A Functional Integral Representation for Many Boson Systems

Tadeusz Balaban
Rutgers University
tbalaban@math.rutgers.edu

Joel Feldman
University of British Columbia
feldman@math.ubc.ca

Horst Knörrer, Eugene Trubowitz
ETH-Zürich
trub@math.ethz.ch
knoerrer@math.ethz.ch

(P)reprints: http://www.math.ubc.ca/~feldman/

Abstract

Functional integrals have long been used, formally, to provide intuition about the behaviour of quantum field theories. For the past several decades, they have also been used, rigorously, in the construction and analysis of those theories. I will talk about the rigorous derivation of some functional integral representations for the partition function and correlation functions of (cutoff) many Boson systems that provide a suitable starting point for their construction.

Outline

- The Physical Setting
- The Goal
- One Result
- The Main Ideas Leading to the Results

The Physical Setting

Consider a gas of small (i.e. quantum mechanical effects are important) particles that are bosons (i.e. integer spin; e.g. photons, Helium-4 atoms).

- Each particle has a kinetic energy. The corresponding quantum mechanical observable is an operator h. For example, if the energy is $\frac{\mathbf{p}^2}{2m}$, then $h = -\frac{1}{2m}\Delta$.
- The particles interact with each other through a repulsive two-body potential, $v(\mathbf{x}, \mathbf{y})$.
- The system is in thermodynamic equilibrium through contact with a heat bath. The gas and the heat bath can exchange both energy and particles. The probability distributions for the energy and for the number of particles in the gas are controlled by the temperature $T = \frac{1}{k\beta}$ and chemical potential μ respectively.
- The space of all states of this system is

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}_n \text{ with } \mathcal{F}_n = L^2_s(\mathbb{R}^{3n})$$

- Two interesting observables (=self–adjoint operators) for this system are the Hamiltonian H (built from h and v) and the number operator N (defined by $N \restriction \mathcal{F}_n = n\mathbb{1}$).

- I will concentrate on one quantity of interest, the partition function

$$Z = \mathrm{Tr}\, e^{-\beta(H-\mu N)}$$

The Goal

Formally,

$$\operatorname{Tr} e^{-\beta(H-\mu N)} = \int_{\phi_\beta=\phi_0} \mathcal{D}(\phi^*,\phi) \quad e^{\mathcal{A}(\phi^*,\phi)} \tag{1}$$

where

$$\mathcal{D}(\phi^*,\phi) = \prod_{\substack{\mathbf{x}\in\mathbb{R}^3 \\ 0\le\tau\le\beta}} \frac{d\phi_\tau^*(\mathbf{x})\ d\phi_\tau(\mathbf{x})}{2\pi i}$$

and

$$\mathcal{A}(\phi^*,\phi) = \int_0^\beta d\tau \int_{\mathbb{R}^3} d^3\mathbf{x} \quad \phi_\tau^*(\mathbf{x}) \tfrac{\partial}{\partial\tau} \phi_\tau(\mathbf{x}) \\ - \int_0^\beta d\tau\ K\big(\phi_\tau\big)$$

and

$$K(\phi) = \iint d\mathbf{x}d\mathbf{y}\ \phi(\mathbf{x})^* \mathrm{h}(\mathbf{x},\mathbf{y})\phi(\mathbf{y}) \\ -\mu \int d\mathbf{x}\ \phi(\mathbf{x})^*\phi(\mathbf{x}) \\ + \iint d\mathbf{x}d\mathbf{y}\ \phi(\mathbf{x})^*\phi(\mathbf{y})^* v(\mathbf{x},\mathbf{y})\,\phi(\mathbf{x})\phi(\mathbf{y})$$

Both sides of (1) require careful definition.

The definition of the left hand side is similar in spirit to the definition of the Riemann integral. You take a limit of obviously well–defined approximations. One way to get (pretty) obviously well–defined approximation is to replace space $\mathbb{R}^3$ by a finite number of points X. Then

- The space of all states of this system is

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}_n \text{ with } \mathcal{F}_n = L^2_s(X^n) = \mathbb{C}^{|X|^n}/S_n$$

- $\mathrm{Tr}\, e^{-\beta(H-\mu N)}$ is well–defined because

$$\begin{aligned}
&H, N : \mathcal{F}_n \rightarrow \mathcal{F}_n \\
&N \restriction \mathcal{F}_n = n\mathbb{1} \\
&H \restriction \mathcal{F}_n \geq (Cn - D)n\mathbb{1} \\
&\dim \mathcal{F}_n \leq |X|^n \\
&\implies \mathrm{Tr}_{\mathcal{F}_n} e^{-\beta(H-\mu N)} \leq e^{-\beta(Cn^2 - Dn - \mu n)}|X|^n
\end{aligned}$$

- The analog of (1) is again

$$\mathrm{Tr}\, e^{-\beta(H-\mu N)} = \int_{\phi_\beta = \phi_0} \mathcal{D}(\phi^*, \phi) \;\; e^{\mathcal{A}(\phi^*,\phi)} \tag{2}$$

but with

$$\mathcal{D}(\phi^*, \phi) = \prod_{\substack{\mathbf{x} \in X \\ 0 \leq \tau \leq \beta}} \frac{d\phi^*_\tau(\mathbf{x}) \;\, d\phi_\tau(\mathbf{x})}{2\pi i}$$

and $\int_{\mathrm{I\!R}^3} d^3\mathbf{x}$ replaced by

$$\int d\mathbf{x} = (\text{cell volume}) \sum_{\mathbf{x}\in X}$$

The goal is to get a rigorous version of (2) which can provide a useful starting point for a study of the limit $X \to \mathrm{I\!R}^3$.

WARNING

The exponent $\mathcal{A}(\phi^*, \phi)$ is complex.
$\implies$ $e^{\mathcal{A}(\phi^*,\phi)}$ oscillates wildly
$\implies$ $\frac{1}{\text{const}}\mathcal{D}(\phi^*,\phi)e^{\text{part of}\mathcal{A}(\phi^*,\phi)}$ cannot be turned into an ordinary well–defined complex measure on some space of paths, in contrast to Wiener measure.

For example, if

$$d\mu(\vec{\phi}) = \frac{e^{-\frac{1}{2}\sigma\vec{\phi}\cdot C\vec{\phi}}\ d^n\vec{\phi}}{\int_{\mathrm{I\!R}^n} e^{-\frac{1}{2}\sigma\vec{\phi}\cdot C\vec{\phi}}\ d^n\vec{\phi}}$$

with $C > 0$, $\operatorname{Re}\sigma > 0$ and $\operatorname{Im}\sigma \neq 0$, then

$$\int_{\mathrm{I\!R}^n} \left|d\mu(\vec{\phi})\right| = \left\{\tfrac{\operatorname{Re}\sigma}{|\sigma|}\right\}^{-n/2} \overset{n\to\infty}{\longrightarrow} \infty$$

$\implies$ The rigorous version is not going to be very aesthetically satisfying.

One Result

Notation:

$$\mathcal{T}_p = \left\{ \ \tau = q\frac{\beta}{p} \ \middle| \ q = 1\,, \ \cdots\,, \ p \ \right\}$$

$$\varepsilon_p = \frac{\beta}{p}$$

$$d\mu_{p,r}(\phi^*,\,\phi) = \prod_{\tau\in\mathcal{T}_p}\prod_{\mathbf{x}\in X}\left[\tfrac{d\phi_\tau^*(\mathbf{x})\,d\phi_\tau(\mathbf{x})}{2\pi\imath}\,\chi(|\phi_\tau(\mathbf{x})| < r)\right]$$

Theorem. *Suppose that the sequence* $\mathrm{R}(p) \to \infty$ *as* $p \to \infty$ *at a suitable rate. Then*

$$\begin{aligned}
&\mathrm{Tr}\ e^{-\beta\,(H-\mu N)} \\
&\quad = \lim_{p\to\infty}\int d\mu_{p,\mathrm{R}(p)}(\phi^*,\phi) \\
&\qquad\qquad \prod_{\tau\in\mathcal{T}_p} e^{-\int d\mathbf{y}\,[\phi_\tau^*(\mathbf{y})-\phi_{\tau-\varepsilon_p}^*(\mathbf{y})]\phi_\tau(\mathbf{y})}\, e^{-\varepsilon_p K(\phi_{\tau-\varepsilon_p}^*,\phi_\tau)}
\end{aligned}$$

with the convention that $\phi_0 = \phi_\beta$ *.*

The Main Ingredients – Coherent States

If $|X| = 1$, then

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}_n \text{ with } \mathcal{F}_n = \mathbb{C}$$

Let $e_n = 1 \in \mathbb{C} = \mathcal{F}_n$. For each $\phi \in \mathbb{C}$ the coherent state

$$|\,\phi\,\rangle = \sum_{n=0}^{\infty} \tfrac{1}{\sqrt{n!}} \phi^n e_n \in \mathcal{F}$$

is an eigenvector for the field (or annihilation) operator.

$$\psi e_n = \sqrt{n}\, e_{n-1}$$

The inner product between two coherent states is

$$\langle\, \alpha \,|\, \gamma \,\rangle = e^{\overline{\alpha}\,\gamma}$$

For general X, there is a similar coherent state $|\,\phi\,\rangle$ for each $\phi \in \mathbb{C}^{|X|}$. The inner product between two coherent states is

$$\langle\, \alpha \,|\, \gamma \,\rangle = e^{\int d\mathbf{y}\ \overline{\alpha(\mathbf{y})}\,\gamma(\mathbf{y})}$$

The Main Ingredients – Resolution of the identity

Formally

$$\mathbb{1} = \int \prod_{\mathbf{x}\in X} \left[\tfrac{d\phi^*(\mathbf{x})\, d\phi(\mathbf{x})}{2\pi\imath}\right] e^{-\int d\mathbf{y}\ |\phi(\mathbf{y})|^2}\ \ |\,\phi\,\rangle\,\langle\,\phi\,|$$

Here

$$|\,\phi\,\rangle\,\langle\,\phi\,| : v \in \mathcal{F} \mapsto \big\{\text{inner product of } v \text{ and } |\,\phi\,\rangle\big\}\ \ |\,\phi\,\rangle$$

Theorem. *For each $r > 0$, let*

$$\mathrm{I}_r = \prod_{\mathbf{x}\in X} \left[\int_{|\phi(\mathbf{x})|<r} \frac{d\phi^*(\mathbf{x})\, d\phi(\mathbf{x})}{2\pi\imath}\right] e^{-\int d\mathbf{y}\ |\phi(\mathbf{y})|^2}\ \ |\,\phi\,\rangle\,\langle\,\phi\,|$$

(a) $0 < \mathrm{I}_r < \mathbb{1}$.

(b) I_r *commutes with* N.

(c) I_r *converges strongly to the identity operator as* $r \to \infty$.

(d) *For all n and r, the operator norm*

$$\big\|(\mathbb{1} - \mathrm{I}_r)\ \restriction \mathcal{F}_n\big\|\ \le\ |X|\, 2^{n+1}\, e^{-r^2/2}$$

Proof: It is easy to guess an orthonormal basis of eigenvectors for I_r and to find all of the eigenvalues:
If $|X| = 1$, then

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}_n \text{ with } \mathcal{F}_n = \mathbb{C}$$

and $\left\{ e_m = 1 \in \mathbb{C} = \mathcal{F}_m \;\middle|\; m = 0, 1, 2, 3, \cdots \right\}$ is an orthonormal basis for $\mathcal{F}$. Recall that

$$|\,\phi\,\rangle = \sum_{n=0}^{\infty} \tfrac{1}{\sqrt{n!}} \phi^n e_n$$

So

$$\begin{aligned}
\mathrm{I}_r e_m &= \int_{|\phi|<r} \tfrac{d\bar\phi\, d\phi}{2\pi\imath}\; e^{-|\phi|^2}\; |\,\phi\,\rangle \left\langle\, \phi \,\middle|\, e_m \,\right\rangle \\
&= \int_{|\phi|<r} \tfrac{d\bar\phi\, d\phi}{2\pi\imath}\; e^{-|\phi|^2}\; |\,\phi\,\rangle\, \tfrac{1}{\sqrt{m!}} \bar\phi^m \\
&= \sum_{n=0}^{\infty} \tfrac{1}{\sqrt{n!}\sqrt{m!}}\; e_n \int_{|\phi|<r} \tfrac{d\bar\phi\, d\phi}{2\pi\imath}\; e^{-|\phi|^2}\; \bar\phi^m \phi^n \\
&= \left\{ \tfrac{1}{m!} \int_{|\phi|<r} \tfrac{d\bar\phi\, d\phi}{2\pi\imath}\; e^{-|\phi|^2}\; |\phi|^{2m} \right\} e_m \\
&= \left\{ 1 - \tfrac{1}{m!} \int_{\sqrt{r}}^{\infty} e^{-t} t^m \; dt \right\} e_m
\end{aligned}$$

■

The Main Ingredients – Trace

Formally,

$$\mathrm{Tr}\, B = \int \prod_{\mathbf{x}\in X} \left[\tfrac{d\phi^*(\mathbf{x})\, d\phi(\mathbf{x})}{2\pi\imath}\right]\ e^{-\int d\mathbf{y}\ |\phi(\mathbf{y})|^2}\ \langle\, \phi \mid B \mid \phi \,\rangle$$

Proposition. *Let B be a bounded operator on $\mathcal{F}$ that commutes with N. For all $r > 0$, $B\mathrm{I}_r$ is trace class and*

$$\mathrm{Tr}\, B\mathrm{I}_r = \prod_{\mathbf{x}\in X}\Big[\int_{|\phi(\mathbf{x})|<r} \frac{d\phi^*(\mathbf{x})\, d\phi(\mathbf{x})}{2\pi\imath}\Big] e^{-\int d\mathbf{y}\ |\phi(\mathbf{y})|^2} \langle\, \phi \mid B \mid \phi \,\rangle$$

Proof: Let P_n be the orthogonal projection from $\mathcal{F}$ onto the direct sum $\bigoplus\limits_{0\le m\le n} \mathcal{F}_m$. If $|X| = 1$,

$$\begin{aligned}
\mathrm{Tr}\ B\mathrm{I}_r\mathrm{P}_n &= \sum_{m\le n} \langle\, e_m \mid B\mathrm{I}_r \mid e_m \,\rangle \\
&= \sum_{m\le n} \int_{|\phi|<r} \frac{d\bar\phi\, d\phi}{2\pi\imath}\, e^{-|\phi|^2}\, \langle\, e_m \,|\, B \,|\, \phi \,\rangle \big\langle \phi \,\big|\, e_m \big\rangle \\
&= \int_{|\phi|<r} \frac{d\bar\phi\, d\phi}{2\pi\imath} e^{-|\phi|^2} \sum_{m\le n} \big\langle \phi \,\big|\, e_m \big\rangle \langle\, e_m \,|\, B \,|\, \phi \,\rangle \\
&= \int_{|\phi|<r} \frac{d\bar\phi\, d\phi}{2\pi\imath}\ e^{-|\phi|^2}\ \langle\, \phi \mid \mathrm{P}_n B \mid \phi \,\rangle
\end{aligned}$$

Now take limits. ∎

Combining the previous two results,

$$
\begin{aligned}
\mathrm{Tr}\; e^{-\beta(H-\mu N)} &= \mathrm{Tr}\prod_{\tau\in\mathcal{T}_p} e^{-\frac{\beta}{p}(H-\mu N)} \\
&= \lim_{p\to\infty} \mathrm{Tr}\prod_{\tau\in\mathcal{T}_p} e^{-\frac{\beta}{p}(H-\mu N)} \mathrm{I}_{\mathrm{R}(p)} \\
&= \lim_{p\to\infty} \prod_{\substack{\mathbf{x}\in X\\ \tau\in\mathcal{T}_p}} \Big[\int_{|\phi_\tau(\mathbf{x})|<\mathrm{R}(p)} \frac{d\phi_\tau^*(\mathbf{x})\, d\phi_\tau(\mathbf{x})}{2\pi\imath}\; e^{-|\phi_\tau(\mathbf{x})|^2}\Big] \\
&\qquad\qquad \prod_{\tau\in\mathcal{T}_p} \Big\langle \phi_\tau \,\Big|\, e^{-\frac{\beta}{p}\,(\mathrm{H}-\mu N)} \,\Big|\, \phi_{\tau+\frac{\beta}{p}} \Big\rangle
\end{aligned}
$$

Proposition. *For each $\varepsilon > 0$, there is an analytic function $F(\varepsilon, \alpha^*, \phi)$ such that*

$$
\Big\langle \alpha \,\Big|\, e^{-\varepsilon(H-\mu N)} \,\Big|\, \phi \Big\rangle = e^{F(\varepsilon,\alpha^*,\phi)}
$$

on the domain $\|\alpha\|_\infty,\ \|\phi\|_\infty < C\frac{1}{\sqrt{\varepsilon}}$. Write

$$
F(\varepsilon,\alpha^*,\phi) = \int_X d\mathbf{x}\,\alpha(\mathbf{x})^*\phi(\mathbf{x}) - \varepsilon K(\alpha^*,\phi) + \mathcal{F}_0(\varepsilon,\alpha^*,\phi)
$$

There is a constant const *such that for all* $0 < \varepsilon \le 1$

$$
|\mathcal{F}_0(\varepsilon,\alpha^*,\phi)| \le \mathrm{const}\,\varepsilon^2(\Phi^2 + \|v\|_{1,\infty}^2\,\Phi^6)
$$

for all $\|\alpha\|_\infty,\ \|\phi\|_\infty \le \Phi \le \frac{1}{2}C\frac{1}{\sqrt{\varepsilon}}$.

Idea of Proof: $\left\langle\, \alpha \,\middle|\, e^{-\varepsilon(H-\mu N)} \,\middle|\, \phi \,\right\rangle$ is an entire function of α^* and ϕ and a C^∞ function of ε for $\varepsilon \ge 0$. Since $\left\langle\, \alpha \,\middle|\, \phi \,\right\rangle = e^{\int \alpha^*(\mathbf{x})\phi(\mathbf{x})\, d\mathbf{x}} \neq 0$, the matrix element has the representation

$$\left\langle\, \alpha \,\middle|\, e^{-\varepsilon(H-\mu N)} \,\middle|\, \phi \,\right\rangle = e^{F(\varepsilon,\alpha^*,\phi)}$$

in a neighbourhood of 0, with $F(\varepsilon,\alpha^*,\phi)$ is analytic in α^*, ϕ. F satisfies the differential equation

$$\begin{aligned} \tfrac{\partial}{\partial\varepsilon} F = & -\mathcal{K}(\alpha^*, \tfrac{\partial}{\partial\alpha^*})F \\ & \quad - \iint_X d\mathbf{x} d\mathbf{y}\ \alpha(\mathbf{x})^* \alpha(\mathbf{y})^*\, v(\mathbf{x},\mathbf{y})\, \tfrac{\partial\ F}{\partial\alpha(\mathbf{x})^*} \tfrac{\partial\ F}{\partial\alpha(\mathbf{y})^*} \end{aligned}$$

with the initial condition

$$F(0,\alpha^*,\phi) = \ln \left\langle \alpha \,\middle|\, \phi \right\rangle = \int_X d\mathbf{x}\, \alpha(\mathbf{x})^* \phi(\mathbf{x})$$

It is tedious but straight forward to convert this into a system of integral equations for coefficients in the Taylor expansion of $F(\varepsilon,\alpha^*,\phi)$ in powers of α^* and ϕ. The system can be solved and bounded by iteration.

So we now have

$$\begin{aligned}
&\mathrm{Tr}\ e^{-\beta\,(H-\mu N)} \\
&\quad = \lim_{p\to\infty} \int d\mu_{p,\mathrm{R}(p)}(\phi^*,\phi) \\
&\qquad\qquad \prod_{\tau\in\mathcal{T}_p} e^{-\int d\mathbf{y}\,[\phi^*_\tau(\mathbf{y})-\phi^*_{\tau-\varepsilon_p}(\mathbf{y})]\phi_\tau(\mathbf{y})}\, e^{-\varepsilon_p K(\phi^*_{\tau-\varepsilon_p},\phi_\tau)} \\
&\qquad\qquad \prod_{\tau\in\mathcal{T}_p} e^{-\mathcal{F}_0(\varepsilon_p,\phi^*_{\tau-\varepsilon_p},\phi_\tau)}
\end{aligned}$$

and we just have to eliminate the $\mathcal{F}_0$'s. The sum

$$\sum_{\tau\in\mathcal{T}_p} \mathcal{F}_0(\varepsilon_p,\phi^*_{\tau-\varepsilon_p},\phi_\tau)$$

- has p terms.
- Each term is bounded by $\dfrac{1}{p^2}$

  times an unbounded function of the $|\phi_\tau|$'s.