A combinatorial moduli space for polynomials of degree n.

François Bergeron

LaCIM, Université du Québec à Montréal

Gauss first "proof" of the fundamental theorem of algebra

For any degree n complex polynomial $f(z)$, consider the plane algebraic curves:

$$\begin{aligned} R(f) &:= \{(x,y) \mid Re(f(x+i\,y)) = 0\}, \qquad \text{and} \\ I(f) &:= \{(x,y) \mid Im(f(x+i\,y)) = 0\}. \end{aligned}$$

then

$$Z(f) = R(f) \cap I(f)$$

Gauss first "proof" of the fundamental theorem of algebra

For any degree n complex polynomial $f(z)$, consider the plane algebraic curves:

$$\begin{aligned} R(f) &:= \{(x,y) \mid Re(f(x+i\,y)) = 0\}, \qquad \text{and} \\ I(f) &:= \{(x,y) \mid Im(f(x+i\,y)) = 0\}. \end{aligned}$$

then

$$Z(f) = R(f) \cap I(f)$$

For example, if $f(z) = z^5 - z^4 - 2\,z^3 + 2\,z^2 + z + 1$ then

$$\begin{aligned} R(f) &= \{(x,y) \mid x^5 - 10\,x^3y^2 + 5\,xy^4 - x^4 + 6\,x^2y^2 - y^4 \\ &\qquad -2\,x^3 + 6\,xy^2 + 2\,x^2 - 2\,y^2 + x + 1 = 0\} \\ I(f) &= \{(x,y) \mid 5\,x^4y - 10\,x^2y^3 + y^5 - 4\,x^3y + 4\,xy^3 \\ &\qquad -6\,x^2y + 2\,y^3 + 4\,xy + y = 0\} \end{aligned}$$

Example

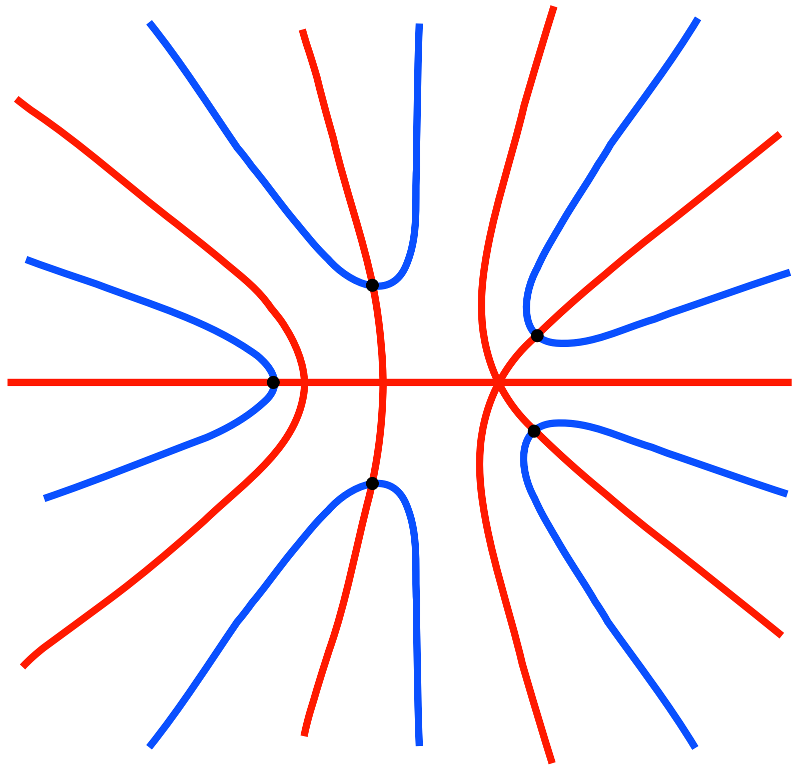


Figure 1: $f(z) = z^5 - z^4 - 2\,z^3 + 2\,z^2 + z + 1.$

Example

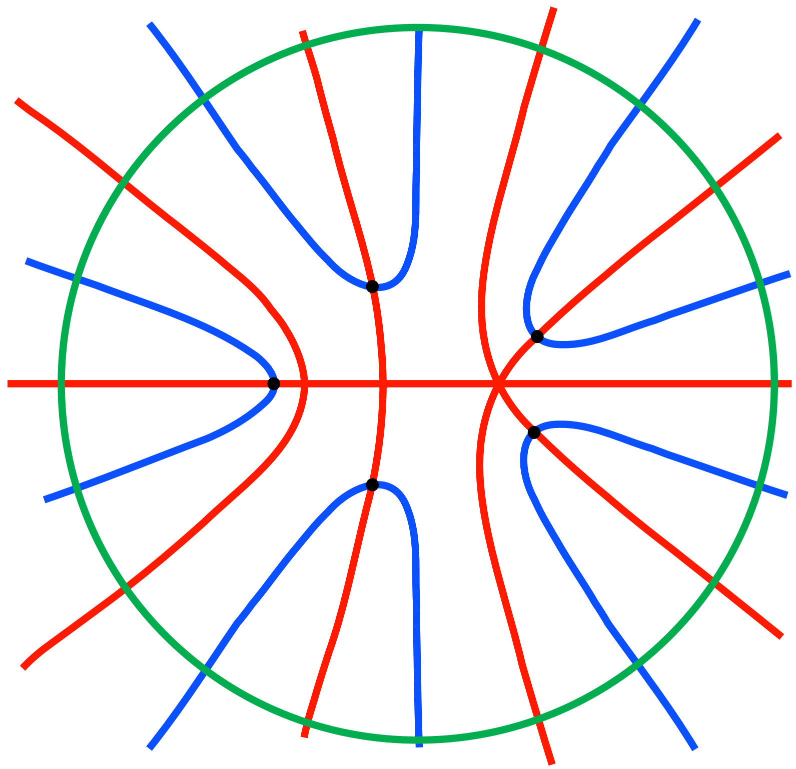


Figure 1: $f(z) = z^5 - z^4 - 2\,z^3 + 2\,z^2 + z + 1.$

$$x^2 + y^2 = \rho^2 \gg 0$$

Imaginary part

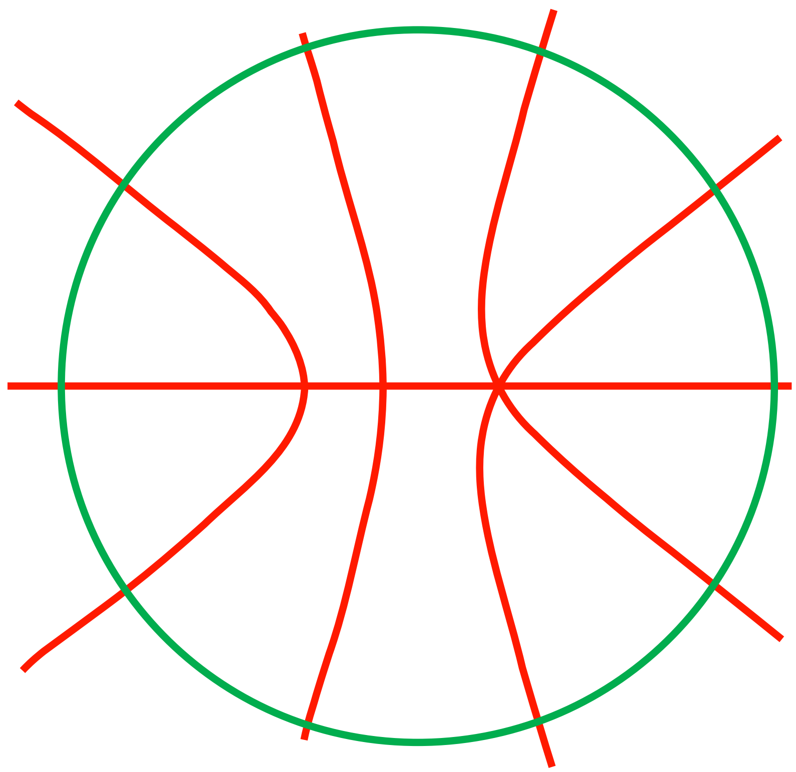


Figure 2: $5\,x^4y-10\,x^2y^3+y^5-4\,x^3y+4\,xy^3-6\,x^2y+2\,y^3+4\,xy+y=0.$

Imaginary part

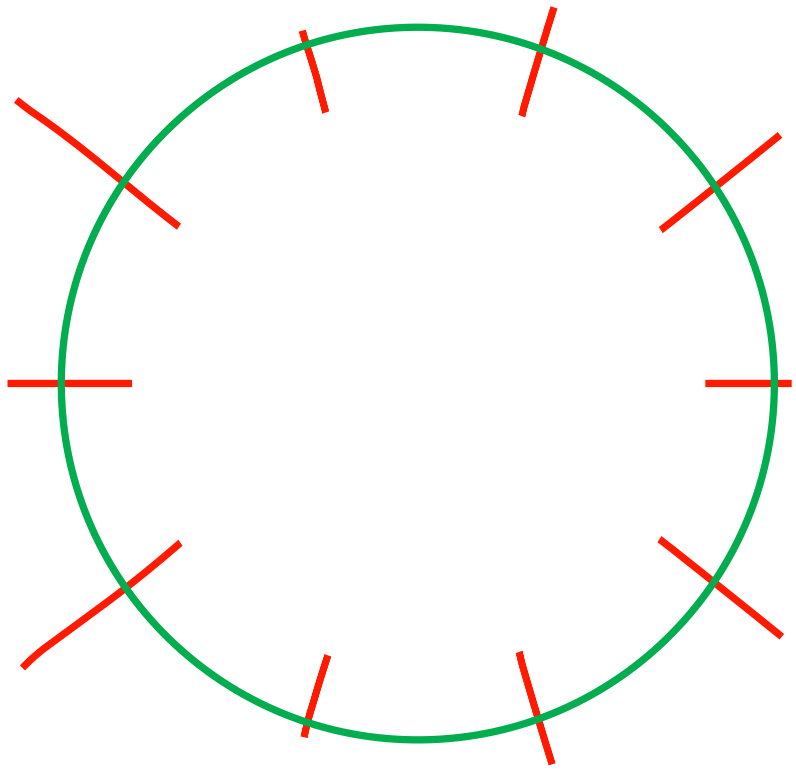


Figure 2: $\sqrt{x^2+y^2} \to \infty$.

Imaginary part

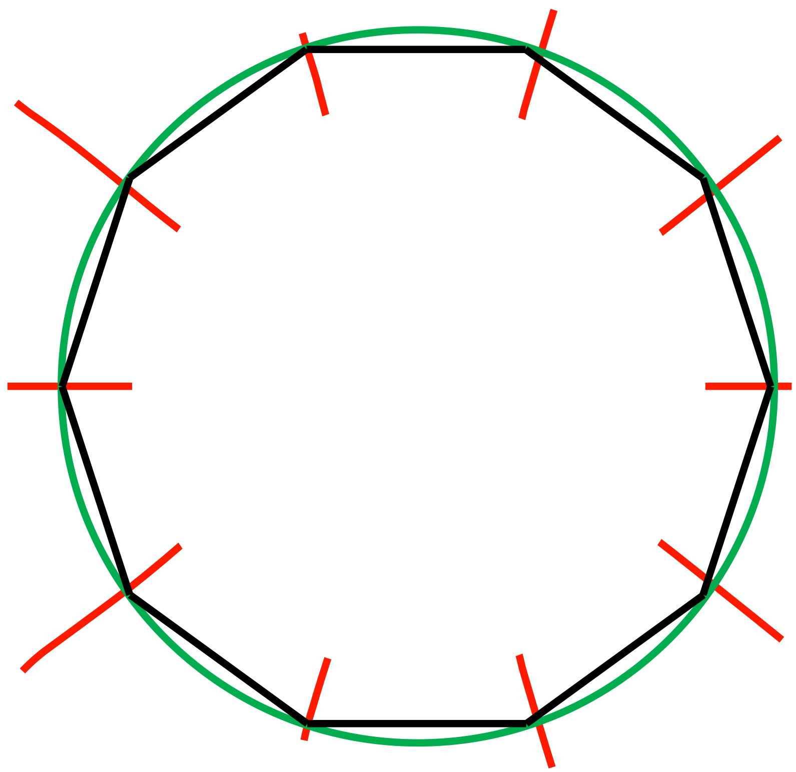


Figure 2: $e^{5\,\theta\,i} - 1 \approx 0$.

Imaginary part

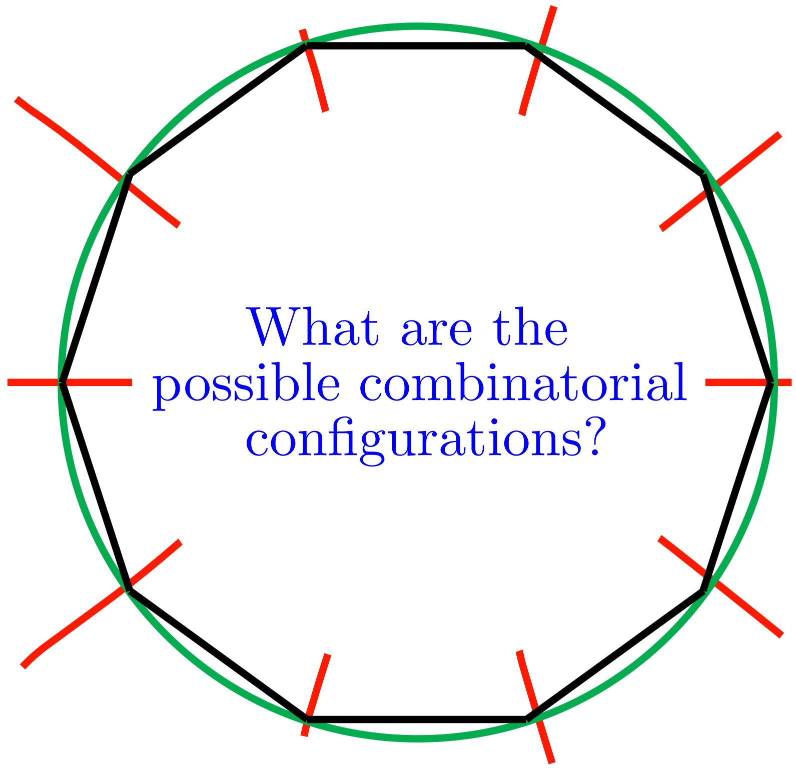


Figure 2: $e^{5\,\theta\, i} - 1 \approx 0$.

Basketball

Figure 3: A basketball.

Basketball

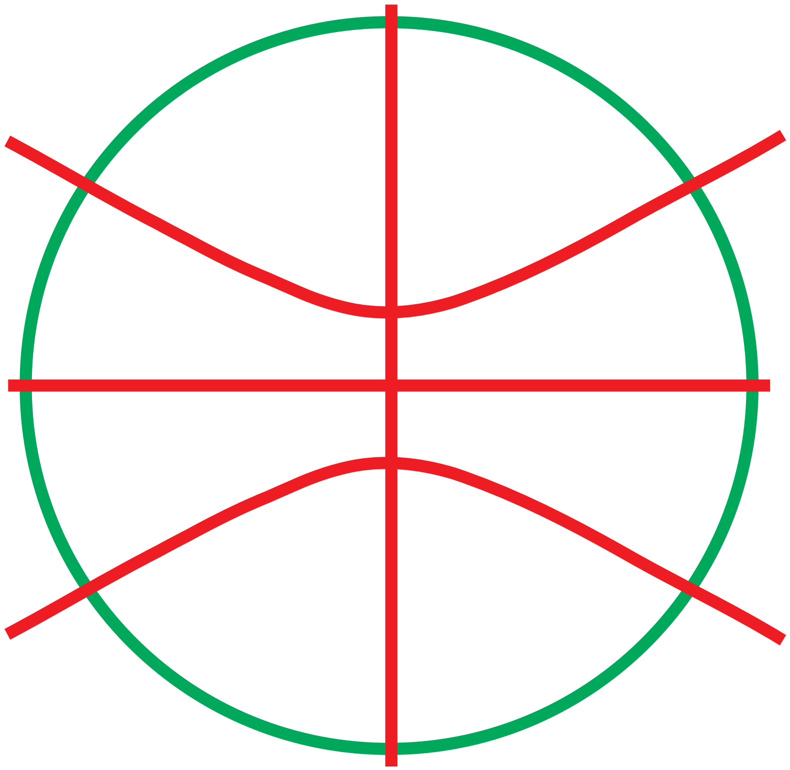


Figure 4: Imaginary part of $f(z) = z^3 + 6\,z^2 + 1$.

Non singular Basketballs

Jeremy Martin, David Savitt, and Ted Singer:

"Harmonic Algebraic Curves and Noncrossing Partitions"

To appear in: Discrete and Computational Geometry.
arXiv:math.CO/0511248

Simple reduction

We assume that f is of the form

$$f(z) = z^n + c_2 z^{n-2} + \ldots + c_n$$

In other words, f is monic with the average of its roots equal to 0.

Evident symmetries

Letting ρ_n be the rotation of the plane by an angle of π/n

$$\begin{aligned} R(f(e^{-\pi i/n}\, z) &= \rho_n(R(f(z))), \\ I(f(e^{-\pi i/2n}\, z) &= \rho_n(I(f(z))) \end{aligned}$$

also

$$R(\overline{f(\overline{z})}) = \tau(R(f(z)))$$

with τ being the reflection in the real axis.

Reformulation

$$\begin{aligned} R(f) &= \{z \mid f(z) = i\,t, \text{ with } t \in \mathbb{R}\}, \qquad \text{and} \\ I(f) &= \{z \mid f(z) = t, \text{ with } t \in \mathbb{R}\}. \end{aligned}$$

since

$$\begin{aligned} R(f) &= R(g), \qquad \text{iff} \qquad f(z) - g(z) \in i\,\mathbb{R} \\ I(f) &= I(g), \qquad \text{iff} \qquad f(z) - g(z) \in \mathbb{R} \end{aligned}$$

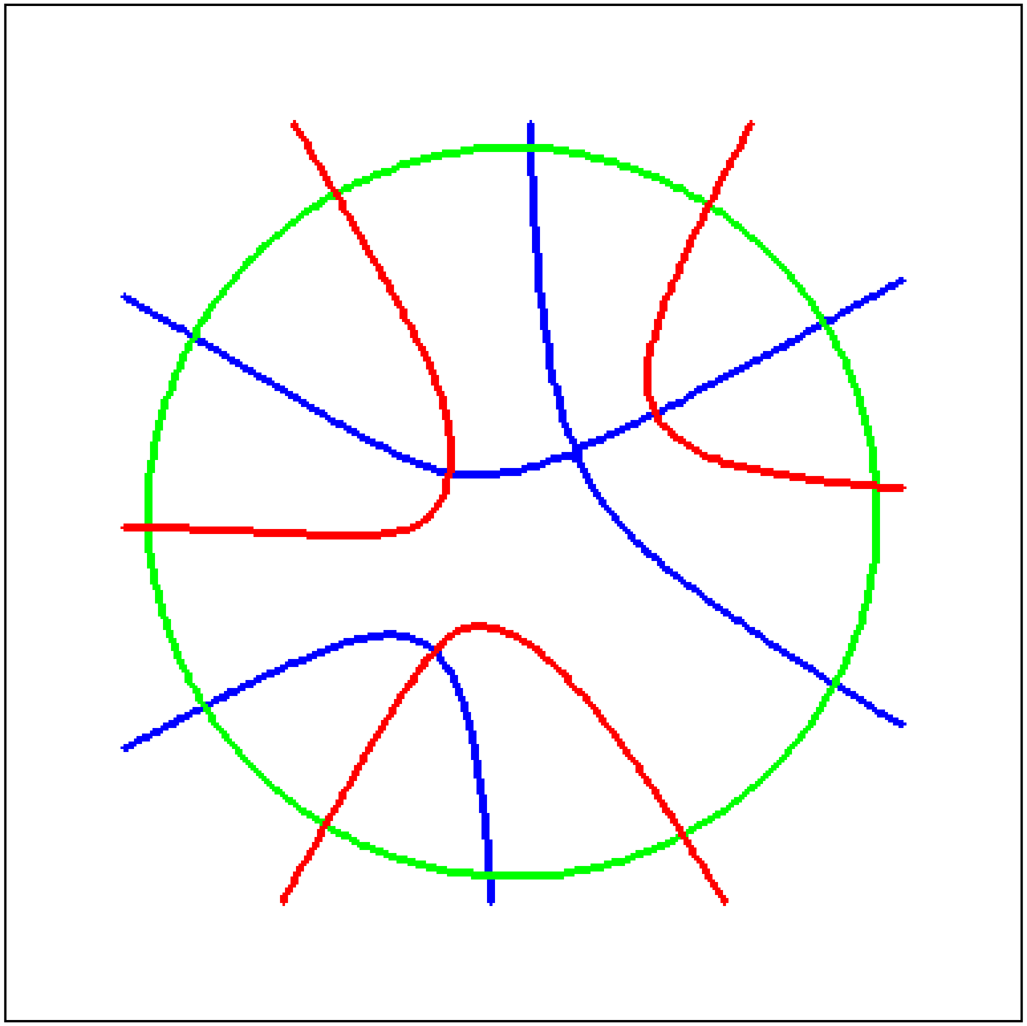
In other words, $R(f)$ decomposes into n branches $\rho_i : \mathbb{R} \longrightarrow \mathbb{C}$, such that

$$f(z) - i\,t = (z - \rho_1(t))(z - \rho_2(t)) \cdots (z - \rho_n(t)),$$

Let us call them the *real branches*. Likewise

$$f(z) - t = (z - \iota_1(t))(z - \iota_2(t)) \cdots (z - \iota_n(t)),$$

$f(z) - t\,i$, with t going from $-\infty$ to $+\infty$



Real/Imaginary Intersections

Observe that:

1) Each real branch intersects one “and only one” imaginary branch.

2) Multiple intersections occur only at singular points of the respective components, with agreeing multiplicities.

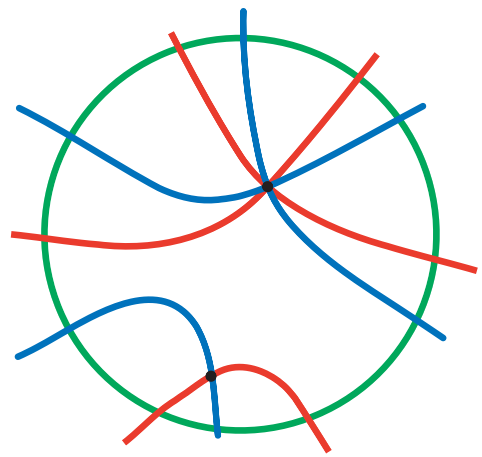


Figure 5: $f(z) = (z + 2q)(z - q)^2$, with $q = e^{i\theta}$.

One branch

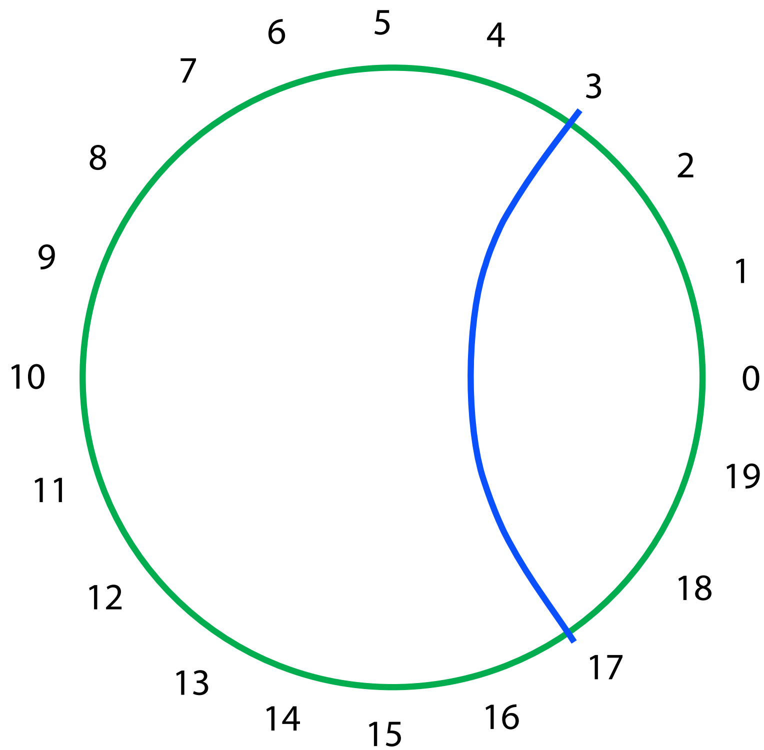


Figure 6: $\rho_2(t)$.

One branch

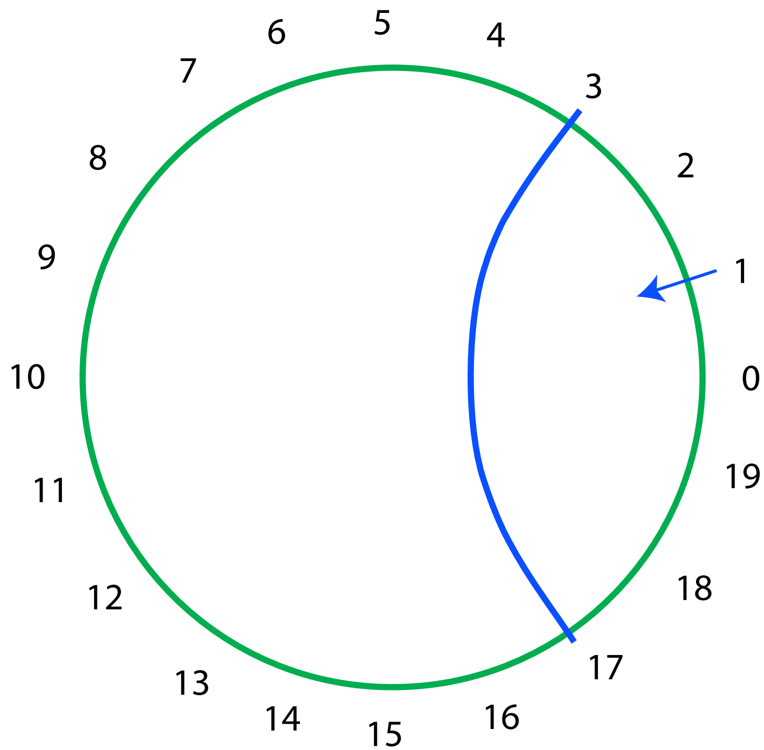


Figure 6: $\rho_2(t)$ and half of $\rho_1(t)$.

No singularity

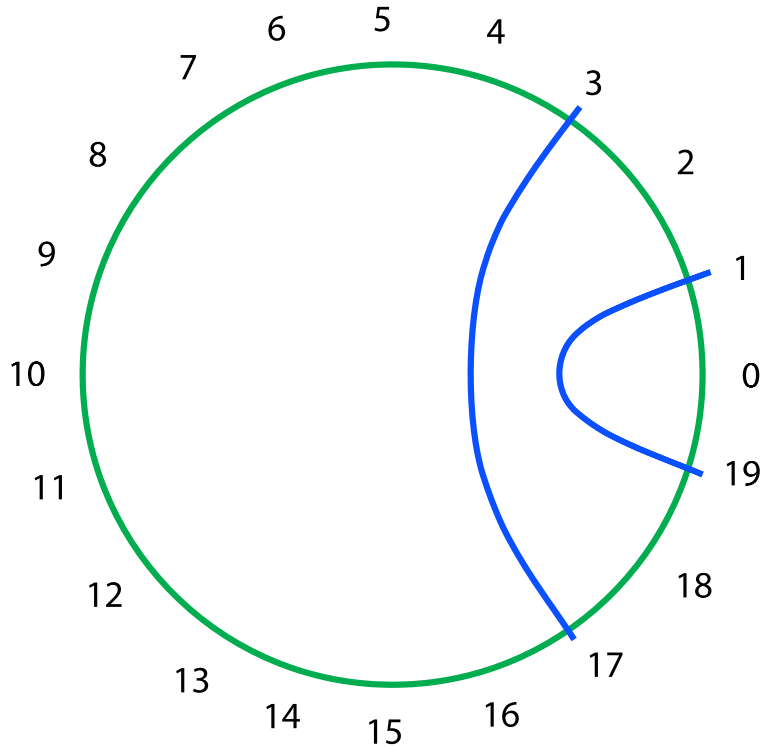


Figure 7: $\rho_1(t)$ and $\rho_2(t)$.

With singularity

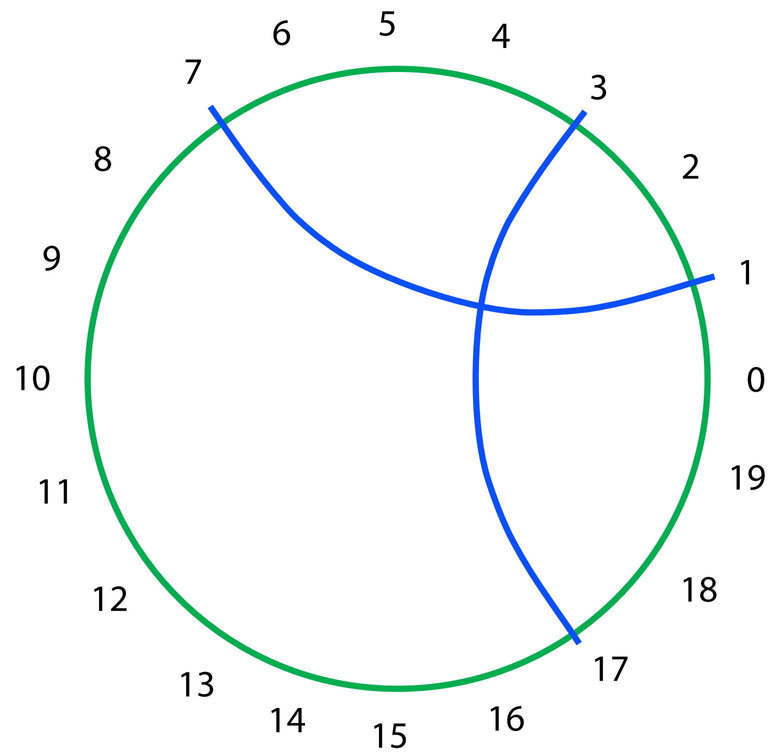


Figure 8: $\rho_1(t)$ crosses $\rho_2(t)$.

The case $n = 2$

The possible basketballs for $f(z) = z^2 - (a + b\,i)$, are readily classified as follows:

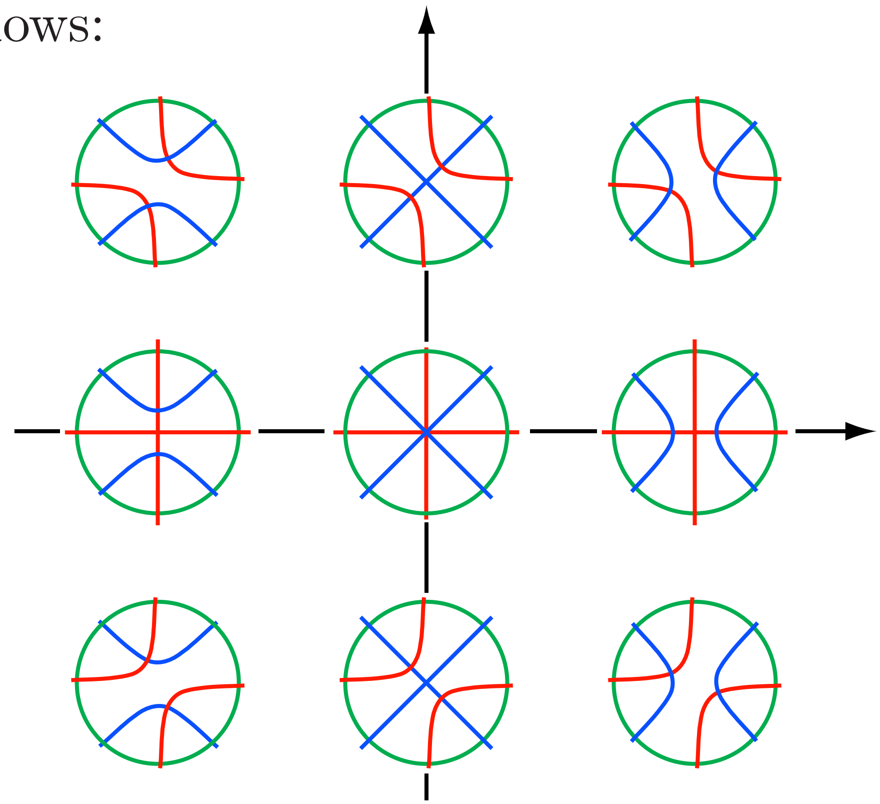


Figure 9: $f(z) = z^2 - (a + b\,i)$

The generic (non singular) case

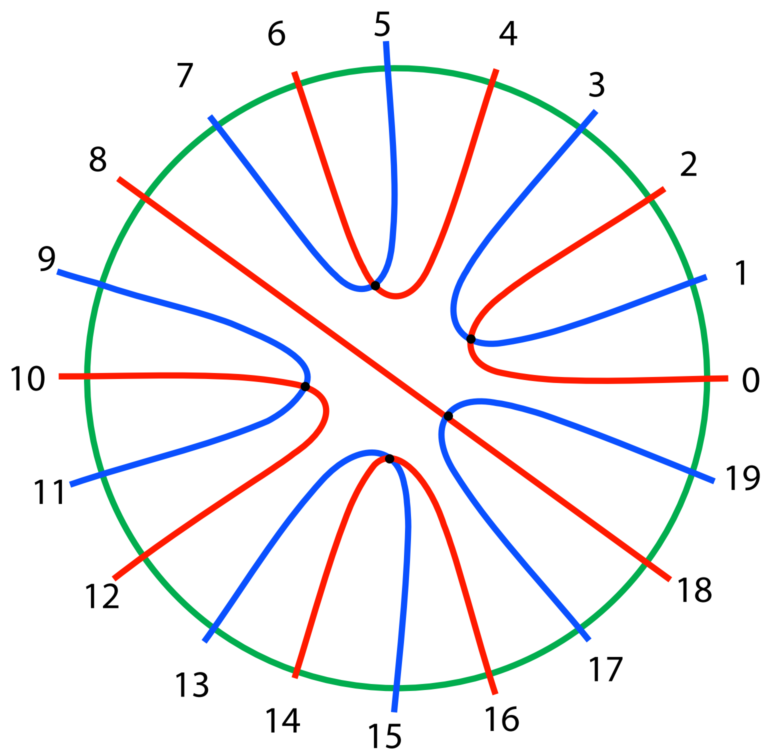
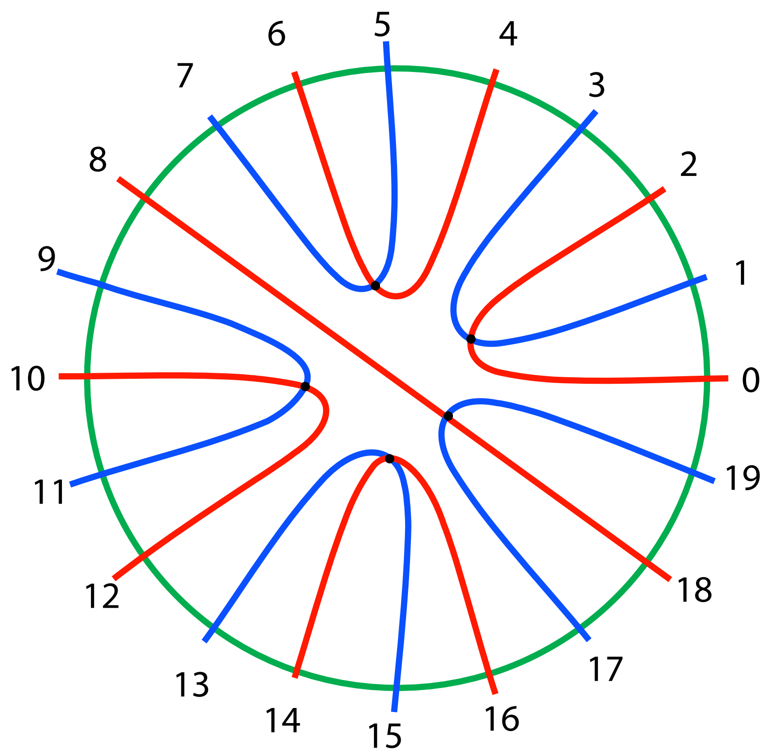


Figure 10: $f(z) = z^5 - e^{\pi\, i/5} z + 1$

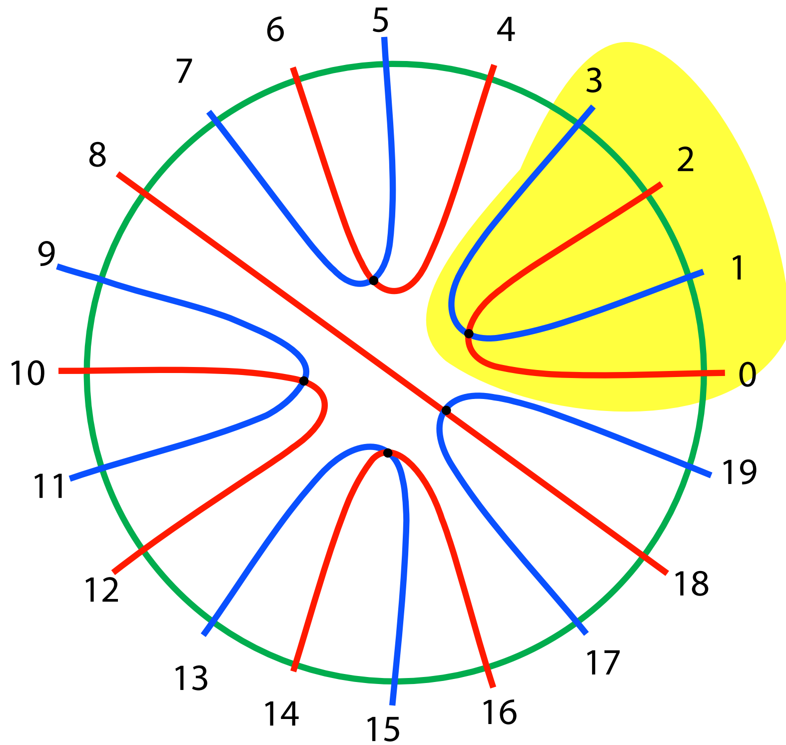
The generic (non singular) case



Noncrossing Bimatching:

$$\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix} \quad \begin{pmatrix} 5 & 4 \\ 7 & 6 \end{pmatrix} \quad \begin{pmatrix} 17 & 8 \\ 19 & 18 \end{pmatrix} \quad \begin{pmatrix} 9 & 10 \\ 11 & 12 \end{pmatrix} \quad \begin{pmatrix} 13 & 15 \\ 14 & 16 \end{pmatrix}$$

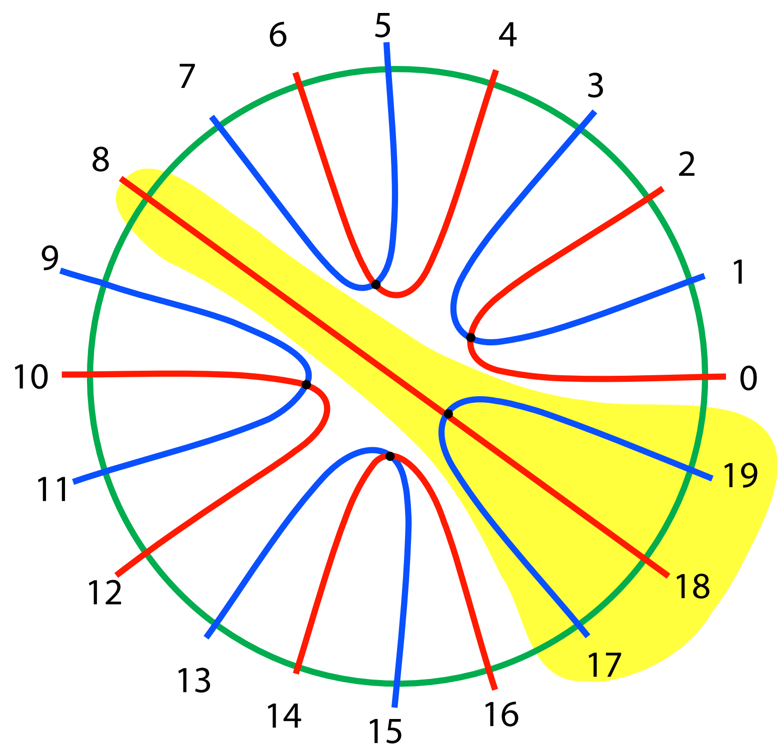
The generic (non singular) case



Noncrossing Bimatching:

$$\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix} \quad \begin{pmatrix} 5 & 4 \\ 7 & 6 \end{pmatrix} \quad \begin{pmatrix} 17 & 8 \\ 19 & 18 \end{pmatrix} \quad \begin{pmatrix} 9 & 10 \\ 11 & 12 \end{pmatrix} \quad \begin{pmatrix} 13 & 15 \\ 14 & 16 \end{pmatrix}$$

The generic (non singular) case



Noncrossing Bimatching:

$$\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix} \quad \begin{pmatrix} 5 & 4 \\ 7 & 6 \end{pmatrix} \quad \begin{pmatrix} 17 & 8 \\ 19 & 18 \end{pmatrix} \quad \begin{pmatrix} 9 & 10 \\ 11 & 12 \end{pmatrix} \quad \begin{pmatrix} 13 & 15 \\ 14 & 16 \end{pmatrix}$$

A Result of Martin, Savitt and Singer

Defining *non singular n-basketballs* as pairs of noncrossing matchings, such that each edge of the first crosses one and only one edge from the second, then

Theorem (MSS-2005). *The non singular n-basketball configurations number*

$$\frac{1}{3n+1}\binom{4n}{n},$$

and each such configuration is realizable as a pair $(R(f), I(f))$, for some polynomial $f(z)$.

Singular Basketballs

Possible shapes for the real and imaginary parts

Observe that

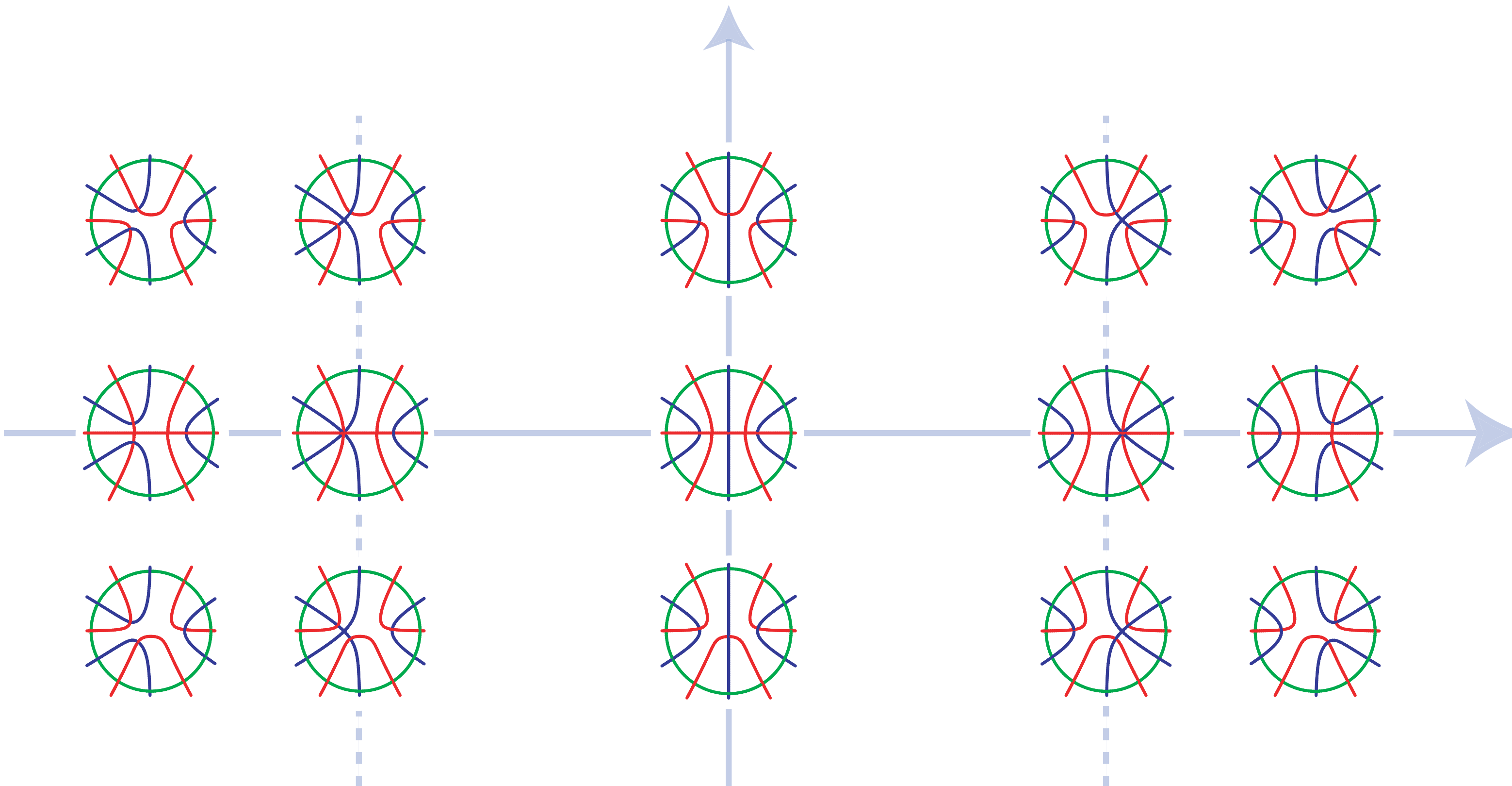
$$\begin{aligned} R(i\,f(e^{-\pi i/2n}\,z) &= \rho_{2n}(I(f)), \\ I(i\,f(e^{-\pi i/2n}\,z) &= \rho_{2n}(R(f)) \end{aligned}$$

In other words, we may exchange the role of the real and imaginary parts by a rotation of $\pi/2\,n$.

Non singular shapes for the real (or imaginary) part are *noncrossing matchings*. Recall that they number

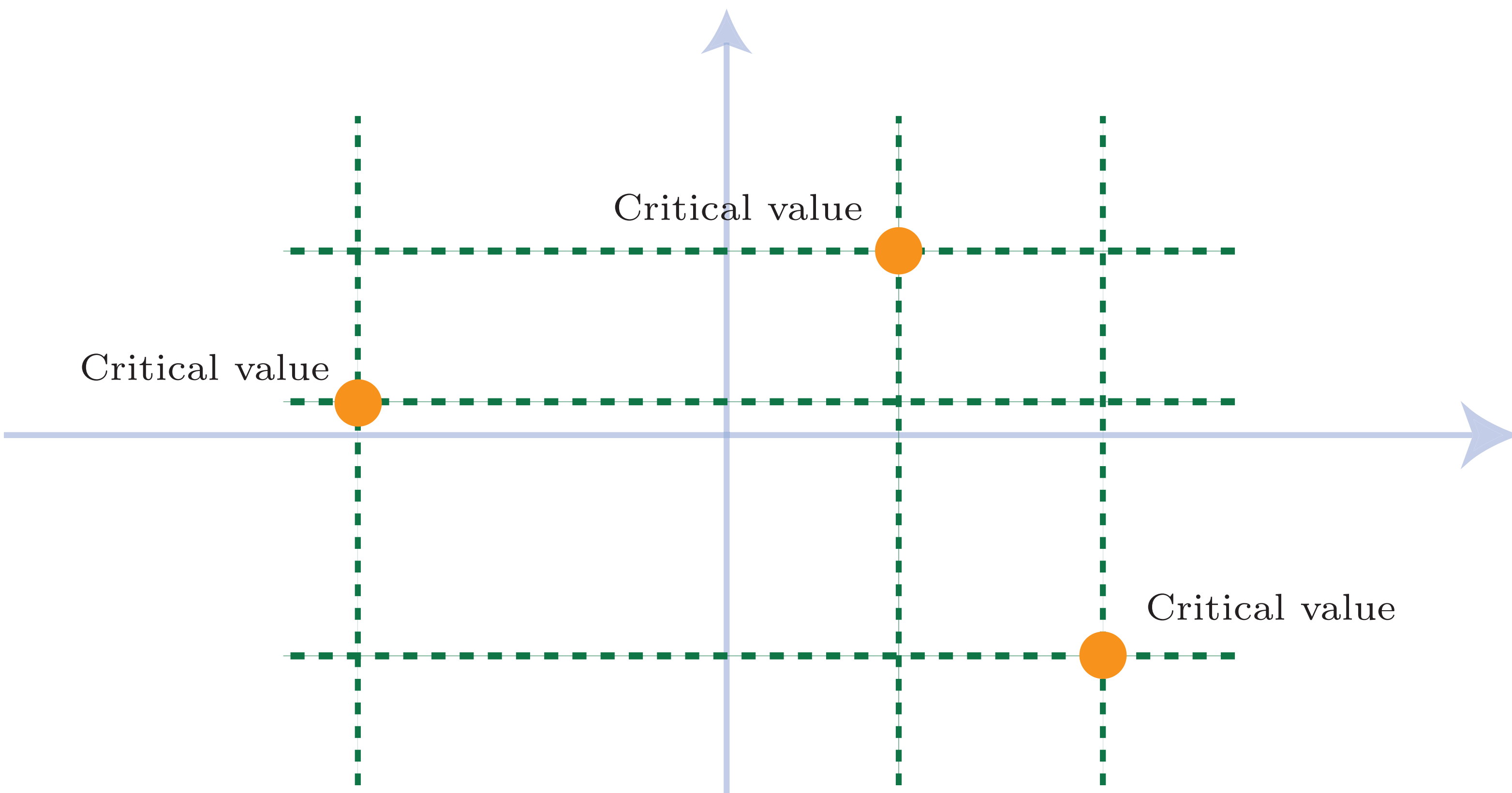
$$\frac{1}{2\,n+1}\binom{2\,n}{n}$$

$$f(z) = z^3 - 3\,z$$



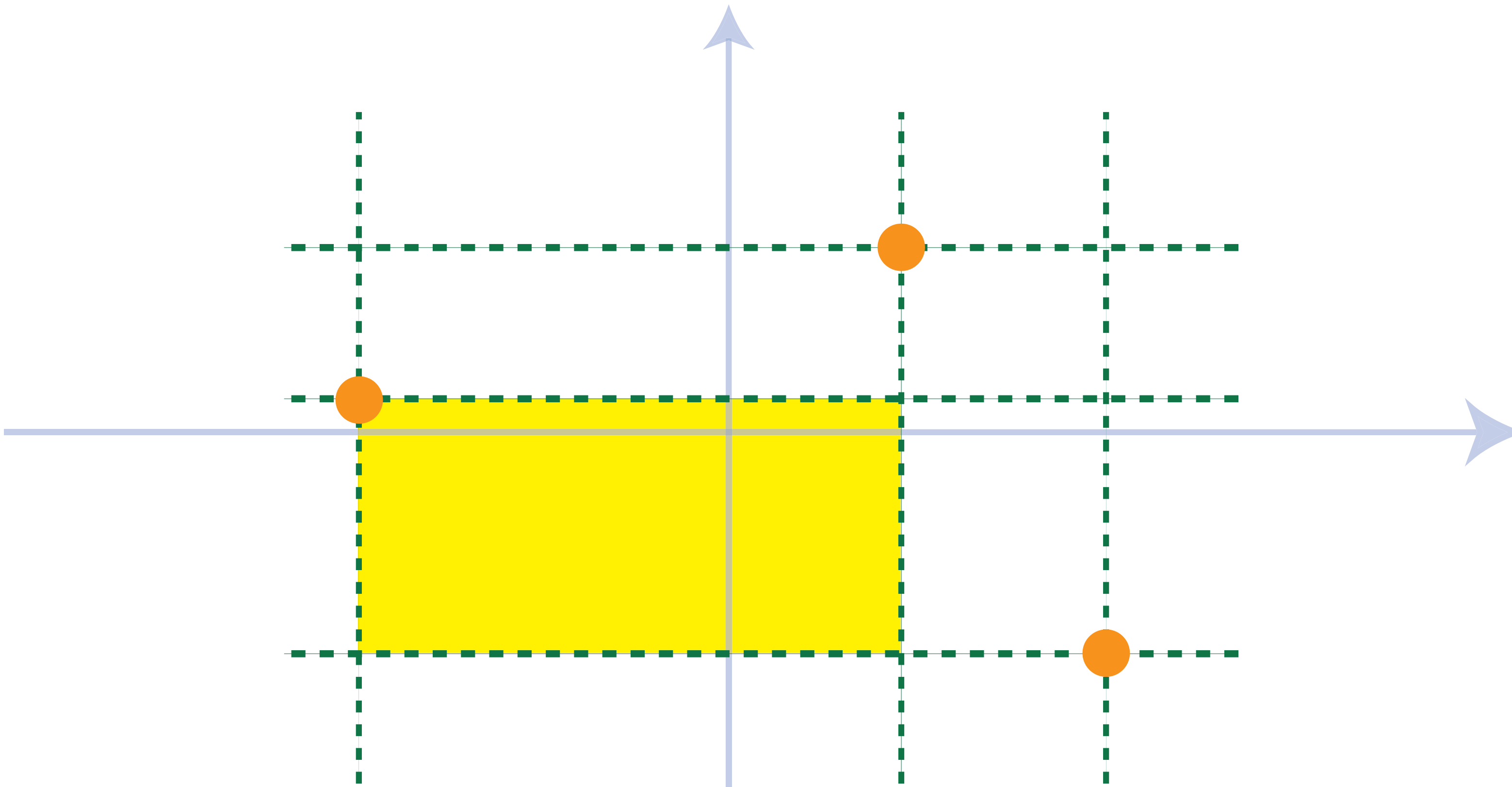
Critical values

If $f'(w) = 0$, it is said that $f(w)$ is a *critical value* for $f(z)$. It this case, $f(z) - f(w)$ has a multiple root.



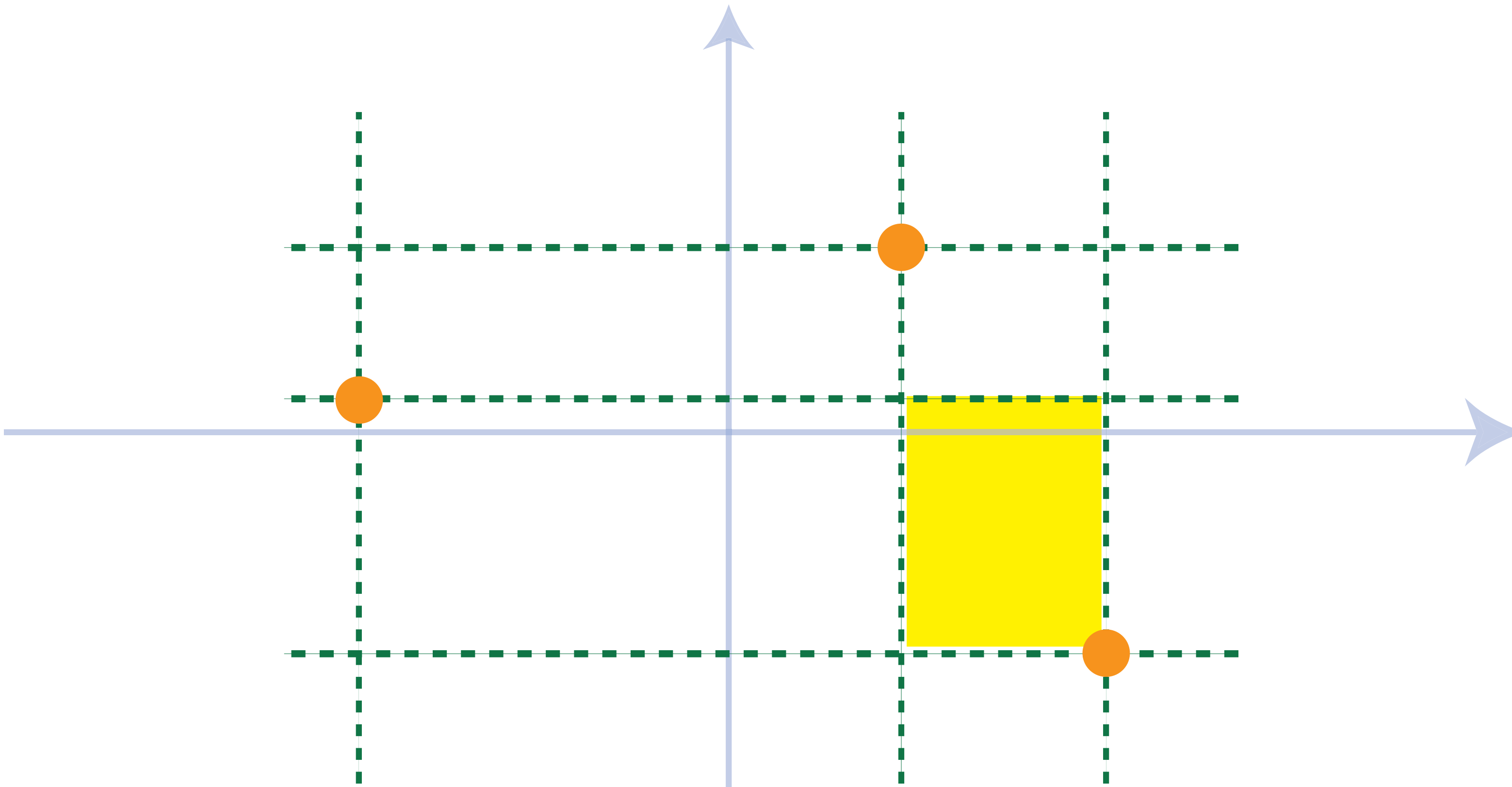
Critical values

If $f'(w) = 0$, it is said that $f(w)$ is a *critical value* for $f(z)$. It this case, $f(z) - f(w)$ has a multiple root.



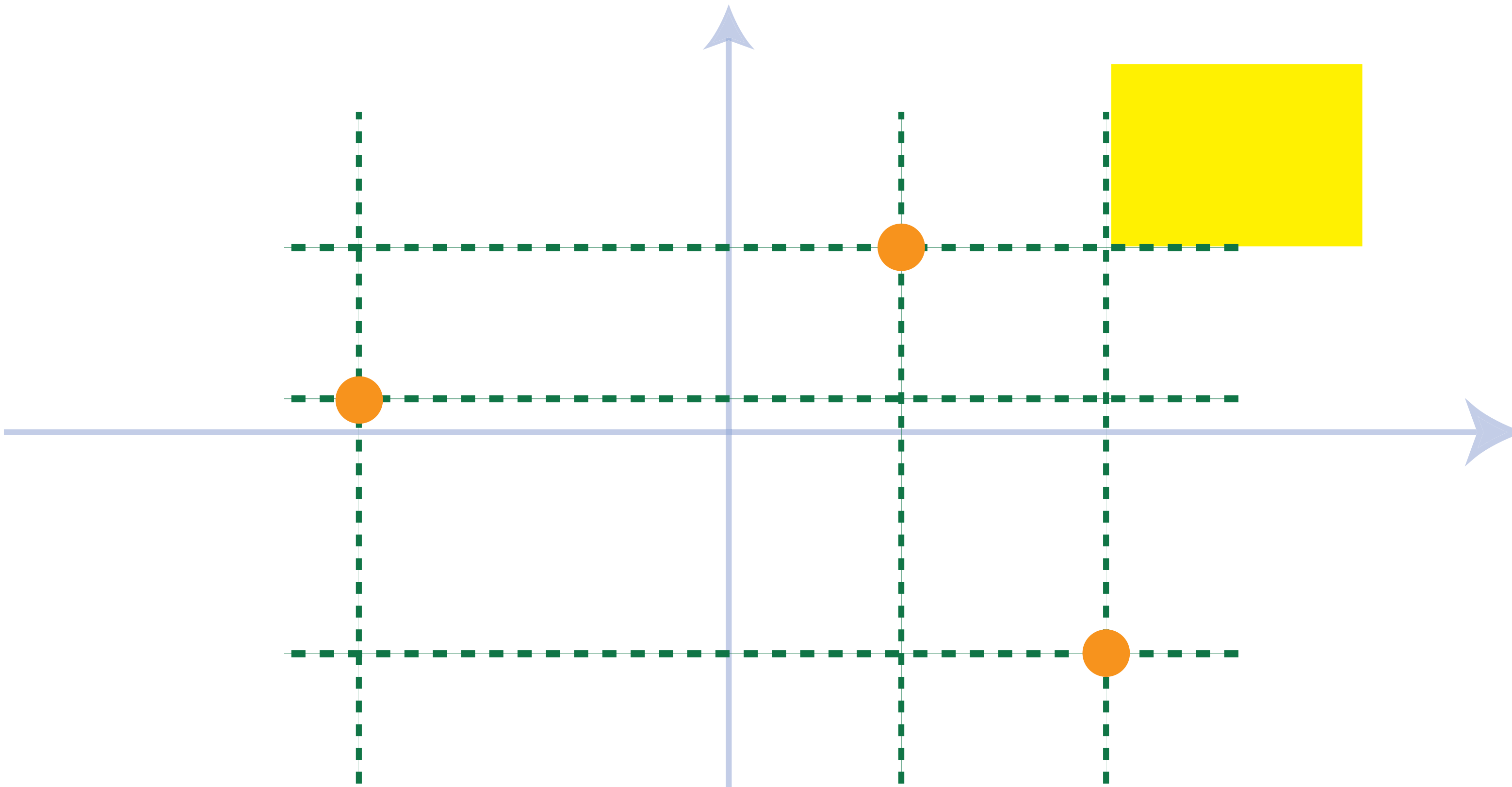
Critical values

If $f'(w) = 0$, it is said that $f(w)$ is a *critical value* for $f(z)$. It this case, $f(z) - f(w)$ has a multiple root.



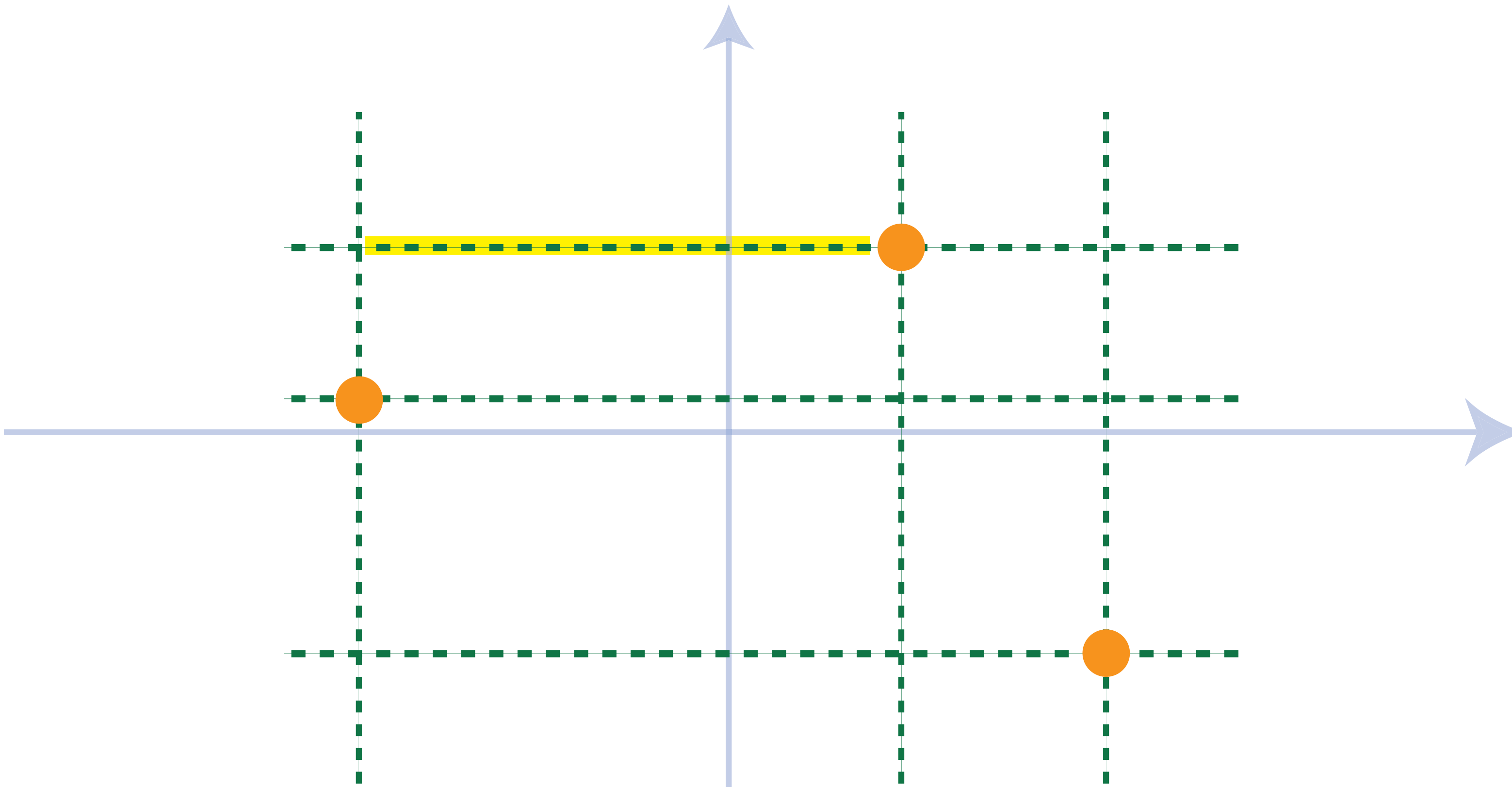
Critical values

If $f'(w) = 0$, it is said that $f(w)$ is a *critical value* for $f(z)$. It this case, $f(z) - f(w)$ has a multiple root.



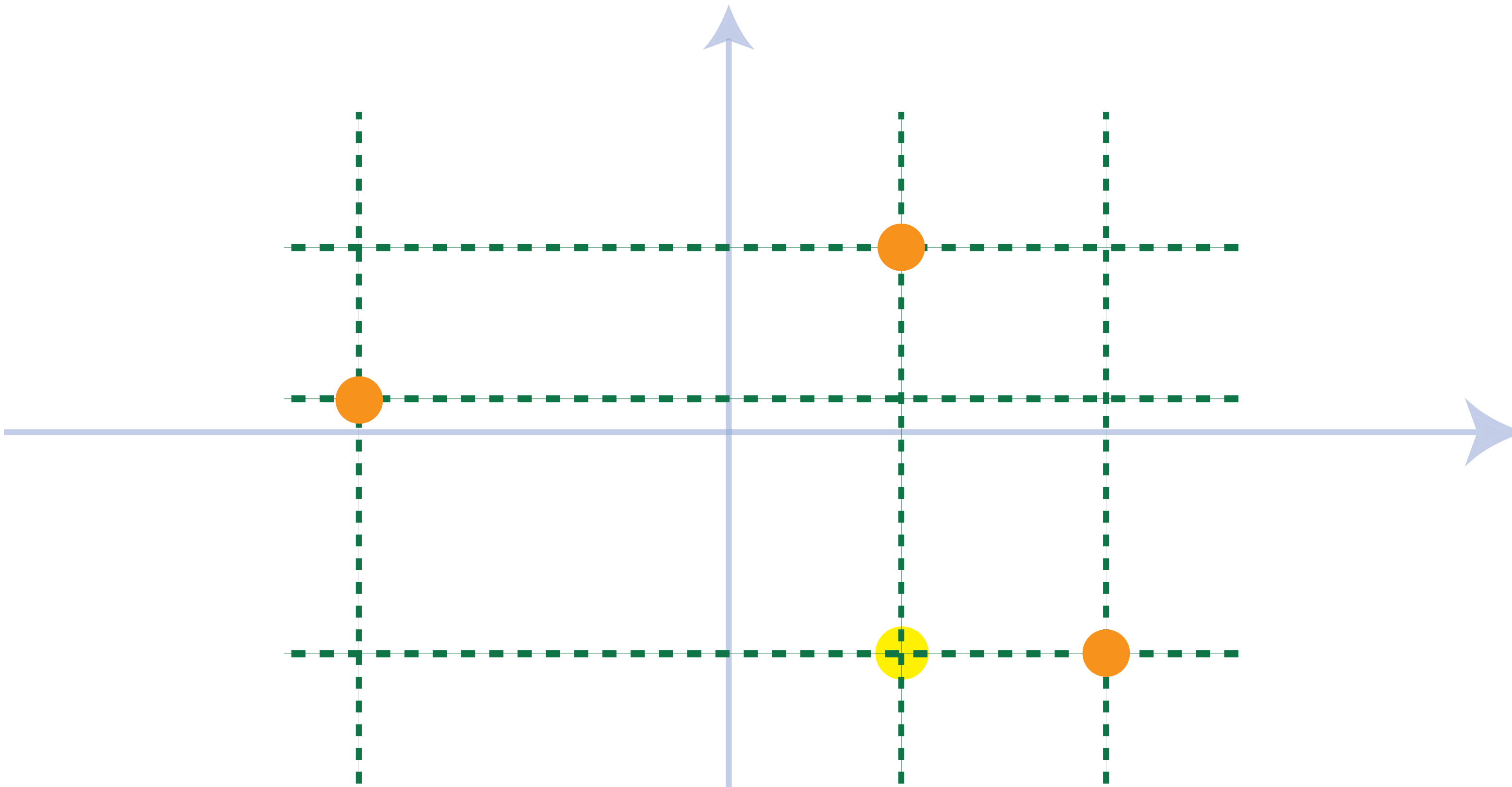
Critical values

If $f'(w) = 0$, it is said that $f(w)$ is a *critical value* for $f(z)$. It this case, $f(z) - f(w)$ has a multiple root.



Critical values

If $f'(w) = 0$, it is said that $f(w)$ is a *critical value* for $f(z)$. It this case, $f(z) - f(w)$ has a multiple root.



Critical values

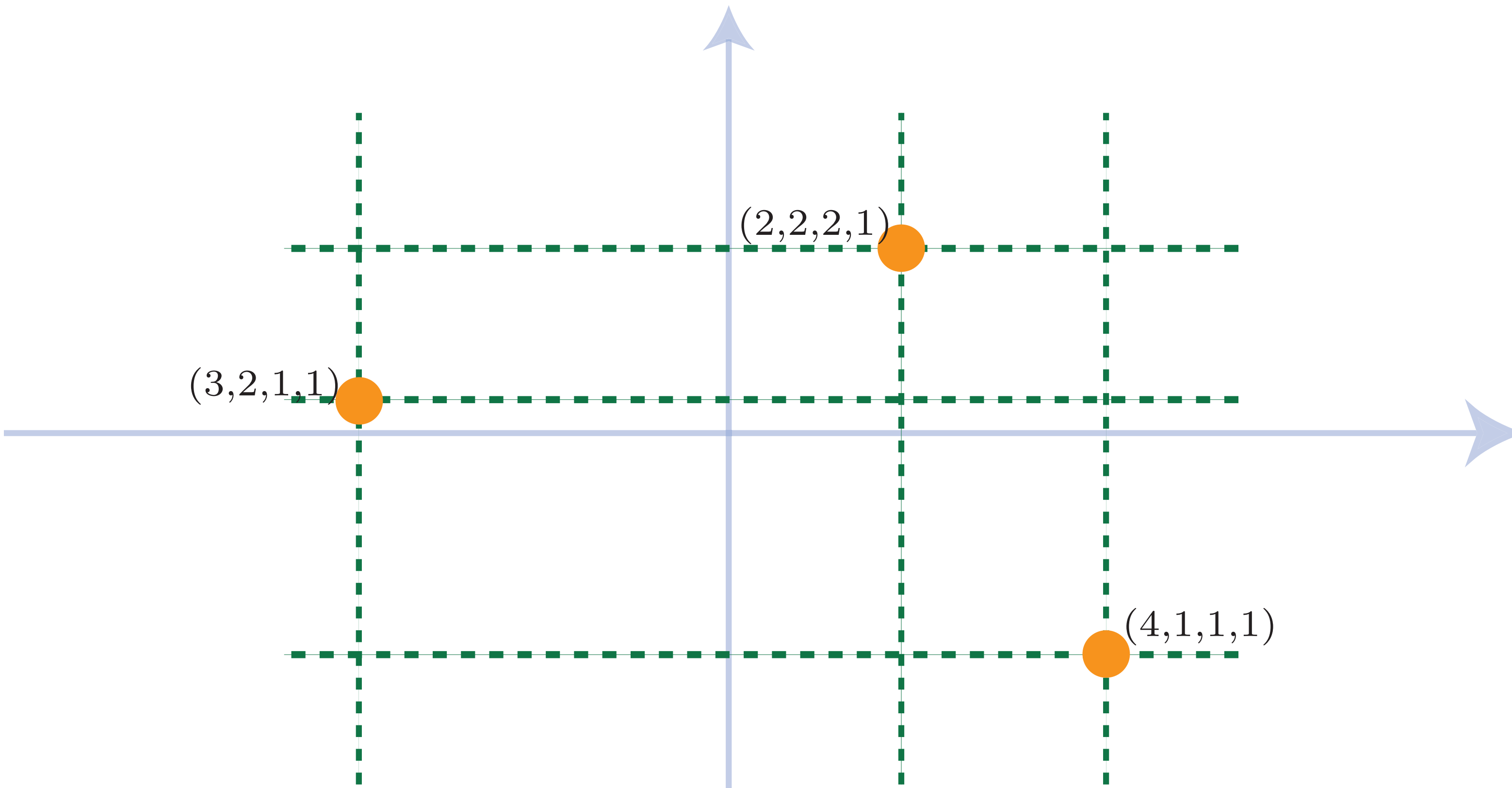
The *passeport* of a polynomial is the sequence of partitions of n giving the multiplicities of the roots of $f(z) - f(w)$; one partition for each critical value. This notion appears in the study of *Hurwitz problem*.

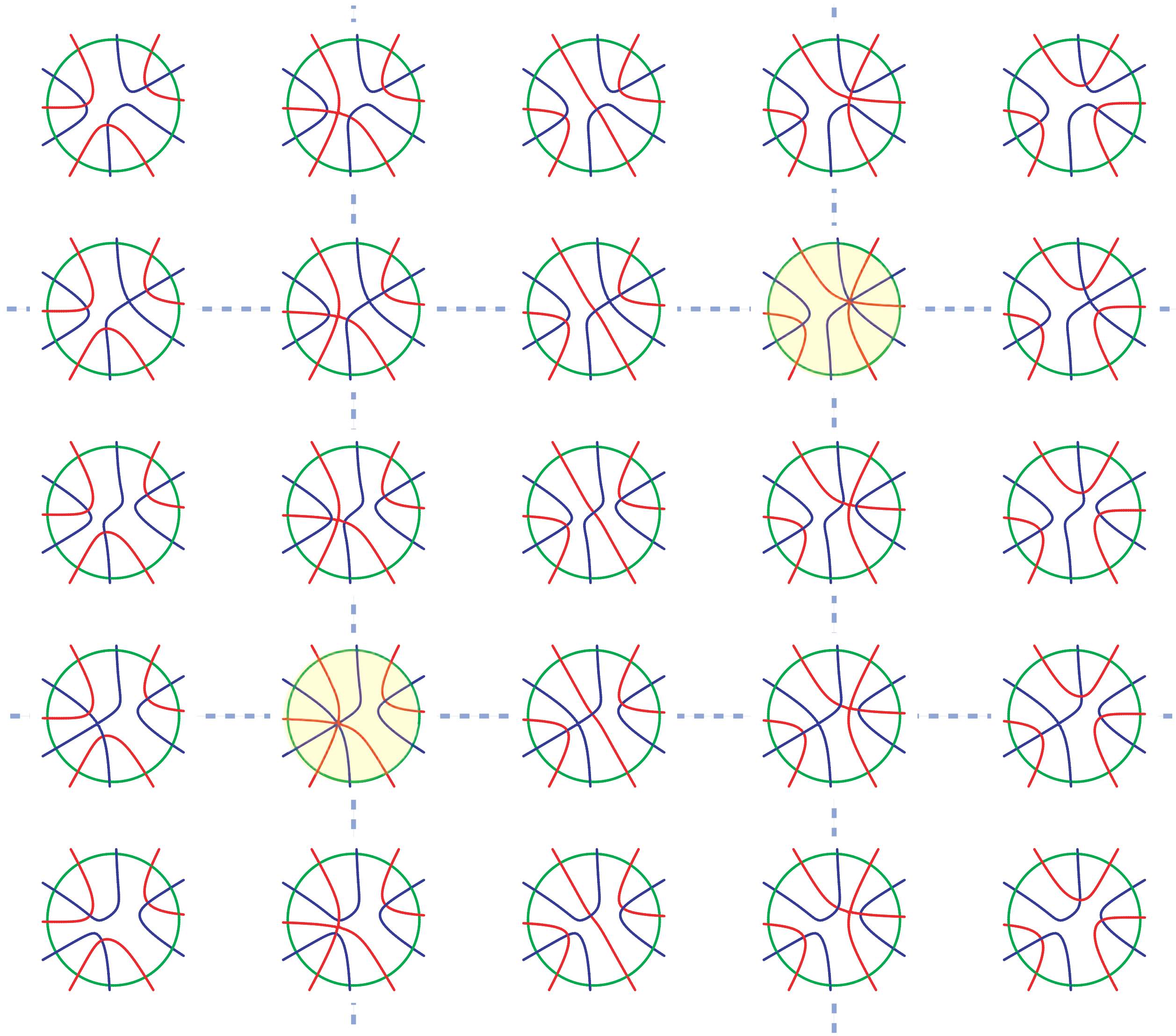
This has given rise to many recent (and less recent) studies. For a very nice description of many aspects of these questions, see

Graphs on Surfaces and Their Applications,
S.K. Lando and A.K. Zvonkin

Part of the originality of the basketball approach lies in the study of the disposition of critical values, as well as giving a combinatorial description of how they relate.

Critical values





Rigid classification of generic polynomials

On the space of degree n complex polynomials consider the *LL-mapping* (Lyaschko-Looinjenga):

$$LL : f(z) \mapsto \mathrm{Disc}_z(f(z) - t)$$

normalized to be monic in t.

For example,

$$\begin{aligned}\mathrm{Disc}_z(z^4 + a_2 z^2 + a_3 z + a_4 - t) \quad = \quad & t^3 + \frac{1}{2}(a_2^2 - 6\,a_4)t^2 \\ & + \frac{1}{16}(a_2^4 + 9\,a_2 a_3^2 - 16\,a_2^2 a_4 + 48\,a_4^2)t \\ & - \frac{1}{256}(16\,a_2^4 a_4 - 4\,a_2^3 a_3^2 - 128\,a_2^2 a_4^2 \\ & \quad +144\,a_2 a_4 a_3^2 - 27\,a_3^4 + 256\,a_4^3)\end{aligned}$$

An Enumeration Result

Theorem (Lyashko–Looijenga, 1974). *The LL-mapping is of degree (number of preimages of a generic point)*

$$n^{n-2}.$$

In other words, taking $n = 4$, the system of equations

$$\begin{aligned}
e_1 &= a_2^2 - 6\,a_4 \\
e_2 &= a_2^4 + 9\,a_2 a_3^2 16\,a_2^2 a_4 + 48\,a_4^2 \\
e_3 &= 16\,a_2^4 a_4 - 4\,a_2^3 a_3^2 - 128\,a_2^2 a_4^2 \\
&\qquad +144\,a_2 a_4 a_3^2 - 27\,a_3^4 + 256\,a_4^3
\end{aligned}$$

has 16 solutions (a_2, a_3, a_4), for generic values of e_1, e_2 and e_3.

Other Results in that direction

(1) A proof (not the simplest) of Cayley formula for the number of labelled trees (Looijenga,1974).

(2) A theorem of Lando–Zvonkine (1999) gives the degree of the LL-mapping restricted to the stratum consisting of polynomials with a given passeport.

(3) The enumerative content of (2) is equivalent to the Goulden-Jackson (1992) enumerative formula for “cacti”.

(4) Formulas for the number of ramified coverings of the sphere by the torus. Formulas for Hurwitz numbers and Hodge integrals, etc.

Singularity locus

$R(f)$ is singular if and only if

$$\mathrm{Disc}_z(f(z) - i\,t) = 0$$

for some real value t. Similarly, $I(f(z))$ is singular if and only if

$$\mathrm{Disc}_z(f(z) - t) = 0,$$

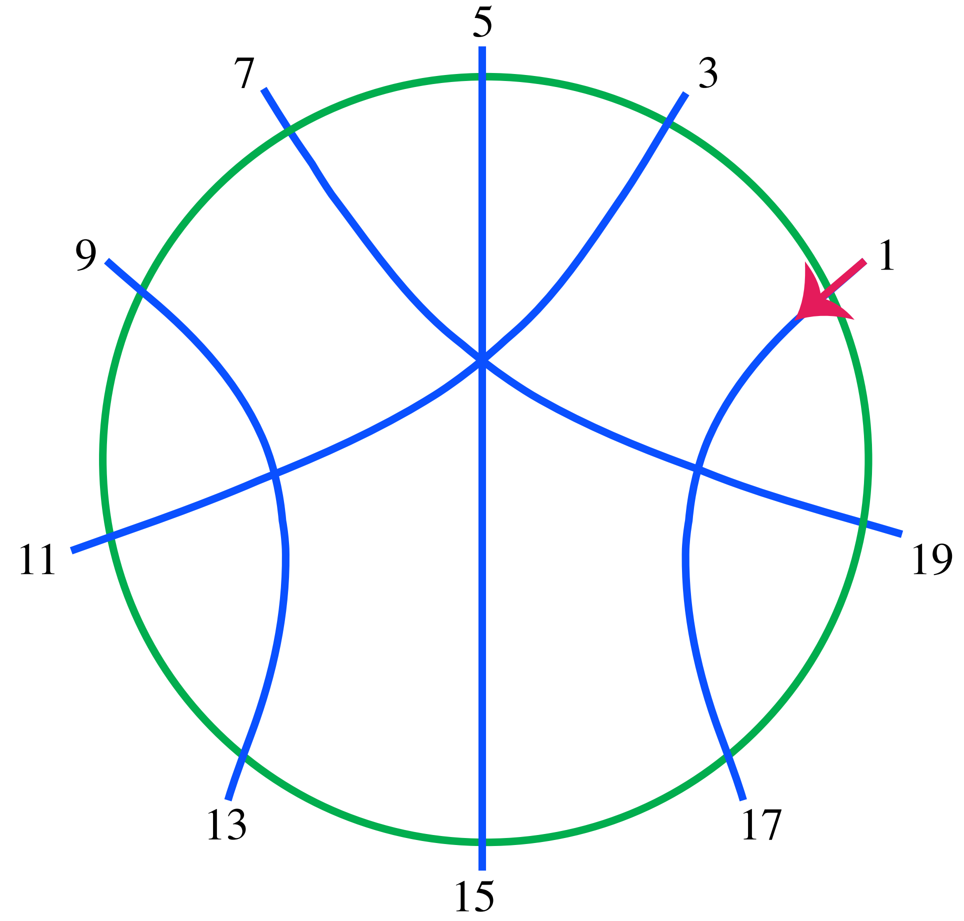
for some $t \in \mathbb{R}$.

We want to classify the possible "shapes" for these singular situations. We will then construct basketballs by combining "compatible" shapes. Observe that

$$I(f^2) = R(f) \cup I(f),$$

so that there is a link between properties of the shapes, and these compatibility conditions.

Even internal degree plane trees



Enumeration, including degree sequence indicator

Let

$$x\,\mathcal{A}(x,\mathbf{d}) := \sum_{n,\lambda} a_{n,\lambda}\mathbf{d}_\lambda x^{2n},$$

where $a_{n,\lambda}$ is the number of plane trees with $2\,n$ leaves on a circumscribed circle ($\mathcal{A}$-trees), and internal vertex of degree sequence $\lambda = (2\lambda_1,\ldots,2\lambda_k)$. Then we have the functionnal equation

$$\mathcal{A} = x + \sum_{k\geq 2} d_{2k}\mathcal{A}^{2\,k-1}$$

Or simply

$$\mathcal{A} = x + \frac{\mathcal{A}^3}{1-\mathcal{A}^2}$$

if we set all $d_k = 1$.

Enumeration, including degree sequence indicator

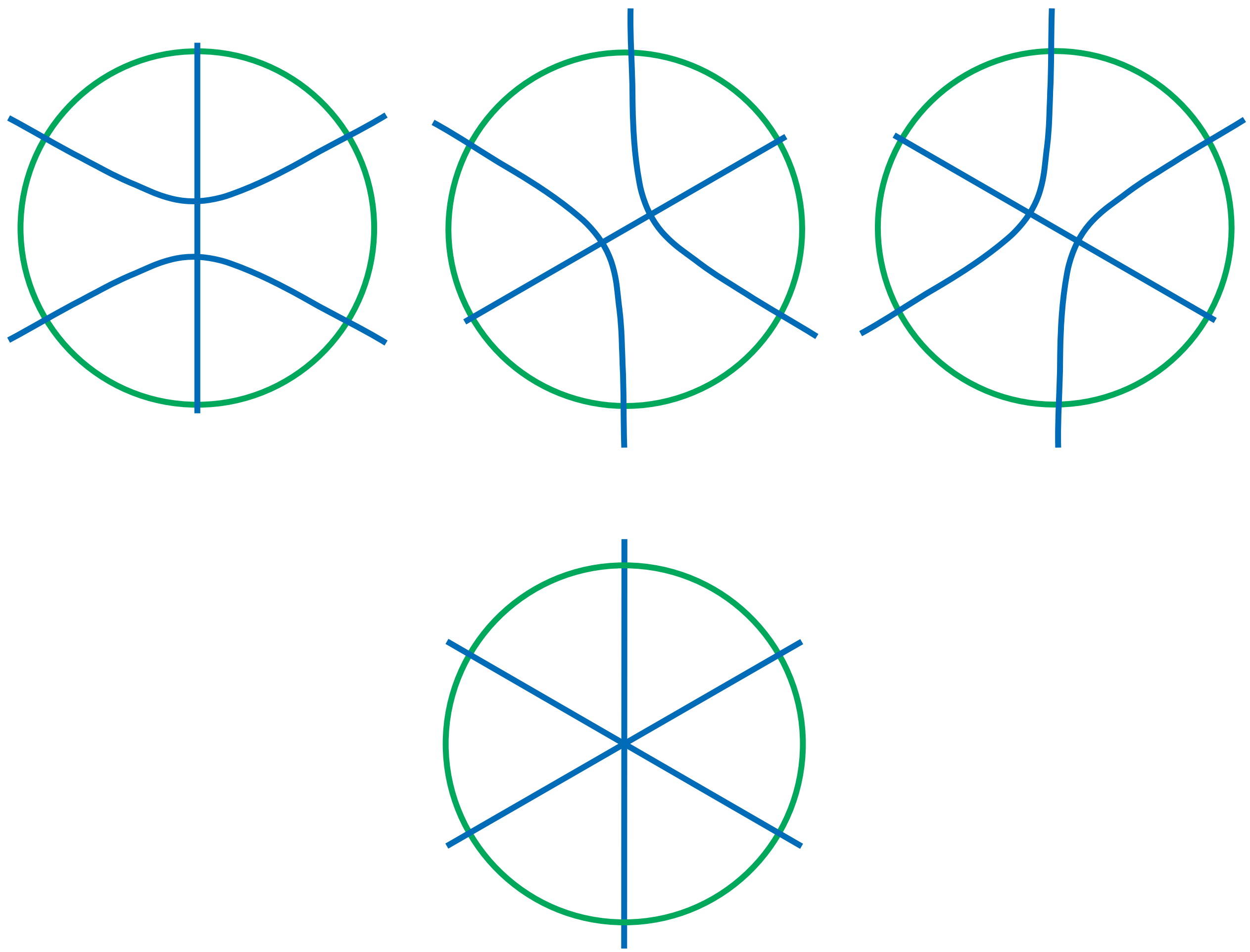
We get

$$x\,\mathcal{A}(x,\mathbf{z}) \;=\; x^2 + d_4x^4 + (3\,d_4^2 + d_6)x^6 + (12\,d_4^3 + 8\,d_6d_4 + d_8)x^8 + (55\,d_4^4 + 55\,d_6\,d_4^2 + 10\,d_8d_4 + 5\,d_6^2 + d_{10})x^{10} + \ldots$$

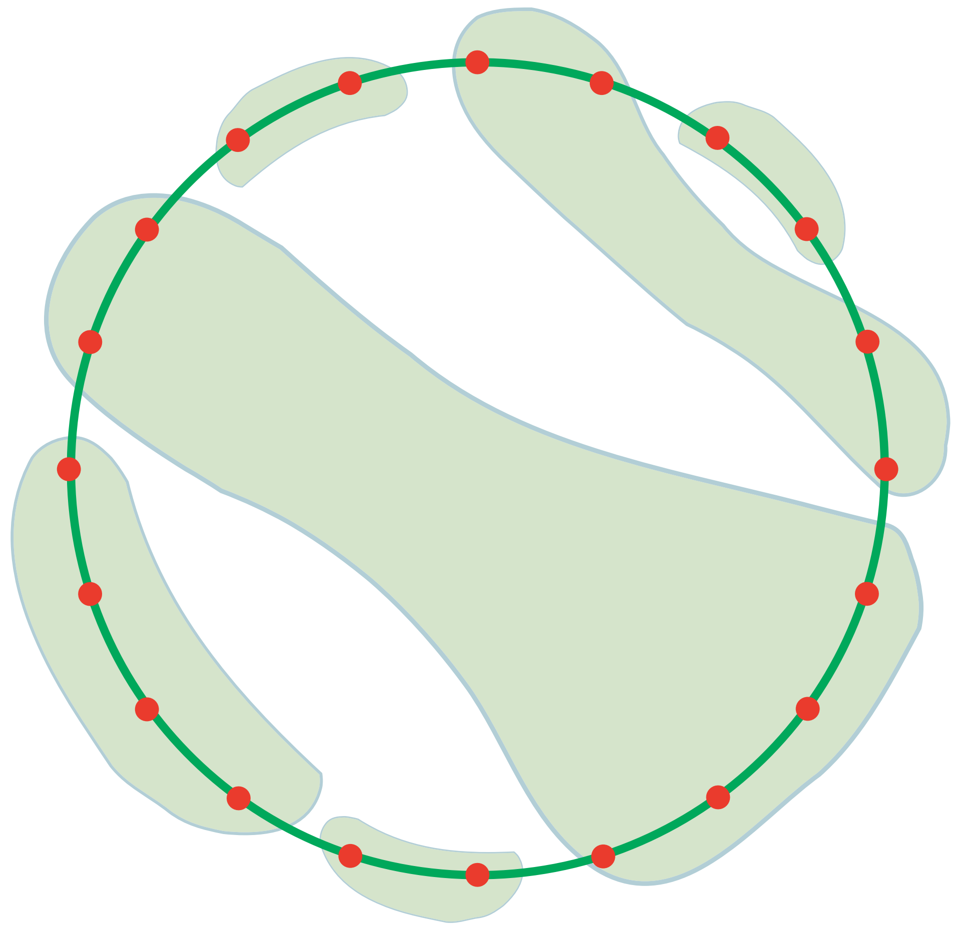
And specializing the d_k parameters to t:

$$x\,\mathcal{A}(x;t) \;=\; x^2 + t\,x^4 + (3\,t^2 + t)x^6 + (12\,t^3 + 8\,t^2 + t)x^8 + (55\,t^4 + 55\,t^3 + 15\,t^2 + t)x^{10} + (273\,t^5 + 364\,t^4 + 156\,t^3 + 24\,t^2 + t)x^{12} + \ldots$$

$$\ldots + (3\,d_4^2 + d_6)x^6 + \ldots$$



Noncrossing partitions with even part size



Noncrossing partitions, with even part sizes

They number:

$$1,\ 1,\ 3,\ 12,\ 48,\ \ldots$$

and taking into account the size of the parts

$$1,\ \pi_2,\ (2\,\pi_2^2+\pi_4),\ (5\,\pi_2^3+6\,\pi_2\pi_4+\pi_6),\\ (14\,\pi_2^4+24\,\pi_2^2\pi_4+3\,\pi_4^2+5\,\pi_2\pi_6+\pi_8),\ \ldots$$

The forests of $\mathcal{A}$-trees are obtained by choosing a noncrossing partition, with even part sizes, and then choosing an $\mathcal{A}$-tree of corresponding size for each part.

Some results

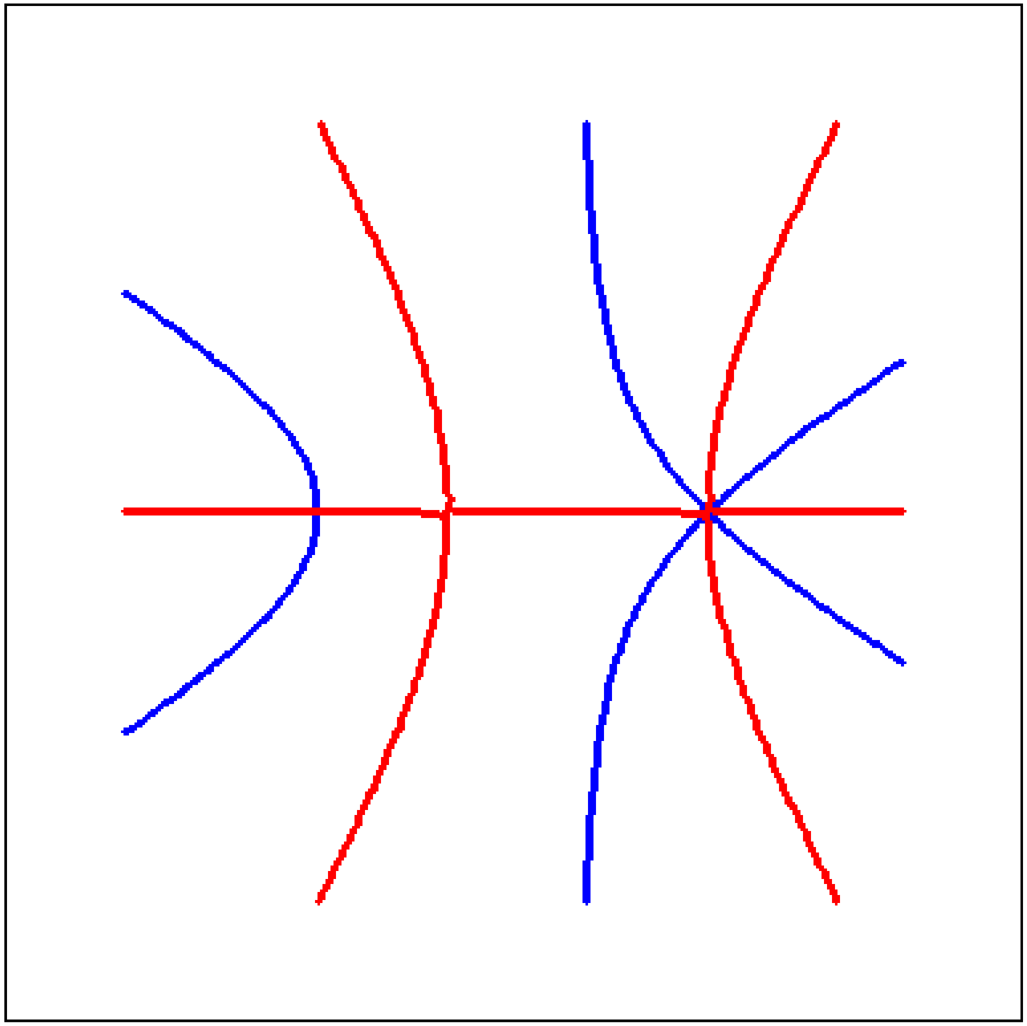
Proposition (1). *The shape of $R(f)$ is a tree, if and only if all critical values of $f(z)$ share the same real part.*

Proposition (2). *In general, the shape of $R(f)$ is a forest of noncrossing $\mathcal{A}$-trees.*

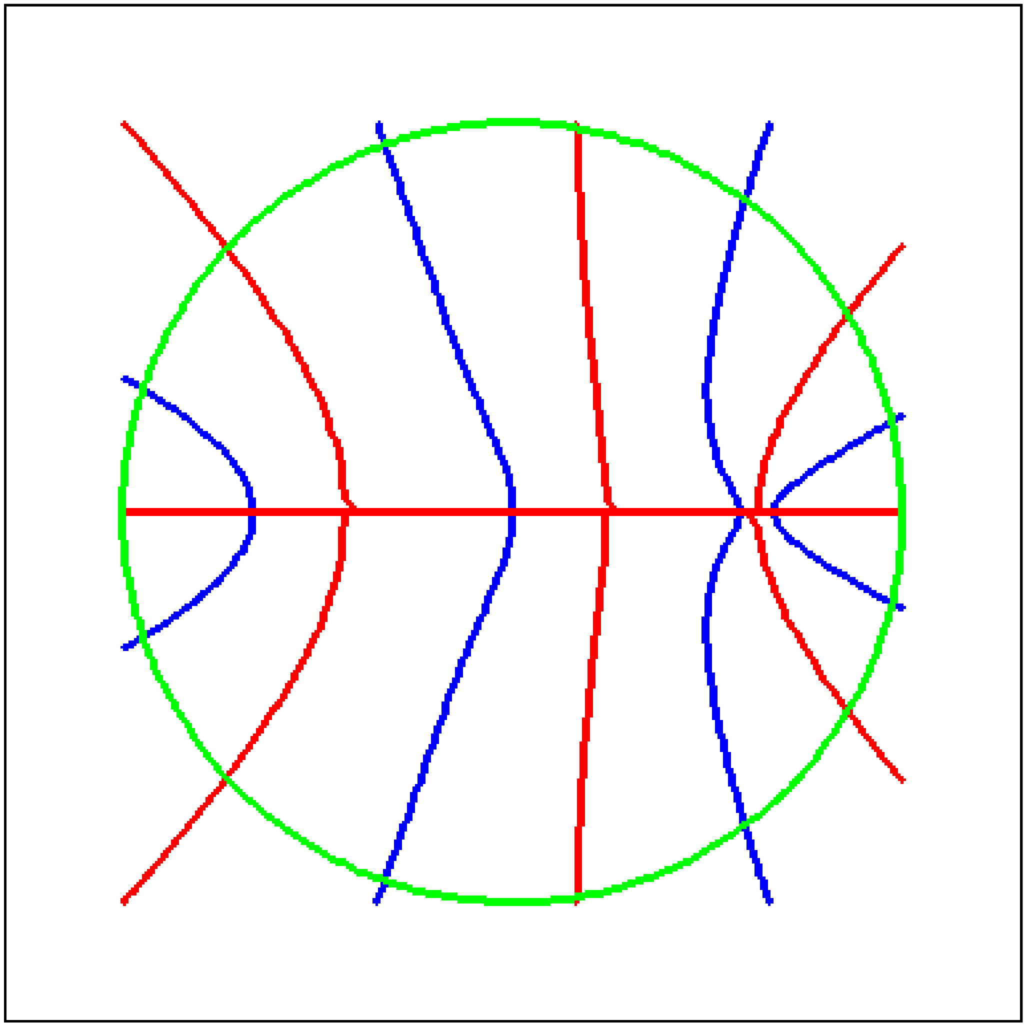
Proposition (3). *The possible shapes of $(R(f), I(f))$, for degree n polynomials, is classified by the set of basketballs. These are the set of "compatible" pairs of forests of $\mathcal{A}$-trees.*

Two forests of $\mathcal{A}$-trees are said to be compatible, if their union is a forest of $\mathcal{A}$-trees.

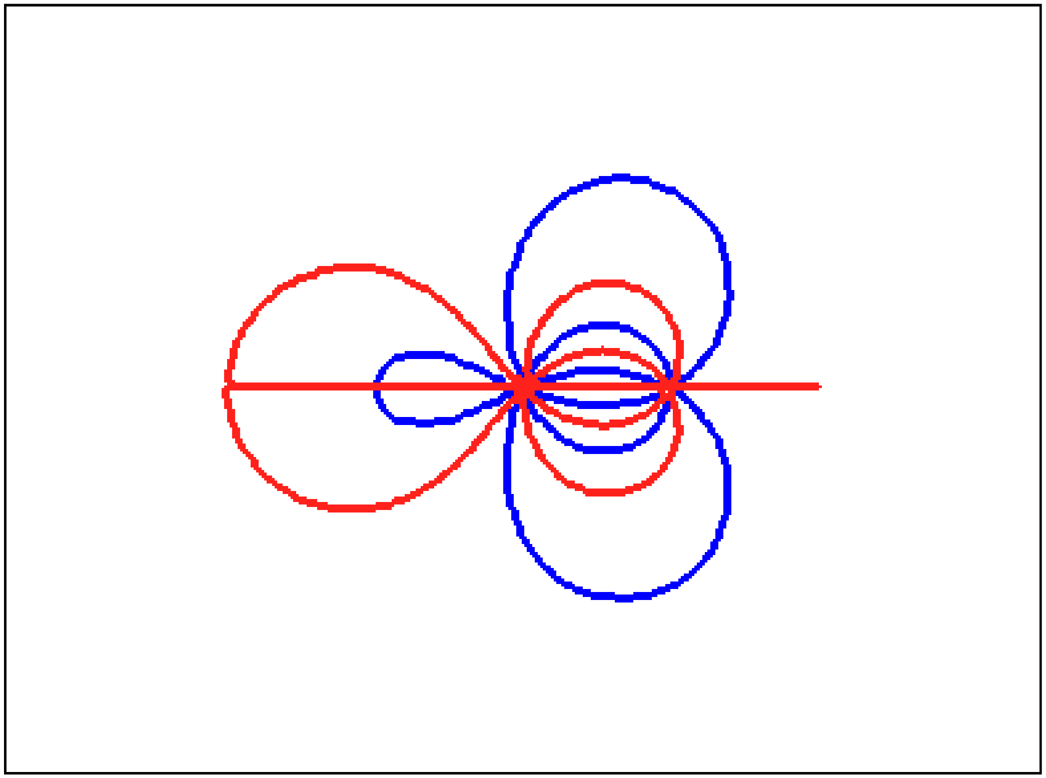
Trois racines



Close to the singular locus



Rational Fraction



Meromorphic Case