

Hermitian K-theory

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Lecture 1 : basic definitions

- 1.1. Let A be an arbitrary ring with unit. There is a well known description of the K-theory of A : we consider the category $\mathcal{C} = \mathcal{P}(A)$ of finitely generated projective (right) A -modules over A and call $\mathcal{F}(A)$ the set of isomorphism classes \hat{E} of objects E of \mathcal{C} . Then $\mathcal{F}(A)$ is a \mathbb{Z} group for the operation rule $\hat{E} + \hat{F} = \widehat{E \oplus F}$. The Grothendieck group $K(A)$ is the associated group; it is the quotient of $M \times A$ by the equivalence relation
 $(a, b) \sim (c, d) \iff \exists e \in M \mid a+d+e = b+c+e$

- 1.2. There are well known examples of $K(A)$

- a) A is the ring of integers in a number field. Then $K(A) \cong \mathbb{Z} \oplus G$, where G is a finite group called the ideal class group of A , a definition due to Kummer in relation with Fermat last theorem.
- b) A is the ring of continuous functions on a compact space X with complex values. Then $K(A) \otimes \mathbb{Q} \cong H^{\text{even}}(X; \mathbb{Q})$ = the direct sum of the even cohomology groups $H^{2k}(X; \mathbb{Q})$ with rational coefficients.

1.2

- 1.3. Before going further, one should notice that this definition of $K(A)$ extends to rings not having necessarily a unit - we assume simply that A is a \mathbb{K} -algebra (for instance $\mathbb{K} = \mathbb{Z}$) and we define $K(A)$ as the kernel of $K(A^+) \rightarrow K(\mathbb{K})$, where $A^+ = A \otimes \mathbb{K}$ is the ring A with ~~the~~ unit added; the multiplication in A^+ is given by $(a, 1)(a', 1) = (aa' + \lambda a' + \lambda' a, 1 \cdot 1)$. One can prove that this definition is independent of \mathbb{K} .

- 1.4. Inspired by Topology, Bass gave a related definition of a K_1 -group, called $K_1(A)$. First we define $GL_2(A)$ as the group of invertible 2×2 matrices with coefficients in A and $GL(A)$ as the direct limit

$$GL(A) = \varinjlim GL_2(A) = \bigcup_{n=1}^{\infty} GL_n(A)$$

The group $K_1(A)$ of Bass is then defined as the abelianization of A , i.e.

$$K_1(A) = GL(A)/[GL(A), GL(A)]$$

one can show (Whitehead's lemma) that the commutator subgroup $[GL(A), GL(A)]$ is generated by "elementary" matrices, i.e. of the type

$$\text{i.e. } \begin{pmatrix} 1 & 0 & & \\ & \ddots & \ddots & \lambda \\ & & \ddots & 0 \\ 0 & & & 1 \end{pmatrix} = e_{ij} \quad (i \neq j)$$

[all the elements are 0 except the diagonal elements ± 1 and the (ij) element $= \lambda$]

1.3

This definition can also be extended to non unital rings as for K_0 . However, this definition of $K_1(A)$ now depends on k if we view A as a k -algebra.

1.5. The reason why we introduced the group K_1 is that - in some sense - it is a ~~the~~ left derived functor of K_0 .

More precisely, let us consider an exact sequence of rings

$$0 \rightarrow A' \xrightarrow{\delta} A \xrightarrow{\beta} A'' \rightarrow 0$$

(δ and β are ring maps and the underlying sequence of abelian groups is exact). We have then the following basic theorem

1.6. Theorem - one has an exact sequence (with $K = K_0$)

$$K_1(A') \rightarrow K_1(A) \rightarrow K_1(A'') \rightarrow K_0(A') \rightarrow K_0(A) \rightarrow K_0(A'')$$

This theorem is implicit in the book of Milnor

(Introduction to Algebraic K-theory). Another formulation is the following : we consider a cartesian square of unitary rings (with q_i surjective)

$$\begin{array}{ccc} A & \xrightarrow{q_1} & A_1 \\ q_2 \downarrow & & \downarrow q_3 \\ A_2 & \xrightarrow{q_2} & A' \end{array} \quad (2)$$

we have then an exact sequence

$$K_1(A) \rightarrow K_1(A_1) \oplus K_1(A_2) \rightarrow K_1(A') \rightarrow K_0(A) \rightarrow K_0(A_1) \oplus K_0(A_2) \rightarrow K_0(A')$$

(-1, 4)

1.7 Sketch of the proof

Let $\varphi: \mathcal{P}(A) \rightarrow \mathcal{P}(B)$ be an additive functor which is "cofinal" in the sense of Bass, which means that any object in $\mathcal{P}(B)$ is a direct summand in some $\varphi(E)$, where E is an object of $\mathcal{P}(A)$. Then one can prove the following exact sequence (as in Algebraic Topology, for relative homology)

$$K_1(A) \rightarrow K_1(B) \xrightarrow{\partial} K(\varphi) \rightarrow K_0(A) \rightarrow K_0(B)$$

Here the "relative group" $K(\varphi)$ is the free group generated by isomorphism class of triples (E, F, α) , where E, F are objects in $\mathcal{P}(A)$ et $\alpha: \varphi(E) \rightarrow \varphi(F)$ is an isomorphism, divided by the following relation

$$(E, F, \alpha) + (F, G, \beta) = (E, G, \beta\alpha)$$

The maps in the sequence are naturally defined, except ∂ which amounts to a matrix $\alpha \in GL_2(B)$ the class of the triple (E, E, α) where $\varphi(E) = B^2 \oplus E'$ and $\alpha = \alpha_E \oplus Id_{E'}$.

The proof of the theorem is then reduced to the following excision lemma (left as an exercise).

1.8. Lemma Under the previous hypothesis, the maps (ψ_1, ψ_2) induce an isomorphism

$$K(\varphi_2) \cong K(\varphi_1)$$

1.9. Note[†] the proof of the exactness of the sequence

$K_1(A') \rightarrow K_1(A) \rightarrow K_1(A'')$ and $K_0(A') \rightarrow K_0(A) \rightarrow K_0(A'')$ have to be done independently -

(1.5)

1.9. Note 2 A key point in the proof is that any f.g. projective A -module is the image of a projection operator p ($p^2=p$) in $M_2(A)$ for a certain r . If we call $\text{Proj}_2(A)$ this set of projection operators, one has to use the following cartesian square

$$\text{Proj}_2(A) \rightarrow \text{Proj}_2(A_r)$$

$$\downarrow \qquad \downarrow$$

$$\text{Proj}_2(A_r) \rightarrow \text{Proj}_2(A')$$

as well as the cartesian square

$$GL_2(A) \rightarrow GL_2(A_r)$$

$$\downarrow \qquad \downarrow$$

$$GL_2(A_r) \rightarrow GL_2(A')$$

These remarks will be crucial in hermitian K-theory.

1.10. This long preliminary in (lower) Algebraic K-theory is I think necessary to understand the corresponding basic definitions in hermitian K-theory. (1,6)

The setting is now slightly changed: we assume that the ring A has an (anti) involution $a \mapsto \bar{a}$ (in particular, we have $\bar{ab} = \bar{b}\bar{a}$) and we give ourselves a "sign of symmetry" $\varepsilon = \pm 1$. If E is a f.g. projective right A -module, a sesquilinear form is a \mathbb{Z} -bilinear map

$$c: E \times E \rightarrow A$$

such that $c(\bar{x}, \bar{y}) = \bar{\lambda} c(x, y)\mu$ if $x, y \in E$; $\lambda, \mu \in A$
we say that c is ε -hermitian if moreover we have
 $c(y, x) = \varepsilon \overline{c(x, y)}$.

1.11. Another way of saying the same thing is to introduce the "dual" module $E^* = \{f: E \rightarrow A \mid f(x\lambda) = \bar{\lambda}f(x)\}$

Then the data c is equivalent to a A -linear map

$$\tilde{c}: E \rightarrow E^* \quad (\text{put } \tilde{c}(y)(x) = c(x, y)) \quad \text{Note that}$$

E^* is a right A -module by $(f.y)(x) = f(x)y$.

As usual, we may identify the "bidual" E^{**} to E

(since E is f.g. projective). Saying that c is ε -hermitian is equivalent to say that the "transpose" of \tilde{c} ,
say ${}^t\tilde{c}: E = (E^*)^* \rightarrow E^*$, is equal to $\varepsilon \tilde{c}$.

1.12. Let c be an ε -hermitian form. We say that c is even iff $c(x, y)$ may be written as $c_0(x, y) + \varepsilon \overline{c_0(y, x)}$. Equivalently, this means that \tilde{c} is of the type $\tilde{c}_0 + \varepsilon {}^t\tilde{c}_0$.

(1.7)

1.13. Let (E, φ) and (F, ψ) two ε -hermitian modules.

An isometry between these two modules is given by a linear map $f: E \rightarrow F$ which is bijective with the additional assumption that $\varphi(f(x), f(y)) = \psi(x, y)$. This last condition is equivalent to the commutativity of the following diagram

$$\begin{array}{ccc} E & \xrightarrow{\tilde{\varphi}} & E^* \\ f \downarrow & & \uparrow f^* \\ F & \xrightarrow{\tilde{\psi}} & F^* \end{array}$$

We define the adjoint of f to be $\tilde{\varphi}^{-1} f^* \tilde{\psi} = f^*$. We have then the identity $f^* f = \text{Id}_E$ and $f f^* = \text{Id}_F$.

1.14. An important example of ε -hermitian module is given by $E = M \oplus M^*$, where M is f.g. projective and φ given by $\varphi((x, u), (x', u')) = u'(x) + \varepsilon \bar{u}(x')$. If we identify again M^{**} with M , it is more convenient to write

$$\tilde{\varphi}: M \oplus M^* \rightarrow (M \otimes M^*) \cong M^* \oplus M$$

as the matrix $\begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}$

This is an even hermitian form if we put

$$\tilde{\varphi}_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

we put $H(M) = M \otimes M^*$, the hyperbolic module associated to M .

An isometry

$$f: H(M) = M \otimes M^* \rightarrow H(N) = N \otimes N^*$$

(1.8)

is given by a 2×2 matrix (called also f)

$$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

which adjoint is

$$f^* = \begin{pmatrix} {}^t d & {}^t b \\ {}^t c & {}^t a \end{pmatrix}$$

1.15. The identity $f f^* = f^* f$ gives the an explicit way to write isometries between hyperbolic modules, especially if $M = A^n$, in which case $\mathcal{U}_{n,n}(A)$ (or $\mathcal{O}_{n,n}^{hyp}(A)$ according to Borel's terminology) denotes the group of isometries of $H(A)$.

As a matter of fact, there is an hyperbolic functor

which amounts to a linear bijection $\alpha: M \rightarrow N$, the isometry $H(\alpha): H(M) \rightarrow H(N)$ defined by

$$H(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & {}^t \alpha^{-1} \end{pmatrix}$$

[Careful: when we are dealing with free modules of the type A^n , morphisms are given by matrices and ${}^t \alpha$ means really conjugate transpose, so that the identity ${}^t (\alpha \beta) = {}^t \beta {}^t \alpha$ is valid]

1.16. Following Tits and Wall, we can define a more subtle notion of quadratic form as follows: such a form on E is given by a morphism $\tilde{\psi}_0: E \rightarrow E^*$

which is well defined modulo mappings of the type

$\tilde{q}_0 - \varepsilon^b \tilde{q}_0$ and such that $q = \tilde{q}_0 + \varepsilon^b \tilde{q}_0$ is a non-degenerate ε -hermitian form

An isometry $(E, \tilde{q}_0) \rightarrow (F, \tilde{q}_0)$ is given by a linear bijection $f: E \rightarrow F$ as before such that
 $f^* \tilde{q}_0 f = \tilde{q}_0 + \tilde{u} - \varepsilon^b \tilde{u}$
for certain $u: E \rightarrow E^*$

1.17. Example - Let us take again the case of an hyperbolic module $E = H(M)$. Then we can choose $\tilde{q}_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ as defining a quadratic form on M . The fact that

$$f: H(M) = M \otimes M^* \rightarrow H(N) = N \otimes N^*$$

is an isometry gives more assumptions than just

$f f^* = f^* f$ - In the notation in 1.14 one has to add the identities $a^b b$ and $c^d d$ of the type $y - \varepsilon^b y$

In the case of $M = A^n$, we obtain the orthogonal group

$$\mathbb{E}^{O_{n,n}}(A) \quad (\text{or } O_{n,n}^{\text{min}}(A) \text{ in Borel's terminology})$$

We have an inclusion $\mathbb{E}^{O_{n,n}}(A) \subset \mathbb{E}^{U_{n,n}}(A)$.

1.18. Remark - If we assume the existence of λ in the center of A such that $\lambda + \bar{\lambda} = 1$, then the two notions of ε -hermitian form and ε -orthogonal form coincide (put $q_0 = \lambda q_0$) -

In particular, one has $\mathbb{E}^{O_{n,n}}(A) = \mathbb{E}^{U_{n,n}}(A)$

(1.40)

The main interest of hyperbolic modules and even hermitian forms is the following basic statement

1.19. Theorem. Let E be a f.g. projective A -module with a ε -quadratic form q_0 . Then the direct sum $(E, q_0) \oplus (E, -q_0)$ is isomorphic to $H(E) = E \otimes E^*$ with the quadratic form $\chi_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ defined above.

Proof. We define $f : (E, q_0) \oplus (E, -q_0) \longrightarrow E \otimes E^*$ by the following matrix

$$f = \begin{pmatrix} 1 - q^* q_0 & q^{-1} q_0 \\ q & -q \end{pmatrix}$$

It is invertible, with $f^{-1} = \begin{pmatrix} 1 & q^{-1} q_0 & q^{-1} \\ 1 & q^{-1} q_0 & q^{-1} - q_0^{-1} \end{pmatrix}$

on the other hand,

$$\begin{aligned} {}^t f \chi_0 f &= \begin{pmatrix} 1 - \varepsilon^6 q_0 q^{-1} & \varepsilon q \\ \varepsilon q & -\varepsilon q \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 - q^* q_0 & q^{-1} q_0 \\ q & -q \end{pmatrix} \\ &= \begin{pmatrix} q_0 & -q_0 \\ \varepsilon^6 q_0 & -\varepsilon^6 q_0 \end{pmatrix} = \underbrace{\begin{pmatrix} q_0 & 0 \\ 0 & -q_0 \end{pmatrix}}_{\theta} + \underbrace{\begin{pmatrix} 0 & -q_0 \\ \varepsilon^6 q_0 & 0 \end{pmatrix}}_{\varepsilon^6 \theta} + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & q_0 - \varepsilon^6 q_0 \end{pmatrix}}_{\theta - \varepsilon^6 \theta} \end{aligned}$$

1.20. Corollary - Any ε -hermitian module E (resp ε -quadratic module) is a direct summand in some $H(A^*)$ with the associated ε -hermitian form (resp ε -quadratic form). (1.11.)

A more concrete way to view this corollary is to say that $E = \text{Im } p$, where $p = p^*$ is a self-adjoint projection operator in $H(A^*)$.

1.21. Finally we want to define a third category of quadratic modules which is "above" it. This category has the same objects of the type (E, \tilde{q}_0) as in 1.16. However, we enlarge the set of morphisms $(E, \tilde{q}_0) \rightarrow (F, \tilde{q}_0)$ by giving us a couple (f, \tilde{u}) where $f: E \rightarrow F$ is a linear bijection and $\tilde{u}: E \rightarrow F^*$ is such that

$${}^t f \tilde{q}_0 f = \tilde{q}_0 + \tilde{u} - \varepsilon {}^t \tilde{u}$$

The composition $(f, \tilde{u}): E \rightarrow F$ with $(g, \tilde{v}): F \rightarrow G$ is given by the following formula

$$(g, \tilde{v}) \circ (f, \tilde{u}) = (g \circ f, \tilde{u} + {}^t f \tilde{v} f)$$

1.22. Summarizing, we have three categories and three functors

$${}_{\varepsilon}Q^{\text{split}}(A) \rightarrow {}_{\varepsilon}Q^{\text{mig}}(A) \longrightarrow {}_{\varepsilon}Q^{\text{max}}(A)$$

which K-theory should be elevated by

$${}_{\varepsilon}KQ^{\text{split}}(A), {}_{\varepsilon}KQ^{\text{mig}}(A), {}_{\varepsilon}KQ^{\text{max}}(A)$$

1.23. From the point of view of group-theory, if we fix (E, q_0) , we have 3 groups

(1.12)

- 1) the unitary group $\mathcal{U}(E, q)$ (cf. 1.13)
- 2) the orthogonal group $\mathcal{O}(E, q_0)$ (cf. 1.16)
- 3) the split orthogonal group $\mathcal{O}^{\text{split}}(E, q_0)$ (cf. 1.21)

a) $\mathcal{O}(E, q_0)$ is a subgroup of $\mathcal{U}(E, q)$

b) $\mathcal{O}^{\text{split}}(E, q_0)$ is a covering of $\mathcal{O}(E, q_0)$

$$1 \rightarrow H \rightarrow \mathcal{O}^{\text{split}}(E, q_0) \xrightarrow{\delta} \mathcal{O}(E, q_0) \rightarrow 1$$

The kernel H of δ is the additive group of morphisms $\tilde{v}: E \rightarrow E^*$ such that $\tilde{v} = \varepsilon v$.

1.24. Remark - This extension is not a semi-direct in general. However, it is so if we assume $\lambda \in \mathbb{Z}(A)$ such that $\lambda + \bar{\lambda}$. Then we define a section s of δ by putting

$$s(f) = (f, \tilde{v}) \text{ where } \tilde{v} = \lambda / ({}^t f q_0 f - q_0)$$

Note that the action of $\mathcal{O}(E, q_0)$ on H is given by the following formula $f \cdot \tilde{v} = {}^t f \tilde{v} f$ $\begin{pmatrix} f \in \mathcal{O}(E, q_0) \\ \tilde{v} : E \rightarrow E^* \end{pmatrix}$

and therefore $\mathcal{O}^{\text{split}}(E, q_0)$ is the semi-direct product of $\mathcal{O}(E, q_0) = \mathcal{U}(E, q)$ by the additive group of such $\tilde{v}: E \rightarrow E^*$ such that $\tilde{v} = \varepsilon \tilde{v}$.

A better way to view this group is to replace it by
 $\epsilon\tilde{v}^{-1}w = w$ - Then we view $\mathcal{E}^{split}(E, q_0)$ as the
semi-direct product of $\mathcal{E}^U(E, q)$ by the $w : E \rightarrow E$ such that
 $w^* = Ew$ with action given by $f \cdot w = f^{-1}w f$ - With
this point of view we see that $\mathcal{E}^{split}(E, q_0)$ is just the
unitary group of $E \otimes_{\mathbb{A}} \mathbb{A}[\epsilon]/\epsilon^2$ with $\bar{\epsilon} = -\epsilon$

- 1.25. In general, we should notice that the hyperbolic
functions lifts to the "split" category. More precisely,
- on the level of objects $E \mapsto E \oplus E^\times$
- on the level of morphisms $\alpha \mapsto (f, u)$ where $f = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$, one has to check the following identities

$$\begin{pmatrix} t_\alpha & 0 \\ 0 & t_{\alpha^{-1}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = X_0$$

$$f \cdot X_0 \cdot f^{-1} = X_0$$

1.25 The kernel of the induced map

$$K(A) \rightarrow_{\epsilon} KQ(A)$$

is called the Witt group $\mathcal{E}^W(A)$. As a matter
of fact there are just 2 Witt groups $\mathcal{E}^{W_{\text{max}}}(A)$ (for
even hermitian forms) and $\mathcal{E}^{W_{\text{min}}}(A)$ (for quadratic
forms). We have of course $\mathcal{E}^{W_{\text{split}}}(A) = \mathcal{E}^{W_{\text{min}}}(A)$.

(1.14)

Examples

1. A is a field F of characteristic $\neq 2$, $\varepsilon = 1$

Then, $W(A) = W(F)$ is the classical Witt group

There is an augmentation ideal $I \simeq \text{Ker}(W(F) \rightarrow \mathbb{Z}_2)$.

According to Milnor's conjecture, proved by Voevodsky,

$$I/\mathbb{I}_{p+1} \simeq k_n^m(F)$$

2) A is the ring of continuous functions on a compact space X with real values -

Then, $W(A) = K(A) \oplus K(A)$

$$W(A) = K(A \otimes_{\mathbb{R}} \mathbb{C})$$

3) A is the field with 2 elements \mathbb{F}_2

Then $W^{\max}(A) = 0$

$W^{\min}(A) = \mathbb{Z}_2$ (detected by the Art invariant)

4) A is the product $B \times B^{op}$ where B^{op} is the opposite ring, the antiinvolution being $(a, b) \mapsto (b, a)$. It is easy to see that the category of ε -hermitian modules over A is isomorphic to the category of projective modules over B (the morphisms being isomorphisms). Therefore, $KQ(A) \simeq K(B)$ etc.

$$W(A) = 0$$