HERMITIAN K-THEORY

Max Karoubi

Lecture 5 : The cup-product of F.J.-B.J. Clauwens and the proof of Theorem 3.8

5.1. We are going to introduce a fundamental idea, due to Clauwens [C] [1979], unfortunately not known until now by the author[1].

We consider two rings A and B with involution and their associated categories of split quadratic modules. More precisely, we consider first the category $\eta Q^{split}(B)$ and secondly the subcategory ${}_{\varepsilon}Q'^{split}(A[s])$ of ${}_{\varepsilon}Q^{split}(A[s])$ consisting of $A[s]$-modules extended from A (the involution of $A[s]$ being induced by the involution of A and the transformation $s \mapsto 1 - s$).

An object of ${}_{\varepsilon}Q'^{split}(A[s])$ may be written as a couple (E, γ), where E is a projective finitely generated A-module and γ is a non degenerate ε-quadratic form on $E \otimes_Z \mathbf{Z}[s]$ which we may write as $\sum \gamma_n s^n$, where γ_n is a morphism from E to E^*.
Let us consider now an object (F, δ) of ${}_{\eta}Q^{split}(B)$ (where δ is a non degenerate η-quadratic form on F with Δ as associated hermitian form). On $E \otimes F$, we then consider the $\varepsilon\eta$-quadratic form defined by the following formula (note that we dont define a functor, but just a pairing between objects) :

$$\kappa = \sum \gamma_n \otimes \Delta\ (\Delta^{-1}\delta)^n$$

In some fundamental lemmas, Clauwens showed that this pairing

$$Obj\ ({}_{\varepsilon}Q'^{split}(A[s]))\ x\ Obj\ ({}_{\eta}Q^{split}(B)) \rightarrow Obj\ ({}_{\varepsilon\eta}Q^{split}(A \otimes B))$$

is well defined on isomorphism classes of quadratic modules. In particular, we have a cup-product

$${}_{\varepsilon}KQ'^{\ split}(A[s])\ x\ {}_{\eta}KQ^{split}(B) \rightarrow {}_{\varepsilon\eta}KQ^{split}(A \otimes B)$$

where ${}_{\varepsilon}KQ'^{\ split}(A[s])$ is the subgroup of ${}_{\varepsilon}KQ^{split}(A[s])$ generated by extended modules over A (this is automatically the case when A is regular noetherian for instance).

5.2. At the beginning of his paper (theorem 1, p. 42), Clauwens showed that up to additions with hyperbolics, we can reduce us to the case where γ is "linear", i.e. of the type $\gamma = g\ s$. In other words, $\gamma_n = 0$, except γ_1 which is equal to g. Since the associated hermitian form $g\ s + \varepsilon {}^t g\ (1 - s)$ is an isomorphism, this implies that ${}^t g = \varepsilon\ g\ (1 + N)$ where N is a nilpotent endomorphism of E (this is called an "almost symmetric form" by Clauwens). In this case, the formula for the quadratic form κ above is quite simple : one finds

[1] I would like to thank A. Ranicki for this reference.

$$\kappa = g \otimes \delta$$

In other words, the above pairing on the KQ^{split} -groups generalizes the usual pairing between hermitian forms and quadratic forms (see Theorem 2, p. 43).

5.3. A careful study of Clauwens lemmas shows that it is important to work in the split category. For instance, if we want to prove that the isomorphism class of the quadratic form κ above only depends of the class of the quadratic form φ, one has to perform delicate arguments using precisely the "splitting" of the quadratic form (see the lemma p. 44 in Clauwens's paper).

5.4. In order to extend the previous considerations to higher KQ^{split}-groups, one may interpret the elements of ${}_{\varepsilon}KQ_n^{split}(A)$ as equivalence classes of suitable flat bundles over homology spheres of dimension n (as explained with great details in [K1] § 3 for usual K-theory). For instance, if we have a flat bundle E (resp F) over an homology sphere X (resp Y), they are equivalent if we can find a third flat bundle G over Z and two homology equivalences

$$X \to Z \leftarrow Y$$

such that the pull-back of G by the first map (resp. the second map) is isomorphic to E (resp. F). We can put on these bundles some extra structures like split quadratic forms. By general non sense, as explained in [K1], we can then extend Clauwens's cup-product as a bilinear pairing (now for n and $p \in \mathbf{Z}$)

$${}_{\varepsilon}KQ_n^{'split}(A[s]) \times {}_{\eta}KQ_p^{split}(B) \to {}_{\varepsilon\eta}KQ_{n+p}^{split}(A \otimes B)$$

In order to show that this pairing is well defined (on the category of flat bundles provided with a quadratic form), the "split" structure is essential as it was shown by Clauwens in his lemma p. 44. More precisely, we have to show that if E (resp. F) is equivalent to E' (resp. F'), then $E \otimes F$ is equivalent to $E' \otimes F'$, which is essentially the contents of this lemma, adapted to the category of flat bundles with the appropriate structure.

5.5. There is another piece of information which is important for us but more easy to define. It is the usual bilinear pairing

$${}_{\varepsilon}KQ_n^{\max}(A) \times {}_{\eta}KQ_n^{split}(B) \to {}_{\varepsilon\eta}KQ_n^{split}(A \otimes B)$$

5.6. THEOREM. *The cup-product of Clauwens is partially associative in the following sense. For two rings C and D, one has the following commutative diagram (with $n = n_1 + n_2$, $\varepsilon = \varepsilon_1\varepsilon_2$ and $A = C \otimes D$)*

$$\begin{array}{ccc}
{}_{\varepsilon_1}KQ_{n_1}^{\max}(C) \times {}_{\varepsilon_2}KQ_{n_2}^{'split}(D[s]) \times {}_{\eta}KQ_p^{split}(B) & \to & {}_{\varepsilon_1}KQ_{n_1}^{\max}(C) \times {}_{\varepsilon_2\eta}KQ_{n_2+p}^{split}(D \otimes B) \\
\downarrow & & \downarrow \\
{}_{\varepsilon_1\varepsilon_2}KQ_{n_1+n_2}^{'split}((C \otimes D)[s]) \times {}_{\eta}KQ_p^{split}(B) & \to & {}_{\varepsilon\eta}KQ_{n+p}^{split}(A \otimes B)
\end{array}$$

5-3

Proof. All what we have to do is to look at the algebraic formula written in 5.1. What we are doing is multiplying the two sides of the formula by the same even hermitian form before of after taking the tensor product with $\Delta \ (\Delta^{-1}\delta)^n$.

5.7. If we look again at the proof of theorem 3.7, we see that we have used a Bott element in ${}_{-1}D_0^{\max}(Z)$ (cf. 4.6.2) and another one is ${}_{-1}E_{-2}^{split}(\bar{Z})$ (cf. 4.6.3) where $\bar{Z} = Z[s]$. The partial associativity showed in 5.6 is used to prove that the two maps between ${}_{\varepsilon}V^{split}(SA)$ and ${}_{-\varepsilon}U^{split}(A)$ are homotopically inverse to each other (on the level of cohomology theories or spectra) : see 5.10 for more details.

5.8. We should remark that if we are just interested in Witt groups as in 3.12, the previous considerations leads to morphisms (for the split Witt groups for any ring)

$$ {}_{\varepsilon}W_n(A) \to {}_{-\varepsilon}W_{n+2}(A) \text{ and } {}_{-\varepsilon}W_{n+2}(A) \to {}_{\varepsilon}W_n(A) $$

which composites in both directions are the multiplication by 4. This improves somewhat the control of the 2-primary torsion of the higher Witt groups (for any ring A).

5.9. Note that for negative degrees, we dont need flat bundles since we have just to consider modules over iterated suspensions S^nA, $n \geq 0$. In this range of degrees, we can restrict ourselves to usual quadratic forms (for $n \geq 0$) or to even hermitian forms (for $n > 0$).

5.10 We are now ready to prove theorem 3.8. As a matter of fact, we just have to copy the proof of theorem 3.7 until a certain point. This point was exactly in 4.6.3 when we want to define (in the split category) a map

$$\theta : {}_{\varepsilon}\mathbb{K}Q^{split}(A) \longrightarrow {}_{-\varepsilon}\mathbb{E}^{split}(S^2A)$$

This is exactly where we need Clauwens's cup-product.

$${}_{\varepsilon}\mathbb{K}Q^{split}(A) \times {}_{\eta}\mathbb{K}Q^{split}(\mathbb{Z}) \to {}_{\varepsilon\eta}\mathbb{K}Q^{split}(A)$$

By taking suitable htpy fibers, one can also define formally a cup-product

$${}_{\varepsilon}\mathbb{K}Q^{split}(A) \times {}_{\eta}\mathbb{E}^{split}(\mathbb{Z}) \longrightarrow {}_{\varepsilon\eta}\mathbb{E}^{split}(A)$$

where $\mathbb{E}^{split}(A)$ is the homotopy fiber of

$$\mathbb{W}^{split}(SA) \longrightarrow \mathbb{W}^{split}(SA \times SA^{op}) \simeq \mathbb{K}(A)$$

The explicit element constructed in ${}_{-1}\mathbb{E}_{-2}^{split}(\mathbb{Z}) = {}_{-1}\mathbb{E}_{-2}(\mathbb{Z})$ enables us to define the required map

$${}_{\varepsilon}\mathbb{K}Q^{split}(A) \longrightarrow {}_{-\varepsilon}\mathbb{E}^{split}(S^2A)$$

The rest of the argument follows the same scheme as the proof of theorem 3.7.

REFERENCES

[Bak] BAK A. : K-theory of forms. Annals of Math Studies 98 (1981).

[B] BALMER P. : Triangular Witt groups. K-theory 19, pp. 311-363 (2000) and Math. Z. 236, pp. 351-382 (2001). See also "Handbook of K-theory" pp. 539-576. Springer (2005).

[BL] J. BARGE et J. LANNES : Suites de Sturm, indice de Maslov et périodicité de Bott (preprint).

[Bass] BASS H. : Algebraic K-theory. Benjamin (1968).

[Bott] BOTT R. : The stable homotopy of the classical groups. Ann. of Math. 70, pp. 313-337 (1959).

[C] CLAUWENS F.J.-B.J : The K-theory of almost symmetric forms. Proceedings of the symposium in Amsterdam, October 31-November 2 (1978). Topological structures II. Mathematical Centre Tracts 115, p. 41-49 (1979)

[K1] KAROUBI M. : Homologie cyclique et K-théorie. Astérisque N° 149 (1987).

[K2] KAROUBI M. : Théorie de Quillen et homologie du groupe orthogonal. Le théorème fondamental de la K-théorie hermitienne. Annals of Maths 112, pp. 207-282 (1980).

[K3] KAROUBI M. : Foncteurs dérivés et K-théorie. Séminaire Heidelberg-Saarbrücken-Strasbourg 1967/68, exposé 4. Springer Lecture Notes in Math. Nr 136 (1970).

[K4] KAROUBI M. : Stabilization of the Witt group. Comptes Rendus Acad. Sci. Paris 362, pp. 165-168 (2006).

[KV] KAROUBI M. et VILLAMAYOR O. : K-théorie algébrique et K-théorie topologique II. Math. Scand. 32, pp. 57-86 (1973).

[M] MILNOR J. : Introduction to Algebraic K-theory. Annals of Math. Studies Nr 72. Princeton University Press (1974).

[O] OJANGUREN M. : On Karoubi's theorem : W(A) = W(A[t]). Arch. Math. Basel 43, pp. 328-331 (1984).

[S] SHARPE R. : On the structure of the unitary Steinberg group. Annals of Math. 96, pp. 444-479 (1972).

[W] WAGONER J.B. : Delooping classifying spaces in Algebraic K-theory. Topology 11, pp. 349-370 (1972).

Note : most of the papers of M. Karoubi may be downloaded from the following Web site :

http://www.math.jussieu.fr/~karoubi/

TOPOLOGICAL STRUCTURES II, MATHEMATICAL CENTRE TRACTS 115 (1979) 41-49

THE K-THEORY OF ALMOST SYMMETRIC FORMS

F.J.-B.J. Clauwens

INTRODUCTION

To motivate this paper we first recall a few facts.

According to [W1 , chapter 5] a normal map f between manifolds of dimension 2k and fundamental group π gives rise to a (so-called quadratic) form ψ defined on some finitely generated free left B module V, where B denotes the integral group ring $Z[\pi]$. The appropriate equivalence class of ψ in $L_{2k}(B)$ is the obstruction s(f) for changing f into a homotopy equivalence by surgery (for $k > 2$).

According to [C] a closed manifold P of dimension 2q and fundamental group ρ gives rise to a (so-called almost symmetric) form σ defined on some finitely generated free left A module K, where A is $Z[\rho]$. The main theorem there states that $\sigma \otimes \psi$ represents the obstruction for doing surgery on $id_p \times f$ if ψ does so for f.

In this paper we will study the algebra of almost symmetric forms; therefore we first recall the main things about quadratic forms from [W2].

Orientability considerations give rise to a homomorphism $w: \pi \to \{\pm 1\}$. The map $-: B \to B$ defined by the formula $\overline{\Sigma n_g g} = \Sigma n_g w(g) g^{-1}$ satisfies $\overline{x+y} = \overline{x}+\overline{y}$, $\overline{xy} = \overline{y}\,\overline{x}$ and $\overline{\overline{x}} = x$. For such an involuted ring B the dual $V^d = \mathrm{Hom}_B(V,B)$ of a left B-module V inherits the structure of a left B-module by $(af)(v) = f(v)\overline{a}$; the canonical map $\hat{}: V \to V^{dd}$ defined by $\hat{x}(f) = \overline{f(x)}$ is an isomorphism provided V is finitely generated projective. A form ζ on V can be viewed as a homomorphism $V \to V^d$; then $\zeta^* = \zeta^d \circ \hat{}: V \to V^{dd} \to V^d$ is one such too.

<u>DEFINITION</u>. Let ε be a sign. An ε-quadratic form over B consists of a finitely generated free left B-module V and a class of forms ψ on V defined up to the equivalence $\psi \sim \psi + \zeta - \varepsilon\zeta^*$. It is called nonsingular if the symmetrisation $\lambda = \psi + \varepsilon\psi^*$ is an isomorphism $V \to V^d$. We call $(W, \phi^d\psi\phi)$ isomorphic to

(V,ψ) if ϕ is a module isomorphism $W \to V$.

If F is f.g. free the quadratic form ψ on $F \oplus F^d$ defined by $\psi_F(x,f) = (f,0)$ is nonsingular; any quadratic form of this isomorphism type is called standard. Now $L_{2k}(B)$ is defined as the quotient of the Grothendieck group of nonsingular $(-1)^k$ quadratic forms over B by the subgroup generated by standard such forms.

DEFINITION. Let η be a sign, A an involuted ring. A nonsingular almost η-symmetric form over A consists of a finitely generated free left A module K and an isomorphism $\sigma\colon K \to K^d$ such that $\sigma^* = \eta\sigma(1+N)$, where N is nilpotent (compare [C; §9]). Again $\phi^d\sigma\phi$ is considered to be isomorphic to σ for any module isomorphism ϕ.

ALMOST SYMMETRIC FORMS ARE QUADRATIC

Let A be an involuted ring, $\eta = (-1)^q$. We consider quadratic forms over the polynomial ring $A[s]$ over A equipped with the involution $\overline{}$ such that $\overline{\Sigma a_j s^j} = \Sigma \overline{a_j}\,(1-s)^j$.

THEOREM 1. *Any element in $L_{2q}(A[s])$ can be represented by a quadratic form Ψ which is linear in s. Any such linear Ψ can be viewed as an almost $(-1)^q$ symmetric form.*

PROOF. Let the element be represented by a quadratic form $\Psi = \Sigma\Psi_i s^i$ of degree M in s. By the addition of a standard form and the use of an isomorphism we get (in matrix notation)

$$\begin{pmatrix}1 & -s & \Psi_M^*(1-s)^{M-1}\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\Psi & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ -1+s & 1 & 0\\ \Psi_M s^{M-1} & 0 & 1\end{pmatrix} = \begin{pmatrix}\Psi-\Psi_M s^M & 0 & -s\\ \Psi_M s^{M-1} & 0 & 1\\ 0 & 0 & 0\end{pmatrix}$$

a form of degree $M-1$ if $M \geq 2$; so we can make that $M = 1$. We can get rid of the constant term by using the equivalence

$$\Psi_0 + \Psi_1 s \sim \Psi_0 + \Psi_1 s - \Psi_0(1-s) + \eta\Psi_0^* s = (\Psi_1 + \Psi_0 + \eta\Psi_0^*)s.$$

To prove the last clause we consider the linear $\Psi = \Psi_1 s$ and write σ for $\eta\Psi_1^*$. Then the symmetrisation $\Lambda = \Psi + \eta\Psi^*$ of Ψ becomes $\sigma + (\eta\sigma^* - \sigma)s$ which is invertible exactly if σ is invertible and $\eta\sigma^{-1}\sigma^{*}-1$ is nilpotent. Q.E.D.

THEOREM 2. *There is a well defined biadditive pairing*

$$L_{2q}(A[s]) \times L_{2k}(B) \to L_{2q+2k}(A \otimes B)$$

which assigns to the quadratic form $\Psi = \Sigma\Psi_i s^i$ *over* $A[s]$ *with symmetrisation* Λ *and the quadratic form* ψ *over* B *with symmetrisation* λ *the quadratic form*

$$\lambda\Psi(\lambda^{-1}\psi) = \Sigma\Psi_i \otimes \lambda\cdot(\lambda^{-1}\psi)^i$$

over $A \otimes B$. *In particular it extends the familiar product of a symmetric form with a quadratic form.*

PROOF. Again write $\eta = (-1)^q$, $\varepsilon = (-1)^k$.

We start with the observation that for a general form $\Gamma = \Sigma\Gamma_i s^i$ over $A[s]$ we have

$$\begin{aligned}\lambda\Gamma^{*}(\lambda^{-1}\psi) &= \Sigma\Gamma_i^{*} \otimes \lambda(1-\lambda^{-1}\psi)^i = \Sigma\Gamma_i^{*} \otimes \lambda(\varepsilon\lambda^{-1}\psi^{*})^i \\ &= \varepsilon\Sigma\Gamma_i^{*} \otimes \lambda^{*}((\lambda^{*})^{-1}\psi^{*})^i = \varepsilon\Sigma\Gamma_i^{*} \otimes (\psi^{*}(\lambda^{*})^{-1})^i\lambda^{*} \\ &= \varepsilon\{\Sigma\Gamma_i \otimes \lambda(\lambda^{-1}\psi)^i\}^{*} = \varepsilon\{\lambda\Gamma(\lambda^{-1}\psi)\}^{*}\end{aligned}$$

Hence the symmetrisation of the image is

$$\lambda\Psi(\lambda^{-1}\psi) + \varepsilon\eta\{\lambda\Psi(\lambda^{-1}\psi)\}^{*} = \lambda\Psi(\lambda^{-1}\psi) + \eta\lambda\Psi^{*}(\lambda^{-1}\psi) = \lambda\Lambda(\lambda^{-1}\psi)$$

which is invertible since both λ and Λ are. Furthermore if we change Ψ into the equivalent $\Psi + Z - \eta Z^{*}$ the image changes into

$$\begin{aligned}&\lambda\Psi(\lambda^{-1}\psi) + \lambda Z(\lambda^{-1}\psi) - \eta\lambda Z^{*}(\lambda^{-1}\psi) = \\ &= \lambda\Psi(\lambda^{-1}\psi) + \lambda Z(\lambda^{-1}\psi) - \eta\varepsilon\{\lambda Z(\lambda^{-1}\psi)\}^{*}\end{aligned}$$

which is equivalent to $\lambda\Psi(\lambda^{-1}\psi)$.

If we change Ψ into the isomorphic $\Phi^{d}\Psi\Phi$ the image changes into

$$\lambda\phi^d(\lambda^{-1}\psi)\Psi(\lambda^{-1}\psi)\Phi(\lambda^{-1}\psi) = \{\Phi(\lambda^{-1}\psi)\}^d\lambda\Psi(\lambda^{-1}\psi)\Phi(\lambda^{-1}\Psi)$$

which is isomorphic to $\lambda\Psi(\lambda^{-1}\psi)$. Finally if Ψ is standard then $\lambda\Psi(\lambda^{-1}\psi)$ is also standard: in fact such a Ψ is induced from a quadratic form over A for which this statement is well-known. Since our pairing obviously respects direct sums we have proven that the class of the image in $L_{2q+2k}(A \otimes B)$ is independent of the choice of the representing element for the class in $L_{2q}(A[s])$.

Now by Theorem 1 we may from now on assume that Ψ is of the type σs, where σ is nonsingular, almost η symmetric; so $\lambda\Psi(\lambda^{-1}\psi)$ is just $\sigma \otimes \psi$.

Firstly if we change ψ by an isomorphism ϕ into $\phi^*\psi\phi$ then $\sigma \otimes \psi$ changes by the isomorphism $1 \otimes \phi$.

Secondly the isomorphism $K \otimes (F \oplus F^d) \cong (K \otimes F) \oplus (K \otimes F)^d$ which maps $a \otimes (x,f)$ to $(a \otimes x, \sigma(a) \otimes f)$ lets $\sigma \otimes \psi_F$ correspond with $\psi_{K\otimes F}$. So standard forms are mapped to standard forms.

It remains to be shown that the equivalence $\psi \sim \psi + \zeta - \varepsilon\zeta^*$ changes $\sigma \otimes \psi$ into something in the same class; this will be a consequence of the following lemma.

LEMMA. *For every integer $p \geq 0$ there is an isomorphism Φ_p and there are forms Z_p and H_p over $A \otimes B$ such that*

$$\Phi_p^d(\sigma \otimes \psi)\Phi_p = \sigma \otimes (\psi + \zeta - \varepsilon\zeta^*) + Z_p - \varepsilon\eta Z_p^* + H_p(N^{p+1} \otimes 1)$$

where N is $\eta\sigma^{-1}\sigma^* - 1$ and thus nilpotent.

PROOF. We apply induction. For $p = 0$ we take

$$\Phi_0 = 1, \quad Z_0 = -\sigma \otimes \zeta \quad H_0 = -\varepsilon\sigma \otimes \zeta^*.$$

In general Φ_p will be of the form $1 + N \otimes \phi_1 + \ldots + N^p \otimes \phi_p$ and H_p of the form $\sigma \otimes \theta_{p0} + \sigma N \otimes \theta_{p1} + \ldots$. If we assume all this for p then $\Phi_{p+1}^d(\sigma \otimes \psi)\Phi_{p+1}$ becomes

$$\sigma \otimes (\psi + \zeta - \varepsilon\zeta^*) + Z_p - \varepsilon\eta Z_p^* + H_p(N^{p+1} \otimes 1) +$$
$$+ (N^d)^{p+1}\sigma \otimes \phi_{p+1}^d\psi + \sum_{j=1}^{p} (N^d)^{p+1}\sigma N^j \otimes \phi_{p+1}^d\psi\phi_j +$$
$$+ \sigma N^{p+1} \otimes \psi\phi_{p+1} + \sum_{j=1}^{p+1} (N^d)^j\sigma N^{p+1} \otimes \phi_j^d\psi\phi_{p+1}.$$

Now we rewrite $(N^d)^{p+1}\sigma$

$\{(N^d)^{p+1}\sigma \otimes \phi$

$+ \varepsilon(\eta\sigma^* - \sigma)N^p$

and we want the last ter
of $H_p(N^{p+1} \otimes 1)$ hence we
Z by defining $Z_{p+1} = Z_p$
The remaining term
as are the remaining ter
sible because $N^d\sigma$ can be
and H_{p+1} of the right fc

By viewing almost s
classifying the latter u
lence relation on them.
According to Theore
formulation of the produ
sufficiently coarse to c
Poincaré complexes in th
plexes: As explained in
σ to a 2q-dimensional al
class in $L_{2q}(A[s])$. The
as taking the tensor pro
$\sigma = 1$.
Both the inherent p
able for L_{2q} make it pro
tions then $L^{2q}(A)$ is.
One could hope that
to an honest $(-1)^q$ symme
shows that this is not t
the ring A contains a ce
Dedekind domain.
The two-dimensional
and hence to an element
$\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$. Suppose
symmetric form σ; then

Now we rewrite $(N^d)^{p+1}\sigma \otimes \phi^d_{p+1}\psi + \sigma N^{p+1} \otimes \psi\phi_{p+1}$ as

$$\{(N^d)^{p+1}\sigma \otimes \phi^d_{p+1}\psi - \varepsilon\eta\sigma^* N^{p+1} \otimes \psi^*\phi_{p+1}\} +$$

$$+ \varepsilon(\eta\sigma^* - \sigma)N^{p+1} \otimes \psi^*\phi_{p+1} + \sigma N^{p+1} \otimes (\psi + \varepsilon\psi^*)\phi_{p+1}$$

and we want the last term $\sigma N^{p+1} \otimes \lambda\phi_{p+1}$ to cancel the first term $\sigma N^{p+1} \otimes \theta_{p0}$ of $H_p(N^{p+1} \otimes 1)$ hence we define $\phi_{p+1} = -\lambda^{-1}\theta_{p0}$. The first term we absorb in Z by defining $Z_{p+1} = Z_p + (N^d)^{p+1}\sigma \otimes \phi^d_{p+1}\psi$.

The remaining term $\varepsilon\sigma N^{p+2} \otimes \psi^*\phi_{p+1}$ will be absorbed in $H_{p+1}(N^{p+2} \otimes 1)$, as are the remaining terms of $H_p(N^{p+1} \otimes 1)$ and the Σ-terms. The last is possible because $N^d\sigma$ can be rewritten as $-\sigma N(1+N)^{-1}$. So there exists ϕ_{p+1}, Z_{p+1} and H_{p+1} of the right form. Q.E.D.

By viewing almost symmetric forms A as quadratic forms over A[s] and classifying the latter up to stable isomorphism we have defined an equivalence relation on them.

According to Theorem 2 this relation is sufficiently fine to admit the formulation of the product formula (for surgery obstructions). It is also sufficiently coarse to define a bordism invariant of algebraic symmetric Poincaré complexes in the sense of [R], hence one of geometric Poincaré complexes: As explained in [C] we can associate an almost $(-1)^q$ symmetric form σ to a 2q-dimensional algebraic symmetric Poincaré complex and then take its class in $L_{2q}(A[s])$. The result is well-defined on $L^{2q}(A)$ since it can be seen as taking the tensor product with the element of $L_0(Z[s])$ represented by $\sigma = 1$.

Both the inherent periodicity in q and the wealth of techniques available for L_{2q} make it probable that $L_{2q}(A[s])$ is better suited for calculations then $L^{2q}(A)$ is.

One could hope that an almost $(-1)^q$ symmetric form is always equivalent to an honest $(-1)^q$ symmetric one; the following example, due to A. Ranicki shows that this is not the case. However we will see that it is the case if the ring A contains a central element t such that $t + \bar{t} = 1$ or if it is a Dedekind domain.

The two-dimensional torus $T^2 = S^1 \times S^1$ gives rise to an element in $L^2(A)$, and hence to an element in $L_2(A[s])$, where A is the integral group ring of $\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$. Suppose that this element could be represented by an anti-symmetric form σ; then σ could be written as $\phi - \phi^*$; the result $\sigma \otimes \psi$ of its action on a (-1)-quadratic form ψ would be equivalent to $\phi \otimes (\psi - \psi^*)$, hence would depend only on the symmetrisation $\psi - \psi^*$ of ψ. In particular it would kill the Arf nontrivial element in $L_2(\mathbb{Z})$. On the other hand it follows from [SH] that multiplication with a circle induces a split injection on L-groups and hence the product with T^2 gives a split injection $L_2(\mathbb{Z}) \to L_4(A)$.

SOME CALCULATIONS

THEOREM 3. *If there exists a central element* t *of* ~~$\mathbb{Z}$~~ A *such that* $t + \bar{t} = 1$ *then the canonical map* $L_{2q}(A) \to L_{2q}(A[s])$ *is an isomorphism.*

PROOF. The map $A[s] \to A$ substituting t for s gives a left inverse so we must show that for any integer $p \geq 0$ there is an isomorphism ϕ_p and there are forms ζ_p and θ_p such that

$$\phi_p^d(\sigma s)\phi_p = \sigma t + \zeta_p - \eta\zeta_p^* + \sigma N^{p+1}\theta_p .$$

For $p = 0$ we take $\phi_0 = 1$, $\zeta_0 = \sigma s(1-t)$, $\theta_0 = (1-s)t$. In general ϕ_p will be of the form $1 + \alpha_1 N + \ldots + \alpha_p N^p$ and $\theta_p = \theta_{p0} + \theta_{p1}N + \theta_{p2}N^2 + \ldots$ where the α_i and θ_{ij} are polynomial in s and t with Z coefficients, hence central.

If we assume all this for p then $\phi_{p+1}^d(\sigma s)\phi_{p+1}$ becomes

$$\begin{aligned} &\sigma t + \zeta_p - \eta\zeta_p^* + \sigma N^{p+1}\theta_p + \\ &+ \bar{\alpha}_{p+1}(N^d)^{p+1}\sigma s + \sum_{j=1}^{p} \bar{\alpha}_{p+1}(N^d)^{p+1}\sigma s\alpha_j N^j + \\ &+ \sigma s\alpha_{p+1}N^{p+1} + \sum_{j=1}^{p+1} \bar{\alpha}_j (N^d)^j \sigma s\alpha_{p+1}N^{p+1} . \end{aligned}$$

We rewrite $\bar{\alpha}_{p+1}(N^d)^{p+1}\sigma s + \sigma s\alpha_{p+1}N^{p+1}$ as

$$\begin{aligned} &\{\bar{\alpha}_{p+1}(N^d)^{p+1}\sigma s - \eta\sigma^* N^{p+1}\alpha_{p+1}(1-s)\} \\ &+ (\eta\sigma^* - \sigma)N^{p+1}\alpha_{p+1}(1-s) + \sigma(s + (1-s))\alpha_{p+1}N^{p+1} \end{aligned}$$

Then we let the last term cancel the first term of $\sigma N^{p+1}\theta_p$ by defining $\alpha_{p+1} = -\theta_{p0}$ and absorb the first term in ζ by defining $\zeta_{p+1} = \zeta_p + \bar{\alpha}_{p+1}(N^d)^{p+1}\sigma s$.

The middle term σN^{p+}
terms and the remaining t

THEOREM 4.

$L_0(Z[s]) \cong Z,$

PROOF. According to Theo
ψ of the type $\eta\sigma^* s$, where
Z-module K. Thus $N = \eta\sigma^{-}$
of finite index h in som

For $x \in L^{\perp} = \{x \mid \sigma($
since $\eta\sigma^* N^{e-1} = (\sigma + \sigma N)N$

$\sigma(N^{e-1}y)(x) =$

Furthermore $L \subset L^{\perp}$ since

$\sigma(N^{e-1}y)(N^{e-1}$

So σ induces a well-defi
$\tilde{N}(x+L) = Nx + L$ hence N^e
Now $L \otimes Z[s]$ is a
If $x = \Sigma x_j s^j \in K \otimes Z[s]$
of ψ then we have for al

$0 = \lambda(\ell \otimes 1,$

hence $x_j \in L^{\perp}$. We see t
obviously the induced q
It is well known t
that the latter is asso
better e. We can go on
η-symmetric form.
It is also well kn
and a (+1)-symmetric fo
(1) of rank one. Finall

The middle term $\sigma N^{p+2}\alpha_{p+1}(1-s)$ is absorbed in $\sigma N^{p+2}\theta_{p+1}$, as are the Σ terms and the remaining terms of $\sigma N^{p+1}\theta_p$. Q.E.D.

THEOREM 4.

$$L_0(Z[s]) \cong Z, \quad L_2(Z[s]) \cong (0).$$

PROOF. According to Theorem 1 we may restrict attention to η-quadratic forms ψ of the type $\eta\sigma^* s$, where σ is an almost η-symmetric form on some f.g. free Z-module K. Thus $N = \eta\sigma^{-1}\sigma^* - 1$ satisfies $N^e = 0$ for some e. Then $N^{e-1}K$ is of finite index h in some direct summand L of K.

For $x \in L^\perp = \{x \mid \sigma(x)(L) = 0\}$ we have also $\sigma(L)(x) = 0$ and vice versa, since $\eta\sigma^* N^{e-1} = (\sigma + \sigma N)N^{e-1} = \sigma N^{e-1}$ implies

$$\sigma(N^{e-1}y)(x) = \eta\,\overline{\sigma(x)(N^{e-1}y)}, \qquad \text{for } y \in K.$$

Furthermore $L \subset L^\perp$ since $\sigma N = -\eta N^d \sigma^*$ implies

$$\sigma(N^{e-1}y)(N^{e-1}x) = -\eta\,\overline{\sigma(N^e x)(N^{e-2}y)} = 0.$$

So σ induces a well-defined form $\tilde\sigma$ on $L^\perp/L$, and $\tilde N = \eta\tilde\sigma^{-1}\tilde\sigma^* - 1$ satisfies $\tilde N(x+L) = Nx + L$ hence $N^{e-1}K \subset L$ implies $\tilde N^{e-1} = 0$.

Now $L \otimes Z[s]$ is a direct summand of $K \otimes Z[s]$ which is isotropic for ψ. If $x = \Sigma x_j s^j \in K \otimes Z[s]$ is in $(L \otimes Z[s])^\perp$ for the symmetrisation $\lambda = \sigma + \sigma N s$ of ψ then we have for all $\ell \in L$ that

$$0 = \lambda(\ell \otimes 1, \Sigma x_j s^j) = \Sigma\sigma(\ell, x_j)s^j + \Sigma\sigma(N\ell, x_j)(1-s)s^j = \Sigma\sigma(\ell, x_j)s^j$$

hence $x_j \in L^\perp$. We see that $(L \otimes Z[s])^\perp/(L \otimes Z[s])$ is just $(L^\perp/L) \otimes Z[s]$ and obviously the induced quadratic form $\tilde\psi$ on it is just $\eta\tilde\sigma^* s$.

It is well known that ψ is stably equivalent to $\tilde\psi$ and we have just seen that the latter is associated to an almost η-symmetric form $\tilde\sigma$ which has a better e. We can go on inductively until $e = 1$ which means that we get an η-symmetric form.

It is also well known [SE] that a (-1)-symmetric form is stably trivial and a (+1)-symmetric form stably isomorphic to some multiple m of the form (1) of rank one. Finally m can be detected by taking the signature of the quadratic form over $\mathbb{R}$ which we get by mapping s to $\frac{1}{2}$. Q.E.D.

Now some general remarks about torsion are necessary. If we start with a finite Poincaré complex P our module K gets a natural basis (see §6 of [C]).

The symmetrization λ of the associated quadratic form is $\sigma(1+Ns)$ and according to Lemma 9 of [C] we have $N^2 = 0$ and $1 + Ns$ has a resolution by automorphisms $1 + ((-1)^1\Sigma^{-1}\Sigma^* - 1)s$ of the E_i which are simple; in particular the isomorphisms involving N in the proofs of Theorems 2 and 3 are simple. So the torsion of λ lives in $\widetilde{K}_1(A) \subset \widetilde{K}_1(A[s])$ and the appropriate L groups $L^X_{2q}(A[s])$ have $X = Wh(\rho)$ in the general case and (0) in the case of simple Poincaré complexes.

At the time this is written we do not have theorems as the above for the odd-dimensional case. Note however, that if we did, we could use the long exact sequence 9.4 of [R] for the L_n groups to calculate $L_n(Z[\rho][s])$ for ρ the cyclic group of prime order $p > 2$. If ω denotes $\exp(2\pi i/p)$ and F_p is the field of p elements, there are maps from $Z[\rho][s]$ to $Z[\omega][s]$ and $Z[s]$ and from these to $F_p[s]$ satisfying all necessary conditions. Since $K_2(F_p[s]) = 0$ according to Theorem 11 of [Q] and 9.13 of [M] the map

$$\widetilde{K}_1(Z[\rho][s]) \to \widetilde{K}_1(Z[\omega][s]) \oplus \widetilde{K}_1(Z[s])$$

is injective, so we may use the "simple" L-groups throughout and we get an exact seqeunce

$$\ldots\ L_{n+1}(F_p[s]) \to L_n(Z[\rho][s]) \to L_n(Z[\omega][s]) \oplus L_n(Z[s]) \to L_n(F_p[s])\ \ldots$$

But $L_n(Z[\omega][s]) \cong L_n(Z[\omega])$ by Theorem 3, hence is known, and similarly $L_n(F_p[s]) \cong L_n(F_p)$.

The author has now calculated $L_n(Z[\rho][s])$ for ρ cyclic.

REFERENCES

[C] CLAUWENS, F.J.B.J., *Surgery on products*, Indag. Math., 82 (1979) 121-144.

[M] MILNOR, J., *Introduction to algebraic K-theory*, Annals of Math. Studies 72, Princeton University Press, Princeton 1971.

[Q] QUILLEN, D., *Hig*
London Mat

[R] RANICKI, A., *An*
1975.

[SE] SERRE, J.P., *Co*
1970.

[SH] SHANESON, J.L.,
of Math.

[W_1] WALL, C.T.C., *S*
London.

[W_2] WALL, C.T.C., *O*
forms, Pr

[Q] QUILLEN, D., *Higher K-theory for categories with exact sequences*, London Math. Soc. Lecture Note Series 11, 1974, 95-103.

[R] RANICKI, A., *An algebraic theory of surgery*, preprint, Cambridge, 1975.

[SE] SERRE, J.P., *Cours d'Arithmetique*, Presses Universitaires de France, 1970.

[SH] SHANESON, J.L., *Wall's surgery obstruction groups for* $G \times Z$, Annals of Math. 90 (1969) 296-334.

[W_1] WALL, C.T.C., *Surgery on compact manifolds*, Academic Press 1970, London.

[W_2] WALL, C.T.C., *On the axiomatic foundation of the theory of hermitian forms*, Proc. Cambridge Philos. Soc. 67 (1970), 243-250.

F.J.-B.J. Clauwens
Department of Mathematics
Catholic University
Toernooiveld
6525 ED Nijmegen
The Netherlands