center for Bits and Atoms–MIT, Cambridge, Massachusetts 02139*
[2]*Solid State and Photonics Laboratory, Stanford University, Stanford, California 94305-4075*
[3]*HP Labs, Palo Alto, California 94304-1126*
[4]*MSRI, Berkeley, California 94720-5070*
[5]*IBM Almaden Research Center, San Jose, California 95120*
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We report the realization of a nuclear magnetic resonance computer with three quantum bits that simulates an adiabatic quantum optimization algorithm. Adiabatic quantum algorithms offer new insight into how quantum resources can be used to solve hard problems. This experiment uses a particularly well-suited three quantum bit molecule and was made possible by introducing a technique that encodes general instances of the given optimization problem into an easily applicable Hamiltonian. Our results indicate an optimal run time of the adiabatic algorithm that agrees well with the prediction of a simple decoherence model.

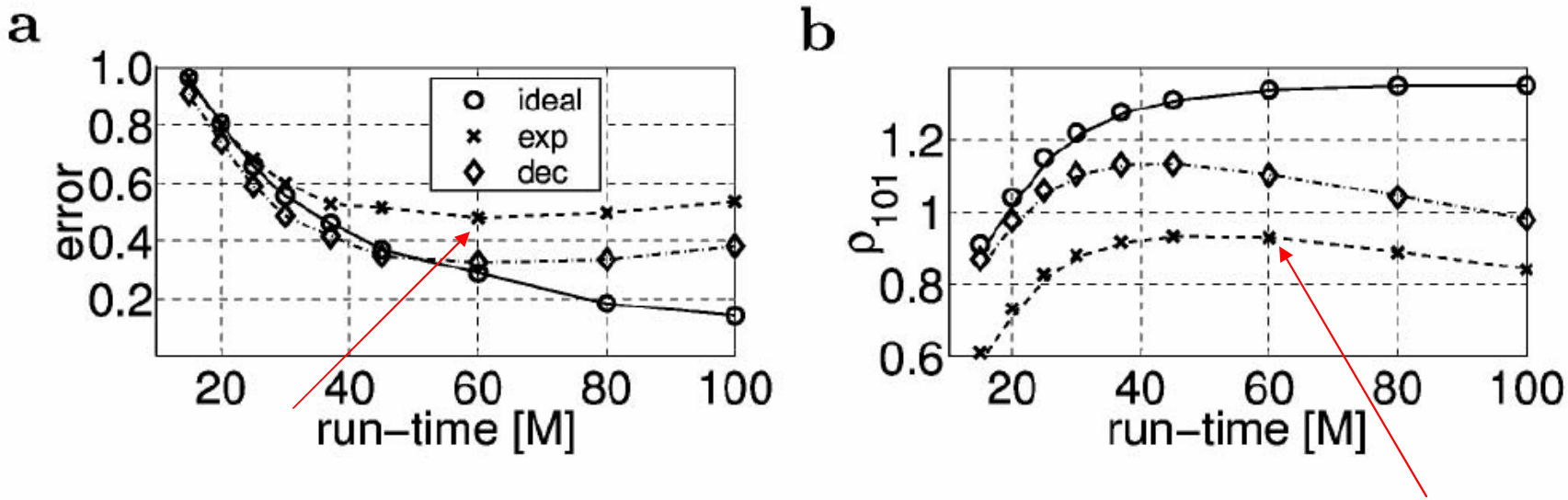
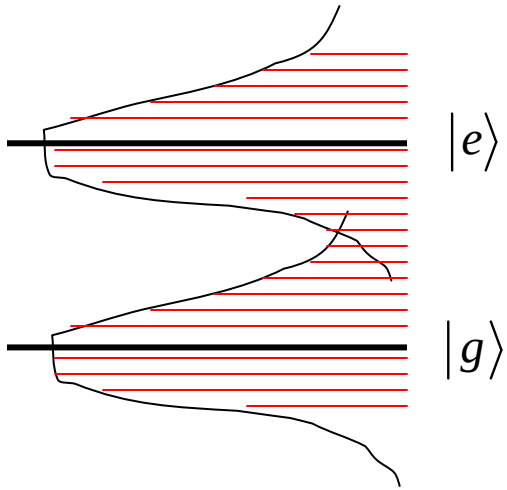


FIG. 4. Experimental performance of the adiabatic algorithm. (a) Plot of the error as a function of M. The error measure is the trace distance $D(\rho, \sigma) = |\rho - \sigma|/2$, where σ is the traceless deviation density matrix for $M = 400$, approximating $M \to \infty$, and ρ equals the ideal expected ($\bigcirc$), the experimentally obtained ($\times$), or the ideal expected traceless deviation density matrix with decoherence effects ($\diamond$) [6]. The minimum error occurs at about $M = 60$ indicating an optimal run time of the algorithm. (b) A similar observation can be made when plotting $|101\rangle\langle 101|$ as a function of M.

Intuition for optimal time:

Decoherence causes broadening of system energy levels (many bath levels accessible),
until they overlap.



Competition between adiabatic time (slow) and need to avoid decoherence (fast)
yields optimal run time.

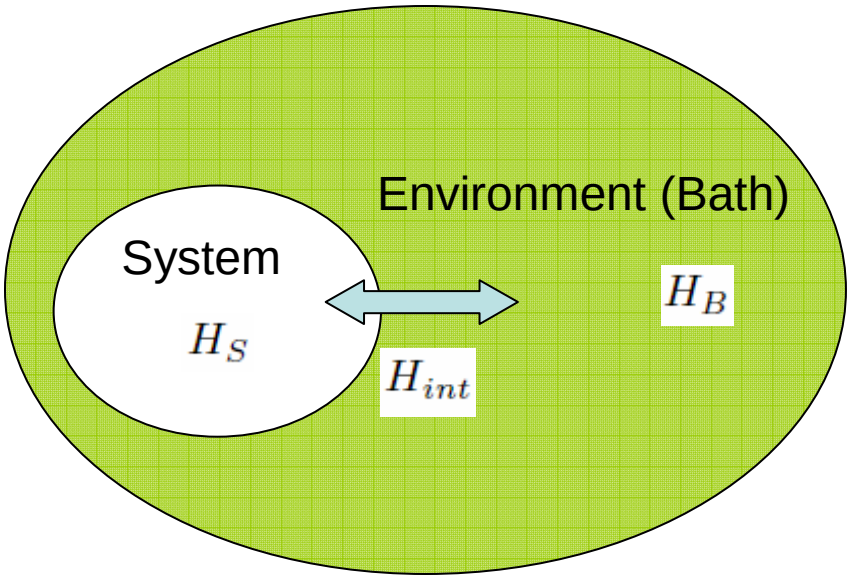
Open Quantum Systems

Every real-life quantum system is coupled to an environment ("bath").

Full Hamiltonian:

$$H = H_S + H_B + H_{\text{int}} \qquad H_{\text{int}} = \sum_\alpha S_\alpha \otimes B_\alpha$$

system bath



Open quantum systems are *not* described by the Schrodinger equation.

They are described quite generally by completely positive maps:

$$\rho(t) = \Phi(\rho(0)) = \sum_i A_i \rho(0) A_i^\dagger \qquad \sum_i A_i^\dagger A_i = I$$

Master Equations for Open Quantum Systems

Under the assumption that Φ is invertible can obtain a class of, generally **non-Markovian**,

convolutionless master equations:

$$\frac{\partial \rho_S}{\partial t} = \mathcal{L}(t)[\rho_S]$$

Hamiltonian part, Dissipative part

$$\mathsf{L}(t) = \mathsf{H}(t) + \mathsf{R}(t)$$

E.g.: $$\mathcal{L}(t)[\rho_S] = -i[H_S'(t), \rho_S] + \frac{1}{2}\sum_\alpha \left(2\Gamma_\alpha(t)\rho_S\Gamma_\alpha^\dagger(t) - \{\rho_S, \Gamma_\alpha^\dagger(t)\Gamma_\alpha(t)\} \right)$$

"Superoperator" $\mathcal{L}(t)$ generally not Hermitian,
does not have a complete orthornormal basis

How should we construct adiabatically decoupled eigenspaces?

Spectrum via the Jordan *block-diagonal* form

Jordan canonical form:

$$\mathcal{L}_J(t) = S^{-1}(t)\,\mathcal{L}(t)\,S(t),$$

where $\mathcal{L}_J(t) = \mathrm{diag}(J_1, ..., J_m)$ is the Jordan form of $\mathcal{L}(t)$, with the Jordan block J_α associated to an eigenvector of $\mathcal{L}(t)$ whose eigenvalue is γ_α

$$J_\alpha = \begin{pmatrix} \gamma_\alpha & 1 & 0 & \dots & 0 \\ 0 & \gamma_\alpha & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \gamma_\alpha & 1 \\ 0 & \dots & \dots & 0 & \gamma_\alpha \end{pmatrix}$$

$$\mathcal{L}_J(t) = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_m \end{pmatrix}$$

One right and one left eigenvector per Jordan block, same eigenvalue

PRA **71**, 012331 (2005)

Left and Right Bases

Apply similarity transformation to left and right eigenvectors, then complete them to a left and right basis:

Right and left bases in the space of linear operators:

$$\left\{|\mathcal{D}_\alpha^{(j)}\rangle\rangle_J\right\} \text{ and } \left\{{}_J\langle\langle\mathcal{E}_\alpha^{(i)}|\right\}.$$

row index inside given Jordan block

Jordan block index

Properties:

- Block structure is preserved under $\mathcal{L}_J(t)$:

$$\mathcal{L}_J(t)\,|\mathcal{D}_\alpha^{(j)}(t)\rangle\rangle = |\mathcal{D}_\alpha^{(j-1)}(t)\rangle\rangle + \lambda_\alpha(t)\,|\mathcal{D}_\alpha^{(j)}(t)\rangle\rangle,$$

$$\langle\langle\mathcal{E}_\alpha^{(i)}(t)|\,\mathcal{L}_J(t) = \langle\langle\mathcal{E}_\alpha^{(i+1)}(t)| + \langle\langle\mathcal{E}_\alpha^{(i)}(t)|\,\lambda_\alpha(t).$$

- Orthonormality condition:

$$\langle\langle\mathcal{E}_\alpha^{(i)}(t)|\mathcal{D}_\beta^{(j)}(t)\rangle\rangle = \delta_{\alpha\beta}\delta^{ij}.$$

Expand the density matrix

$$|\rho(t)\rangle\rangle = \frac{1}{2}\sum_{\beta=1}^{m}\sum_{j=0}^{n_\beta-1} r_\beta^{(j)}(t)|\;\mathcal{D}_\beta^{(j)}(t)\rangle\rangle,$$

PRA **71**, 012331 (2005)

Definition of Adiabaticity in closed/open systems

Adiabaticity in closed quantum systems:
A *closed* quantum system is said to undergo adiabatic dynamics if its evolution is so slow that it proceeds independently in its instantaneous decoupled Hamiltonian-eigenspaces, associated to distinct eigenvalues of $H(t)$.

Adiabaticity in open quantum systems:
An *open* quantum system is said to undergo adiabatic dynamics if its evolution is so slow that it proceeds independently in its instantaneous decoupled superoperator-Jordan blocks, associated to distinct eigenvalues of $\mathcal{L}(t)$.

Can prove, for a closed system:
Adiabatic according to $H(t)$ ç Ł adiabatic according to $\mathcal{L}(t)$

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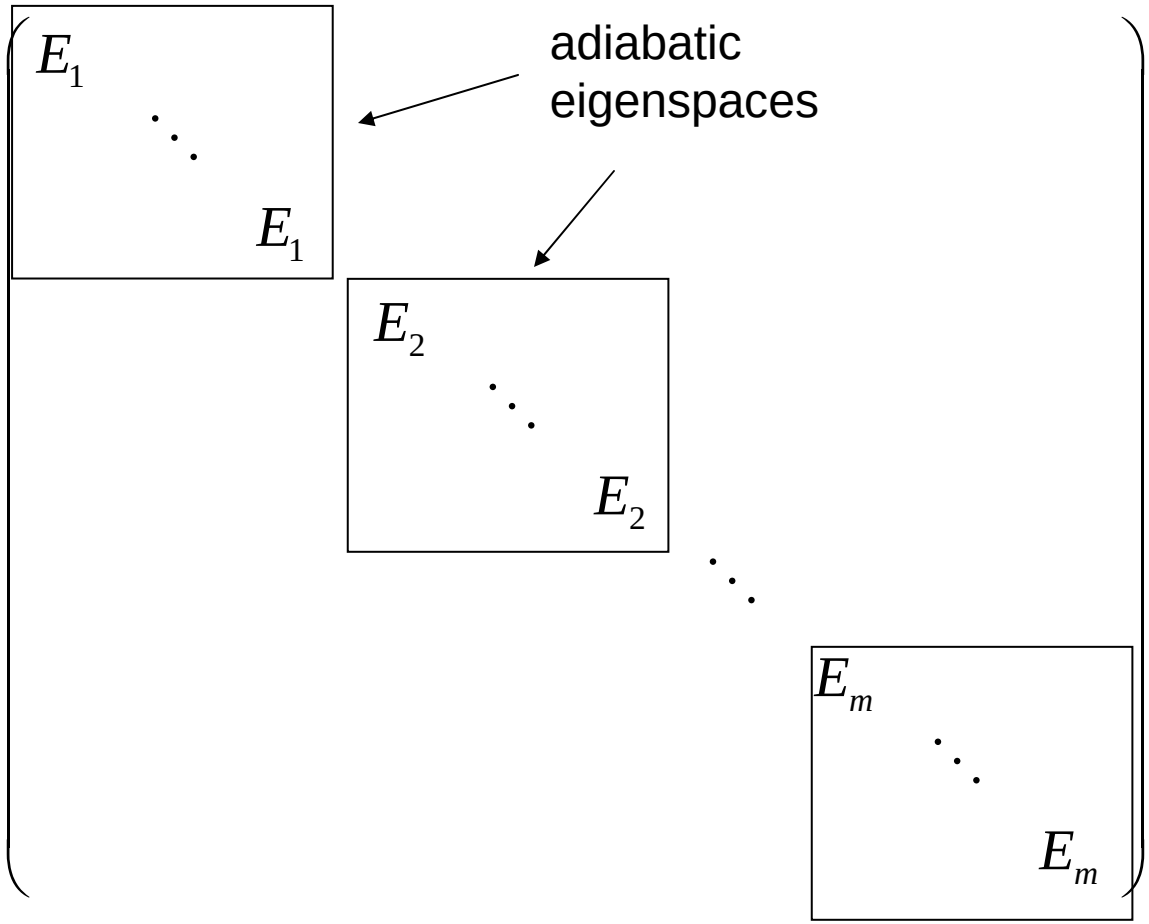
Closed system

adiabatic dynamics takes place in instantaneous decoupled eigenspaces of time-dependent Hamiltonian $\mathcal{H}$

$$\mathcal{H}(t) \mapsto \mathcal{H}_D(t) =$$



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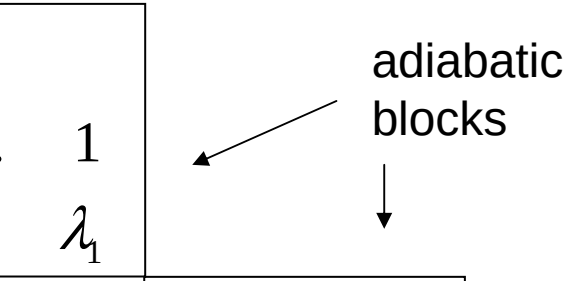
Open system

adiabatic dynamics takes place in instantaneous decoupled Jordan-blocks of dynamical superoperator $\mathcal{L}$

$$\mathcal{L}(t) \;\mapsto\; \mathcal{L}_J(t) = \begin{pmatrix} \lambda_1 & 1 & & & \\ & \ddots & 1 & & \\ & & \lambda_1 & & \\ & & & \lambda_2 & 1 \\ & & & & \ddots \\ & & & & & \ddots \\ & & & & & & \lambda_m & 1 \\ & & & & & & & \ddots & 1 \\ & & & & & & & & \lambda_m \end{pmatrix}$$



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Remark on Order of Operations

In deriving adiabatic condition for open systems we have a choice:

1. $\mathrm{Tr}_B \circ (\text{adiabatic limit})_{S+B}$

2. $\mathrm{Tr}_B \circ (\text{adiabatic limit})_S$

3. $(\text{adiabatic limit})_S \circ \mathrm{Tr}_B$

We chose 3. since

1. System and bath generally subject to different time scales. May also be impractical.
2. Adiabatic limit on system is not well defined when bath degrees of freedom are still explicitly present.

PRA **71**, 012331 (2005)

Time Condition for Adiabatic Dynamics

- Condition for adiabaticity:

  Total evolution time can be lower-bounded by

$$\frac{\text{max. norm of time-derivative of generator } (H)}{\text{min. } \cancel{\text{square}} \text{ power of |spectral gap| (energies)}}$$

$$T \gg \frac{\mathcal{F}}{\mathcal{G}^2},$$

$$\mathcal{F} = \max_{\substack{0 \leq s \leq 1 \\ s = t/T}} \left| \langle k(s) | \frac{dH(s)}{ds} | m(s) \rangle \right|,$$

$$\mathcal{G} = \min_{0 \leq s \leq 1} |g_{mk}(s)|.$$

  - What is the analogous condition for open systems?

PRA **71**, 012331 (2005)

Time Condition for Open Systems Adiabaticity

$$T \gg \max_{0\le s\le 1}\left|\sum_{\beta\neq\alpha}\left(\frac{V_{\beta\alpha}(0)}{\omega^2_{\beta\alpha}(0)}-\frac{V_{\beta\alpha}(s)}{\omega^2_{\beta\alpha}(s)}e^{T\Omega_{\beta\alpha}(s)}+\int_0^s ds' e^{T\Omega_{\beta\alpha}(s')}\frac{d}{ds'}\frac{V_{\beta\alpha}(s')}{\omega^2_{\beta\alpha}(s')}\right)\right| \equiv T^c_\alpha$$

1D Jordan blocks

$$T \gg T^c_\alpha = \max_{0\le s\le 1}\left|\sum_{\beta\,|\,\gamma_\beta\neq\gamma_\alpha}\sum_{j=0}^{n_\beta-1}\sum_{p=1}^{(n_\alpha-i)}\left(\prod_{q=1}^{p}\sum_{k_q=0}^{j-S_{q-1}}\right)(-1)^{S_p}\left[\frac{V^{(ijp)}_{\beta\alpha}(0)}{\omega^{p+S_p+1}_{\beta\alpha}(0)}-\frac{V^{(ijp)}_{\beta\alpha}(s)\,e^{T\Omega_{\beta\alpha}(s)}}{\omega^{p+S_p+1}_{\beta\alpha}(s)}+\int_0^s ds'\,e^{T\Omega_{\beta\alpha}(s')}\frac{d}{ds'}\frac{V^{(ijp)}_{\beta\alpha}(s')}{\omega^{p+S_p+1}_{\beta\alpha}(s')}\right]\right|,$$

where

- $S_q=\sum_{z=1}^{q}k_z$, with $S_0\equiv 0$; numerical factor
- $V^{(ijp)}_{\beta\alpha}(s)=\langle\psi^{(j)}_\beta(s)|\frac{d\mathcal{L}(s)}{ds}|\mathcal{D}^{(i+p-1)}_\alpha(s)\rangle$, with $\mathcal{D}^{(j-S_p)}_\alpha(s)\equiv 0$ for $j-S_p<0$; time derivative of generator
- $\Omega_{\beta\alpha}(t)=\int_0^t\omega_{\beta\alpha}(t')\,dt'$, with $\omega_{\beta\alpha}(t)=\gamma_\beta(t)-\gamma_\alpha(t)$. spectral gap

Remarks:

- The crossover time $T^c_{\alpha s}$ provides a decoupling timescale for each Jordan block
- If there is a growing exponential (Ω real and positive) then adiabaticity persists over a *finite* time interval, then disappears!
- This implies existence of optimal time for adiabaticity
- The exponential cancels in the closed-system case since it is fast oscillating

Application 1: Geometric Phases

- Adiabatic cyclic geometric phase:
  - Berry (Abelian) phase: non-degenerate states
  - Wilczek-Zee (non-Abelian) phase: degenerate states

(Closed System) Berry Phase

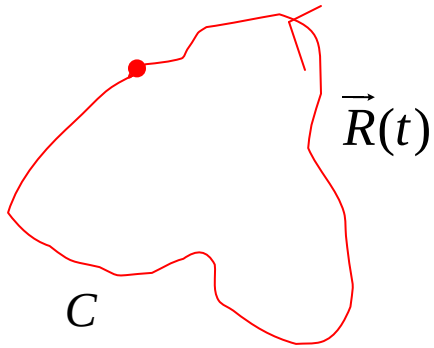
- What is the Berry Phase?

  A purely geometric phase acquired by a quantum state undergoing cyclic adiabatic evolution

  Control parameters (vector field, e.g., magnetic field): $\vec{R}(t)$

  Cyclic evolution along a path C: $H(\vec{R}(0)) = H(\vec{R}(T))$

  $H(\vec{R}(t))\left|\psi_n(t)\right\rangle = E_n(t)\left|\psi_n(t)\right\rangle$



$$\Rightarrow \quad \left|\psi_n(T)\right\rangle = \underbrace{e^{i\gamma_n}}_{\text{geometric}} \underbrace{e^{-i\int_0^T E_n(t)\,dt}}_{\text{dynamic}} \left|\psi_n(0)\right\rangle$$

$$\gamma_n = \oint_C \left\langle \psi_n(\vec{R}) \middle| \vec{\nabla} \middle| \psi_n(\vec{R}) \right\rangle \cdot d\vec{R}$$

Open Systems Geometric Phase

Convolutionless master equation, implicit time-dependence through parameters $\vec{R}(t)$:

$$\partial\rho/\partial t = \mathcal{L}[\vec{R}(t)]\rho(t)$$

Solve by expanding d.m. in right eigenbasis, explicitly factor out dynamical phase

$$|\rho(t)\rangle\rangle = \sum_{\beta=1}^{m}\sum_{j=0}^{n_\beta-1} p_\beta^{(j)}(t)\, e^{\int_0^t \lambda_\beta(t')dt'} |\mathcal{D}_\beta^{(j)}[\vec{R}(t)]\rangle\rangle$$

Substituting:

$$\dot{p}_\alpha^{(i)} = p_\alpha^{(i+1)} - \sum_{\beta\,|\,\lambda_\beta=\lambda_\alpha}\sum_{j=0}^{n_\beta-1} p_\beta^{(j)} \langle\langle \mathcal{E}_\alpha^{(i)}|\dot{\mathcal{D}}_\beta^{(j)}\rangle\rangle$$

Simplification for single, 1D, non-degenerate Jordan block:

$$\dot{p}_\alpha = -p_\alpha \langle\langle \mathcal{E}_\alpha|\dot{\mathcal{D}}_\alpha\rangle\rangle$$

Solution:

$$p_\alpha(t) = p_\alpha(0)\exp\left[i\gamma_\alpha(t)\right] \quad \text{with } \gamma_\alpha(t) = i\int_0^t \langle\langle \mathcal{E}_\alpha(t')|\dot{\mathcal{D}}_\alpha(t')\rangle\rangle dt'$$

Abelian Geometric Phase:

$$\gamma_\alpha(C) = i\oint_C \langle\langle \mathcal{E}_\alpha(\vec{R})|\vec{\nabla}|\mathcal{D}_\alpha(\vec{R})\rangle\rangle \cdot d\vec{R}$$

$$\gamma_n = \oint_C \left\langle \psi_n(\vec{R})\middle|\vec{\nabla}\middle|\psi_n(\vec{R})\right\rangle \cdot d\vec{R}$$

PRA **73**, 062101 (2006)

Abelian Geometric Phase: $\gamma_\alpha(C) = i \oint_C \langle\langle \mathcal{E}_\alpha(\vec{R}) | \vec{\nabla} | \mathcal{D}_\alpha(\vec{R}) \rangle\rangle \cdot d\vec{R}$

- $\gamma_\alpha(C)$ is geometric: depends only on path traversed in parameter space.

- $\gamma_\alpha(C)$ is gauge invariant: cannot modify it by redefining left or right eigenvectors via multiplication by complex factor $\chi(\vec{R})e^{i\nu(\vec{R})}$.

- $\gamma_\alpha(C)$ reduces to usual expression for (difference of) geometric phases in limit of closed system.

- $\gamma_\alpha(C)$ can be complex: real part determines visibility.

PRA **73**, 062101 (2006)

Non-Abelian Open Systems Geometric Phase

Case of degenerate 1D Jordan blocks: $\dot{p}^{(i)}_{\lambda_\alpha} = -\sum_{j=1}^{G} p^{(j)}_{\lambda_\alpha} \langle\langle \mathcal{E}^{(i)}_{\lambda_\alpha} | \dot{\mathcal{D}}^{(j)}_{\lambda_\alpha} \rangle\rangle$

Rewrite: $\dot{\mathbf{P}}_{\lambda_\alpha} = -(\vec{A}_{\lambda_\alpha} \cdot \dot{\vec{R}}) \mathbf{P}_{\lambda_\alpha}$ where $\mathbf{P}_{\lambda_\alpha} = \left(p^{(1)}_{\lambda_\alpha}, \cdots, p^{(G)}_{\lambda_\alpha} \right)^t$

Solution: $$\mathbf{P}_{\lambda_\alpha}(C) = \mathcal{U} \mathbf{P}_{\lambda_\alpha}(0)$$

Wilson loop: $$\mathcal{U} = \mathcal{P}\, e^{-\oint_C \vec{A}_{\lambda_\alpha} \cdot d\vec{R}}$$

non-Abelian Wilczek-Zee gauge potential;
holonomic connection:

$$\vec{A}_{\lambda_\alpha} = \begin{pmatrix} \langle\langle \mathcal{E}^{(1)}_{\lambda_\alpha} | \vec{\nabla} | \mathcal{D}^{(1)}_{\lambda_\alpha} \rangle\rangle & \cdots & \langle\langle \mathcal{E}^{(1)}_{\lambda_\alpha} | \vec{\nabla} | \mathcal{D}^{(G)}_{\lambda_\alpha} \rangle\rangle \\ \vdots & \ddots & \vdots \\ \langle\langle \mathcal{E}^{(G)}_{\lambda_\alpha} | \vec{\nabla} | \mathcal{D}^{(1)}_{\lambda_\alpha} \rangle\rangle & \cdots & \langle\langle \mathcal{E}^{(G)}_{\lambda_\alpha} | \vec{\nabla} | \mathcal{D}^{(G)}_{\lambda_\alpha} \rangle\rangle \end{pmatrix}$$

Geometric, gauge-invariant, correct closed-system limit, complex valued.

PRA **73**, 062101 (2006)

Berry's Example: Spin-1/2 in Magnetic Field Under Decoherence

System Hamiltonian: $H(\vec{B}) = -\mu \vec{S} \cdot \vec{B}$ $\quad \vec{B}(t) = (B_x(t), B_y(t), B_z(t))$

In adiabatic + weak-coupling limit leading to the *Markovian* master equation, Lindblad operators *must* follow Hamiltonian (Davies & Spohn, J. Stat. Phys. '78):

Dephasing: $\Gamma_z = \beta_z W(\vec{B}) \sigma_z W^\dagger(\vec{B})$

Spontaneous emission: $\Gamma_- = \beta_- \underbrace{W(\vec{B})} \sigma_- W^\dagger(\vec{B})$

diagonalizes $H(\vec{B})$

Superoperator: $\mathcal{L}(\vec{B}) = \mathcal{H}(\vec{B}) + \mathcal{R}(\vec{B})$

$[\mathcal{H}, \mathcal{R}] = 0$, diagonalizable, hence $\mathcal{L}$ diagonalizable in $\mathcal{H}$ eigenbasis.

But $\mathcal{H}$ eigenbasis doesn't depend on $\beta_{z,-}$

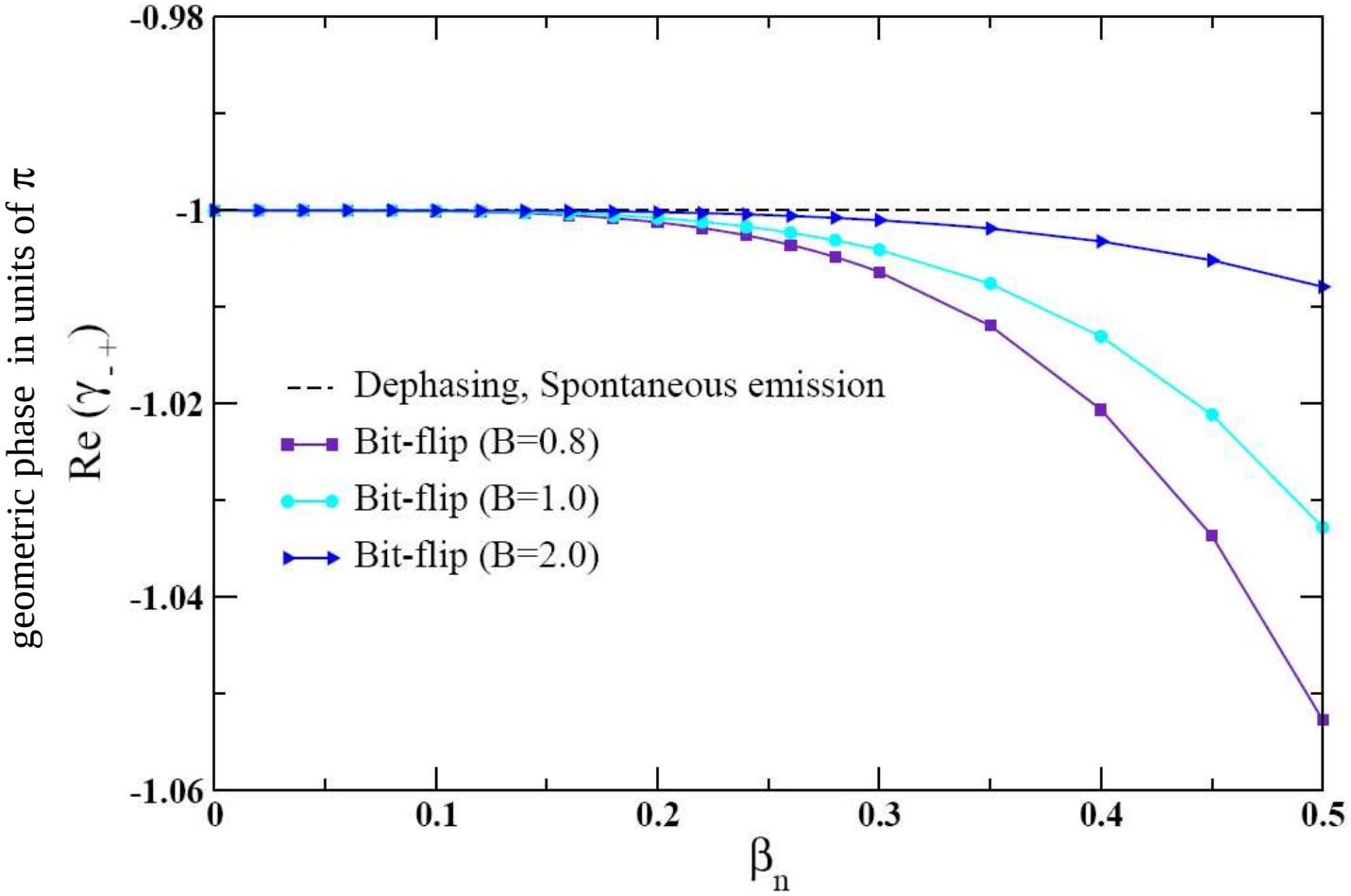
Hence geometric phase immune to dephasing and spont. emission! Closed system result reproduced.

Adiabaticity time does depend on $\beta_{z,-}$. I.e., adiabatic geometric phase disappears when adiabatic approximation breaks down.

PRA **73**, 062101 (2006)

Geometric phase is *not* invariant under bit flip: $\Gamma_x(\vec{B}) = \beta_x W(\vec{B})\sigma_x W(\vec{B})^\dagger$

Results for spherically symmetric B-field, azimuthal angle=π/3



$\mathrm{Im}(\gamma) \approx 10^{-16}$

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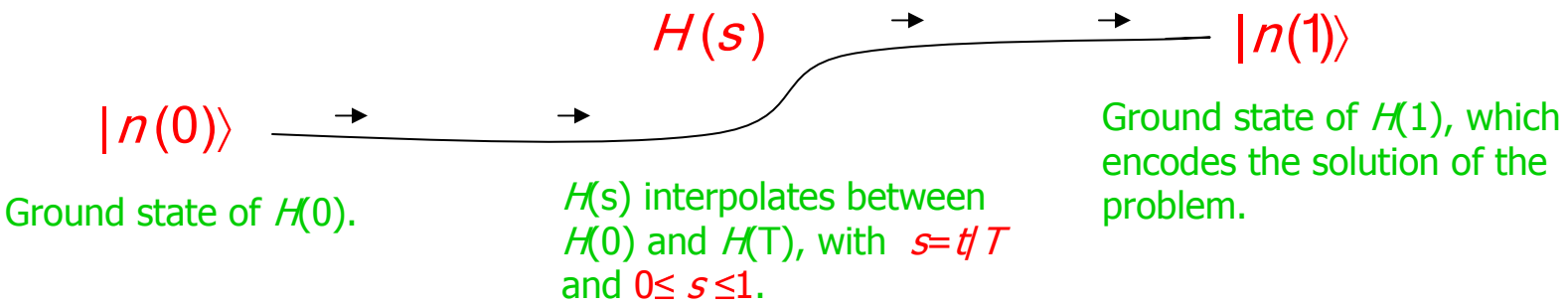
Application 2: Adiabatic Quantum Computing

Adiabatic quantum computation (AQC): Farhi et al., Science (2001)



- Start a system in non-degenerate ground state of easily prepared Hamiltonian,

  e.g., $H(0) = -B(0)\sum_i S_i^z$: $\quad |\psi_{GS}(0)\rangle = |\uparrow\uparrow \ldots \uparrow\rangle$.

- For $t > 0$: $\quad H(t) = -B(t)\sum_i S_i^z - \sum_{i<j} J_{ij}(t)\vec{S}_i \cdot \vec{S}_j$

- Time-evolve adiabatically to new Hamiltonian whose ground state is computationally hard to find, e.g.,

  $H(T) = -\sum_{i<j} J_{ij}(T)\vec{S}_i \cdot \vec{S}_j$, $\quad J_{ij}$ random and fixed: $|\psi_{GS}(T)\rangle = ???$

- Measure individual spin states and find answer to hard computational question!

- Procedure's success depends on gap not being too small: $T \gg \dfrac{\mathcal{F}}{\mathcal{G}^2}$,

$$\mathcal{G} \geq (poly(N))^{-1}$$

Implications for Adiabatic QC

M.S. Sarandy, DAL, Phys. Rev. Lett. **95**, 250503 (2005)

Adiabatic QC can only be performed *while adiabatic approximation is valid*.

Breakdown of adiabaticity in an open system implies same for AQC.

Example: Deutsch-Josza algorithm for a single qubit, under (non-Markovian) dephasing, with unitary interpolation

Adiabatic time interval: *adiabaticity only occurs for a finite time interval*, disappearing afterwards! This behavior is an exclusive feature of open systems, which is illustrated in Fig. 1.

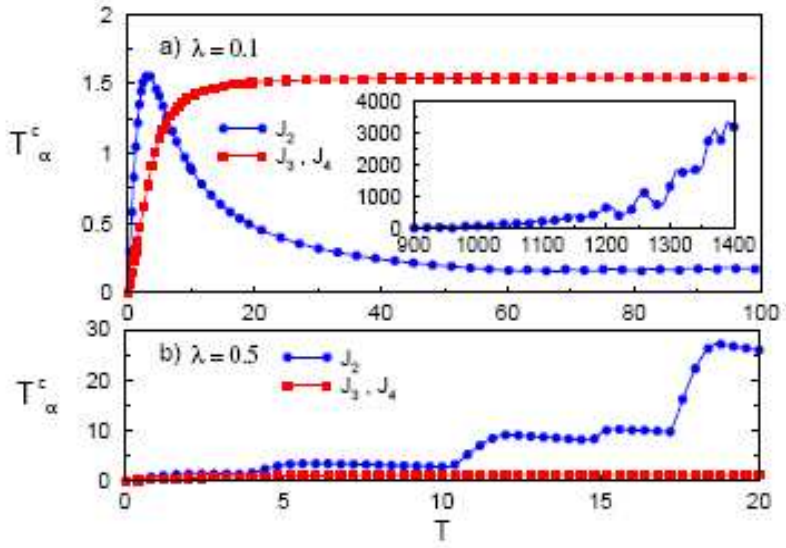


FIG. 1: Crossover time T^c_α as a function of the total evolution time T for the Jordan blocks J_2, J_3, and J_4 (the plot for J_4 is the same as for J_3). The inset of (a) shows that J_2 has non-negligible couplings for $T \sim 10^3$ (in units such that $\omega = 1$).

- Concrete case: take $\lambda = 0.1$ and impose a certainty of 90% (either $p_+ = 0.9$ or $p_- = 0.9$). Then, $T \approx 11$, which is compatible with the adiabaticity, since $T_2^c \approx 0.82$ and $T_3^c = T_4^c \approx 1.43$.

Implications for Adiabatic QC

M.S. Sarandy, DAL, Phys. Rev. Lett. **95**, 250503 (2005)

Robustness of adiabatic QC depends on spectral gap closing slowly (polynomially) with system size. Can we somehow preserve the gap?

Yes, using a “unitary interpolation strategy” [see also M.S. Siu, PRA ’05].

We find:
A constant gap is possible in the Markovian weak-coupling limit
A constant gap is non-generic in non-Markovian case

.

Unitary Interpolation: Adiabatic Open Systems

Sufficient condition for constant gaps in open systems:

Consider a Lindblad super-operator $\mathcal{L}(s) = \mathcal{H}(s) + \mathcal{R}(s)$, where $\mathcal{H}(s)$ denotes the Hamiltonian super-operator and $\mathcal{R}(s)$ the decohering super-operator. If the Hamiltonian changes by a unitary transformation, namely $\mathcal{H}(s) = \mathcal{V}^\dagger(s)\mathcal{H}(0)\mathcal{V}(s)$, then a sufficient condition for a constant spectrum* of $\mathcal{L}(s)$ is $\mathcal{R}(0) = \mathcal{V}(s)\mathcal{R}(s)\mathcal{V}^\dagger(s)$.

In adiabatic + weak-coupling limit leading to the *Markovian* master equation, Lindblad operators *must* follow Hamiltonian (Davies & Spohn, J. Stat. Phys. '78).
But otherwise this condition is non-generic.

*Note: constant super-operator spectrum implies constant gaps in Hamiltonian spectrum

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Example: Deutsch-Josza algorithm under (non-Markovian) dephasing, with unitary interpolation

- Initial state: $|\psi(0)\rangle = |+_1\rangle \otimes \cdots \otimes |+_N\rangle$, where $|\pm_i\rangle = (|0_i\rangle \pm |1_i\rangle)/\sqrt{2}$, with $\{|0_i\rangle, |1_i\rangle\}$ denoting the basis for the i^{th} qubit.

- Initial Hamiltonian:

$$H_0 = \omega \sum_{i=1}^{N} (I_1 \otimes \cdots \otimes I_{i-1} \otimes |-_i\rangle\langle -_i| \otimes I_{i+1} \otimes \cdots \otimes I_N)$$

- Unitary transformation: $U|x\rangle = (-1)^{f(x)}|x\rangle$ ($x \in X_N$, with X_N being the set of natural numbers represented by N bits). Matrix representation: $U = \text{diag}\left((-1)^{f(0)}, ..., (-1)^{f(2^N-1)}\right)$.

- Final state: $|\psi(1)\rangle = U|\psi(0)\rangle$.

- Final Hamiltonian: $H(1) = UH(0)U^\dagger$.

- Interpolation: $H(s) = \tilde{U}(s)H(0)\tilde{U}^\dagger(s)$, with $\tilde{U}(s) = \exp\left(i\frac{\pi}{2}sU\right)$.

- Measurement: if f is constant then all the qubits are in the state $|+\rangle$, otherwise f is balanced. For the simple case $N = 1$ we have:

$$M_\pm = |\pm\rangle\langle\pm|; \;\; p_\pm = \frac{1}{2}\left[1 \pm (-1)^{f(0)+f(1)}\right]$$

- Standard adiabatic theorem: $T \gg \pi/2\omega$ (independently of N).

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Open systems: we illustrate the procedure for the case of a single qubit, i.e., $N = 1$. Let us consider constant dephasing in the computational basis $\{|0\rangle, |1\rangle\}$, given by $\Gamma = \lambda\sqrt{\omega}\,\sigma_z$.

(a) Lindblad super-operator:

$$\mathcal{L}(s) = \omega \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2\,\lambda^2 & 0 & q(s) \\ 0 & 0 & -2\lambda^2 & -r(s) \\ 0 & -q(s) & r(s) & 0 \end{pmatrix}$$

with $r(s) = -\cos\frac{\pi F}{2}s$, $q(s) = \sin\frac{\pi F}{2}s$, and $F \equiv (-1)^{f(0)} - (-1)^{f(1)}$.

- The sufficient condition for constant spectrum is satisfied!

- Eigenvalues: $\gamma_1 = 0$, $\gamma_2 = -2\omega\lambda^2$, $\gamma_3 = \omega(-\lambda^2 - \sqrt{\lambda^4 - 1})$, and $\gamma_4 = \omega(-\lambda^2 + \sqrt{\lambda^4 - 1})$. These eigenvalues are non-degenerate for $0 < \lambda < 1$ and define four *one-dimensional* Jordan blocks for the Lindblad super-operator, denoted by J_α ($\alpha \in \{1, ..., 4\}$).

(b) Adiabatic Lindblad equation: Let us suppose that λ and T are such that the adiabatic approximation can be applied. Then, the solution of the Lindblad equation in the adiabatic regime is

$$\rho(s) = \frac{1}{2}\left[I + e^{-2\lambda^2 sT}(-1)^{f(0)+f(1)}\sigma_x\right]$$

Additional comments:
- Gaps constant in spite of non-Markovian model. True also for *spontaneous emission* in this example.
- Four 1D Jordan blocks; one automatically decoupled. Hence adiabaticity depends on decoupling of other three.

Phys. Rev. Lett. **95**, 250503 (2005)

Example: Deutsch-Josza algorithm under (non-Markovian) dephasing, with unitary interpolation

(c) Adiabatic time interval: *adiabaticity only occurs for a finite time interval*, disappearing afterwards! This behavior is an exclusive feature of open systems, which is illustrated in Fig. 1.

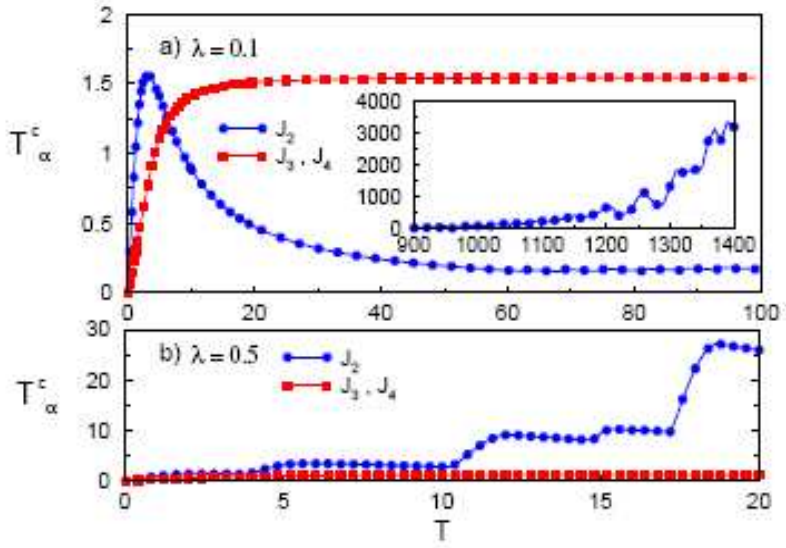

FIG. 1: Crossover time T^c_α as a function of the total evolution time T for the Jordan blocks J_2, J_3, and J_4 (the plot for J_4 is the same as for J_3). The inset of (a) shows that J_2 has non-negligible couplings for $T \sim 10^3$ (in units such that $\omega = 1$).

• Concrete case: take $\lambda = 0.1$ and impose a certainty of 90% (either $p_+ = 0.9$ or $p_- = 0.9$). Then, $T \approx 11$, which is compatible with the adiabaticity, since $T_2^c \approx 0.82$ and $T_3^c = T_4^c \approx 1.43$.

Summary of Conclusions for Adiabatic QC

- Adiabatic QC can only be performed *while adiabatic approximation is valid*.

- However, the adiabatic approximation (typically) breaks down in an open system if the evolution is sufficiently long.

- Breakdown of adiabaticity in an open system implies same for AQC.

- Breakdown is due to vanishing of gaps, due to interaction with environment.

- Gaps can be kept constant via unitary interpolation (at expense of introducing many-body interactions) when bath is Markovian.

- Error correction techniques (e.g., Jordan, Shor & Farhi, quant-ph/0512170) are needed.

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Summary

- Adiabaticity defined for open systems in terms of decoupling of Jordan blocks of super-operator (for another approach based on weak coupling see P. Thunstrom, J. Aberg, E. Sjoqvist, PRA **72**, 022328 (2005))

- Central feature: adiabaticity can be a temporary feature in an open system

- Implications for robustness of adiabatic QC and for geometric phases in open systems

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