

Hermitian K-theory  
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Lecture 2: higher K-groups

2.1. We start by a theorem analogous to 1.6 by considering an exact sequence

$$0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$$

of rings with involutions. Then we have an exact sequence of  $K\mathbb{Q}$ -groups

$$\mathbb{E}K\mathbb{Q}_1(A') \rightarrow \mathbb{E}K\mathbb{Q}_1(A) \rightarrow \mathbb{E}K\mathbb{Q}_1(A'') \rightarrow \mathbb{E}K\mathbb{Q}_0(A') \rightarrow \mathbb{E}K\mathbb{Q}_0(A) \rightarrow \mathbb{E}K\mathbb{Q}_0(A'')$$

Clearly, this statement deserves some comments

- 1) The notation  $K\mathbb{Q}$  is for the 3 theories which interest us:  $K\mathbb{Q}^{\max}$ ,  $K\mathbb{Q}^{\min}$ ,  $K\mathbb{Q}^{\text{split}}$
- 2) For each relevant group, the notation  $\mathbb{E}K\mathbb{Q}_1$  stands for  $G/[G, G]$  where  $G = \varinjlim \mathbb{E}_{m,n}(A)$ , or  $G = \mathbb{E}O(A) = \varinjlim \mathbb{E}_{m,n}^0(A)$ , or  $G = \mathbb{E}O^{\text{split}}(A) = \varinjlim \mathbb{E}_{m,n}^{\text{split}}(A)$   
For short, we write  $G_A$  instead of  $G$ .
- 3) The proof is of the same spirit as the usual proof in algebraic K-theory (see 1.7). All the (technical) details are in [KV] p. 62-72. Let us emphasize the key arguments:

a) If we have a cartesian square of rings (as in 1.9) (2.2.)  
 with  $\varphi_1$  surjective

$$\begin{array}{ccc} A & \longrightarrow & A_1 \\ \downarrow & & \downarrow \varphi_1 \\ A_2 & \longrightarrow & A' \end{array}$$

it induces two cartesian squares

$$\begin{array}{ccc} G_A & \longrightarrow & G_{A_1} \\ \downarrow & & \downarrow \bar{\varphi}_1 \\ G_{A_2} & \longrightarrow & G_{A'} \end{array} \quad \begin{array}{ccc} \text{Proj}^*(H(A^n)) & \longrightarrow & \text{Proj}^*(H(A_1^n)) \\ \downarrow & & \downarrow \\ \text{Proj}^*(H(A_2^n)) & \longrightarrow & \text{Proj}^*(H(A'^n)) \end{array}$$

Moreover,  $\bar{\varphi}_1$  is surjective on the commutator subgroup  $[G_{A'_1}, G_{A'_1}]$

b) Every element of  $[G_A, G_A]$  is a product of  $\varepsilon$ -elementary matrices ([KV] p. 70)

2.2. Let us move now to higher  $K$ -groups, or rather higher  $KQ$ -groups - Again the scheme is the same as the higher  $K$ -theory defined by Quillen. But we shall put some variations for a better handling of the multiplicative structure - If we write  $\varepsilon^0(A)$  for all the three types of groups we are considering, we have the following well-known statement

2.3. Lemma - The commutator subgroup  $\varepsilon^0(A)' = [\varepsilon^0(A), \varepsilon^0(A)]$  is perfect.

Proof : Write  $ghg^{-1}h^{-1}$  as

$$\begin{pmatrix} g & 0 & 0 \\ 0 & g^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} h & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & h^{-1} \end{pmatrix} \begin{pmatrix} g^{-1} & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} h^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & h \end{pmatrix}$$

2.3.

one has then to prove that a matrix of the type  $\begin{pmatrix} g & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & g^{-1} \end{pmatrix}$  is a commutator:

$$\begin{pmatrix} g & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & g^{-1} \end{pmatrix} = \begin{pmatrix} g & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} g^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

2.4. Definition  ${}_{\varepsilon} KQ_n(A) = \pi_n(B_{\varepsilon} O(A)^+)$  ( $n > 0$ )

As we said many times, there are three types of hermitian k-theory  $KQ^{\max}$ ,  $KQ^{\min}$  and  $KQ^{\text{split}}$ .

We have of course  $KQ_0^{\text{split}} = KQ_0^{\min}$  and a surjection  ${}_{\varepsilon} KQ_0^{\min}(A) \rightarrow {}_{\varepsilon} KQ_0^{\max}(A)$ . There is also an exact sequence proved by Bak

$${}_{\varepsilon} KQ_1^{\min}(A) \rightarrow {}_{\varepsilon} KQ_1^{\max}(A) \rightarrow \mathbb{H} \rightarrow {}_{\varepsilon} KQ_0^{\min}(A) \rightarrow {}_{\varepsilon} KQ_0^{\max}(A)$$

where  $\mathbb{H}$  is an explicit group defined in [Bak] p. 191 -

It is precisely  $(\Gamma/\Lambda \otimes \mathbb{H}/\Lambda)/[\alpha \otimes b - b \otimes \bar{\alpha}, \alpha \otimes b - \alpha \otimes bab]$

where  $\Gamma = \{a \in A \mid \bar{a} = \varepsilon a\}$  and  $\Lambda = \{b - \varepsilon \bar{b}\}$ .

2.5. This "Quillen type" definition of  $KQ_n(A)$  is not very enlightening and various other definitions have been proposed (by M. Schlichting for instance). However, it is closer to the homology of

(2.4)

the relevant groups since one can prove that  $BG^+$  is an H-space (for  $G = GL(A)$  or  $\mathcal{E}^0(A)$ ). This implies the following well-known theorem

Theorem -  $H_*(\mathcal{E}^0(A); \mathbb{Q}) \cong S(\mathcal{E}^{1,0}_*(A) \otimes \mathbb{Q})$

where  $S$  stands for the symmetric graded algebra.

2.6.1 At this time, it might be interesting to stop a little bit and give examples of such orthogonal groups and show that we recover most of the classical lie groups if  $A = \mathbb{R}, \mathbb{C}$ . More generally, if  $A$  is a Banach algebra, the previous definitions are still valid in the topological framework (replace  $B_{\mathcal{E}}^0(A)^+$  by the classifying space of  $\mathcal{E}^0(A)$  with its usual topology).

1)  $A = \mathbb{R}, \varepsilon = 1$  - Then  $\mathcal{O}_{n,n}(\mathbb{R})$  is the classical group  $O(n, n)$  which compact maximal subgroup is  $O(n) \times O(n)$ .

2)  $A = \mathbb{R}, \varepsilon = -1$  ;  $\mathcal{O}_{n,n}(\mathbb{R})$  is then the classical symplectic group  $Sp_{2n}(\mathbb{R})$  which compact maximal subgroup is  $U(n)$ .

3)  $A = \mathbb{C}$  with trivial involution -

- Then  $\mathcal{O}_{n,n}(\mathbb{C})$  is  $O_{2n}(\mathbb{C})$  which maximal compact subgroup is  $O(2n)$
- on the other hand  $\mathcal{O}_{n,n}(\mathbb{C})$  is  $Sp_{2n}(\mathbb{C})$  which maximal compact subgroup is called  $Sp(n)$

4)  $A = \mathbb{C}$  with complex conjugation as involution

2.5

${}_{-1}\Omega_{n,n}(\mathbb{C}) \cong {}_{-1}\Omega_{n,n}(\mathbb{C}) \cong \cdots \cong U(n, n)$  which  
maximal subgroup is  $U(n) \times U(n)$

2.6.2 - what about the difference between the max and the min categories? A good example is to take  $E = -1$ ,  $A = \mathbb{Z}$  and look at  ${}_{-1}\Omega_{1,1}^{\text{max}}(\mathbb{Z})$  and  ${}_{-1}\Omega_{1,1}^{\text{min}}(\mathbb{Z})$ . In the first case, we are looking at integral metrics  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  such that  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ . This is just the classical group  $SL_2(\mathbb{Z}) = {}_{-1}\Omega_{1,1}^{\text{max}}(\mathbb{Z})$ . In the second case, we ask moreover that  $ab = -\epsilon(ab)$  that is  $ab = 0$  and symmetrically  $cd = 0$ . Then we get the much smaller group  $\mathbb{Z}/4$  generated by the matrix  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ .

2.7 - As we said before, the space  $BG^+$  is difficult (2.6) to handle, for instance if we want to deal with multiplicative structures - As we have shown in [K1] p. 42, there is a more geometric way to define the function  $X \mapsto [X, BG^+]$  in terms of  $G$ -flat bundle (here  $G$  might be  $GL_n(A)$ ,  $U(A)$  or  $\mathbb{Z}_{m,n}(A)$ ,  $\mathbb{Z}^D(A)$ ) - Since we are dealing with hermitian K-theory, we choose this framework for the definition -

If  $Y$  is a cw-complex (connected), we define a flat  $\varepsilon$ -hermitian  $A$ -bundle to be a covering  $E \rightarrow Y$  such that the fibers are 'finitely' projective  $A$ -module provided with a non degenerate  $\varepsilon$ -hermitian form.

Following [C1] p. 42, we define a virtual  $\varepsilon$ -hermitian bundle on  $X$  by two data

- 1) a fibration  $Y \rightarrow X$  which is acyclic
- 2) a flat  $\varepsilon$ -hermitian  $A$ -bundle on  $Y$

Two virtual bundles  $E \rightarrow Y \rightarrow X$  and  $E' \rightarrow Y' \rightarrow X$  are called equivalent if there exists a virtual bundle  $E_1 \rightarrow Y_1 \rightarrow X$  and a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \alpha \downarrow & f_1 \nearrow & \uparrow f' \\ Y_1 & \xleftarrow{\alpha'} & Y' \end{array}$$

with  $f_1$  acyclic and  $E \cong \sigma^* E_1$ ,  $E' \cong \sigma'^* E_1$  (2.7)

p. 42-50

one can prove, using the same scheme as in [C-1] that the Grothendieck group built out of these virtual bundles is isomorphic to the group of homotopy classes of maps from  $X$  to  $K_0(A) \times B_\varepsilon \mathbb{Q}(A)^+$  (if  $X$  is a finite cw-complex).

2.8. If the previous definition is more or less clear for the groups  $\Omega^{\max}$  and  $\Omega^{\min}$ , it is useful to be more explicit about the group  $\Omega^{\text{split}}$ .  
More precisely, if  $E$  and  $F$  are 2  $A$ -bundles with  $\varepsilon$ -quadratic forms  $q_0$  and  $q_0$  respectively, we define a morphism  $(E, q_0) \rightarrow (F, q_0)$  with the same formulae as in 1.21, except that  $f, \tilde{u}$  are now morphisms of flat bundles.

This remark will be useful later on

2.9. The reader might ask why we are considering this category of flat  $A$ -bundles - one of the reasons (there will be more serious later on) is a clear definition of the cup-product - For instance, in K-theory  $K$ -theory we have a cup-product

$$K(A) \otimes K(B) \longrightarrow K(A \otimes B)$$

(2.8)

If we define  $K_A(X)$  as  $[X, K_0(A) \times \text{Bun}(A)^+]$  with the "flat" interpretation, one has also a cup-product

$$K_A(X) \times K_B(Y) \rightarrow K_{A \otimes B}(X \times Y)$$

defined by tensor product of bundles.

or a "reduced" one for spaces with base points

$$\tilde{K}_A(X) \times \tilde{K}_B(Y) \rightarrow \tilde{K}_{A \otimes B}(X \times Y)$$

For instance, if  $X = S^n$ ,  $Y = S^p$ , there has a pairing

$$K_n(A) \times K_p(B) \rightarrow K_{n+p}(A \otimes B)$$

first defined by Goday in his thesis.

2.10. In the hermitian case, the cup-product is more subtle - It goes schematically in two ways

$$\varepsilon^{(\max)} \times \gamma^{(\max)} \rightarrow \varepsilon_2^{(\min)}$$

$$\varepsilon^{(\max)} \times \gamma^{(\text{split})} \rightarrow \varepsilon_2^{(\text{split})}$$

For instance, if we have an  $\varepsilon$ -hermitian form  $\varphi$  and an  $\gamma$ -hermitian form  $\psi$  which are both even and if

$$\alpha : (E, \varphi) \rightarrow (E, \varphi), \quad \beta : (F, \psi) \rightarrow (F, \psi)$$

are two morphisms, we can consider  $E \otimes F$  with the hermitian form  $\varphi \otimes \psi$  which we can choose to write as  $(\varphi_0 + \varepsilon^\ell \varphi_0) \otimes \psi = \varphi_0 \otimes \psi + \varepsilon_2^\ell (\varphi_0 \otimes \psi)$ .

Then  ${}^\ell(\alpha \otimes \beta)(\varphi_0 \otimes \psi)(\alpha \otimes \beta)$  may be written as

$$\begin{aligned} {}^\ell(\alpha \otimes \beta)(\varphi_0 \otimes \psi)(\alpha \otimes \beta) &= (\varphi_0 + u \cdot) \otimes \psi \quad (\text{where } t_u = -\varepsilon_2^\ell) \\ &= \varphi_0 \otimes \psi + u \otimes \varphi_0 - \varepsilon_2^\ell (u \otimes \varphi_0) \end{aligned}$$

This pairings gives a "cup-product"

$$\mathbb{E} KQ_n^{\max}(A) \times_{\mathbb{Z}} \mathbb{E} KQ_p^{\max}(B) \rightarrow \mathbb{E} KQ_{n+p}^{\min}(A \otimes B)$$

By considering flat bundles over homology spheres we have in the same way a cup-product

$$\mathbb{E} KQ_n^{\max}(A) \times_{\mathbb{Z}} \mathbb{E} KQ_p^{\max}(B) \rightarrow \mathbb{E} KQ_{n+p}^{\min}(A \otimes B)$$

We leave as an exercise to the reader the analogous cup-product

$$\mathbb{E} KQ_n^{\text{split}}(A) \times_{\mathbb{Z}} \mathbb{E} KQ_p^{\max}(B) \rightarrow \mathbb{E} KQ_{n+p}^{\text{split}}(A \otimes B)$$

2.11 - We want to get out of these generalities to state at least one theorem, which we will prove later as a consequence of a more general theorem.

As we said before, the hyperbolic functor defines a map  $K_n(A) \rightarrow \mathbb{E} KQ_n(A)$ , which cokernel is called the higher Witt group  $\mathbb{E} W_n(A)$ . Let us assume

for simplicity the existence of  $\lambda \in \mathbb{Z}(A)$  such that  $\lambda + \bar{\lambda} = 1$ , so that the theories  $\mathbb{E} KQ_n^{\max} \simeq \mathbb{E} KQ_n^{\min}$  and  $\mathbb{E} W_n^{\max} \simeq \mathbb{E} W_n^{\min}$ .

If we choose  $B = \mathbb{Z}$  above with  $p = 2$ ,  $\varepsilon = -1$ , there is a well known element in  $\mathbb{E} W_2(\mathbb{Z})$  which image in  $\mathbb{E} W_2^{\text{top}}(\mathbb{R}) = \widetilde{K}_{\mathbb{C}}(S^2)$  is the Bott generator

(see [K2] p. 247-251 : this is a classical exercise in evaluating  $H_2(Sp(2); \mathbb{Z})$  which goes back to Matsumoto)

2.12. Theorem - Let  $A$  be a ring such that there exists  $\lambda \in Z(A)$  with  $\lambda + \bar{\lambda} = 1$  and let  $u$  be the element in  $W_2(Z)$  chosen above - Then the cup-product with  $u$  induces an isomorphism

$${}_{\mathbb{Z}} W_n(A) \otimes {}_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}] \longrightarrow {}_{\mathbb{Z}} W_{n+2}(A) \otimes {}_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}] \quad (*)$$

More precisely, the kernel and the cokernel of the "periodicity map"

$${}_{\mathbb{Z}} W_n(A) \longrightarrow {}_{\mathbb{Z}} W_{n+2}(A)$$

are killed by the number 2 (i.e. these are just vector spaces over  $\mathbb{F}_2$ ) -

→ (\*) This statement is in fact true for arbitrary rings

2.13. Corollary - The homology of the general orthogonal group  $O(A)$  with rational coefficients can be expressed as the tensor product of two graded symmetric algebras

$$M_x \otimes N_x$$

$$\text{where } M_x = \mathbb{Q} \text{ and } N_x = S({}_{\mathbb{Z}} W_x(A) \otimes \mathbb{Q})$$

where  $x > 0$  and the group  ${}_{\mathbb{Z}} W_x(A) \otimes \mathbb{Q}$  "periodic".

2.14. Example (compatible with A. Borel's computations)

$$H_*(Sp(2), \mathbb{Q}) = \mathbb{Q}[x_2, x_6, x_{10}, \dots]$$

2.15. In order to get a feeling of the proof of this theorem, one has to introduce more "technology" i.e. introduce the classifying space of algebraic K-theory and hermitian K-theory at the same time.

2.16 - If  $A$  is the ring, we define the "cone"  $CA$  to be (2.11)  
 the set of infinite matrices  $(a_{ij}^{\pm})$   $i, j \in \mathbb{N}$ , with  
 the following properties

- 1)  $\forall i$  (resp  $j$ )  $\exists$  a finite <sup>(bounded)</sup> number of  $j$  (resp  $i$ )  
 such that  $a_{ij}^{\pm} \neq 0$
- 2) The  $(a_{ij}^{\pm})$  are chosen among a finite number  
 of elements in  $A$

[ there are variants of this definition where we drop 2)  
 and the boundedness conditions. This will give the  
 same type of statements ]

With this definition,  $CA$  is a ring which contains  
 the ring  $\mathbb{A}$  of finite matrices as a 2-side ideal.

The suspension  $SA$  of  $A$  is defined as  $CA/\mathbb{A}$ . The  
 following definition of the negative  $K$ -groups goes back  
 to 1968 [K3] and coincides with the one given by  
 Bass [B]. It is defined inductively by the  
 following formula

$$K_{-n-1}(A) = K_{-n}(SA) = \dots = K_0(S^{n+1}A)$$

The following theorem is basic in our theory

2.17. Theorem [W] There is an homotopy equivalence

$$K_0(A) \times BGL(A)^+ \sim S^2 BGL(SA)^+$$

This theorem enables us to define the spectrum  
 of algebraic  $K$ -theory  $IK(A)$  as follows

(2.12)

$$|K(A)|_n = \Sigma(BGL(S^{n+1}A)^+) \quad \text{if } n \geq 0$$

$$|K(A)|_n = \bar{\Sigma}^n BGL(A)^+ \quad \text{if } n < 0$$

one should notice that  $|K(A)|_0$  is homotopically equivalent to  $K(A) \times BGL(A)^+$ , but not naturally.

2.18- All what we have done with algebraic K-theory may be extended to hermitian K-theory

so that we have negative  $KQ$ -groups and hermitian K-spectrum  $|KQ|$

2.19. Sketch of the proof of the theorem 2.12.

The theory has now be extended to negative Witt groups - we just have to find an element  $v$  in  ${}_{-1}W_2(\bar{\mathbb{Z}})^{(*)}$  such that the cup-product with  $u$  in  ${}_{-1}W_2(\mathbb{Z})$  is 4 times the generator of  ${}_{-1}W_0(\bar{\mathbb{Z}}) = \mathbb{Z}$  (cf [C]) - The cup-product is by  $u$  and  $v$  give maps between  $\varepsilon^{W_n}$  and  ${}_{-2}W_{n+2}$  which are inverse to each other (up to 2-homotopy)

(\*)  $\bar{\mathbb{Z}}$  is the ring of polynomials  $\mathbb{Z}[s]$  with the involution  $s \mapsto s$

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