

ISAAC Workshop on Pseudo-Differential Operators

**Plancherel Formulas for Integral Transforms in
Time-Frequency Analysis**

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Scope of the Talk

- This is an EXPOSITORY talk leading to some NEW results at the end.

Signals and Images

- Signals and images: f in $L^2(\mathbb{R}^n)$
- Configuration Representation: $f(x)$, $x \in \mathbb{R}^n$
- Frequency Representation: $\hat{f}(\xi)$, $\xi \in \mathbb{R}^n$
- Fourier Transform:

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx, \quad \xi \in \mathbb{R}^n.$$

- Fourier Spectrum: $\{\hat{f}(\xi) : \xi \in \mathbb{R}^n\}$

The Classic Plancherel Formula

- **Theorem** $(f, g) = (\hat{f}, \hat{g})$, $f, g \in L^2(\mathbb{R}^n)$.
- A Useful Reformulation: For each $\xi \in \mathbb{R}^n$, define the function e_ξ on \mathbb{R}^n by

$$e_\xi(x) = e^{ix \cdot \xi}, \quad x \in \mathbb{R}^n.$$

- **Theorem** For all $f, g \in L^2(\mathbb{R}^n)$,

$$(f, g) = (2\pi)^{-n} \int_{\mathbb{R}^n} (f, e_\xi)(e_\xi, g) d\xi.$$

Resolution of the Identity Formula

- We can reconstruct a signal or image f in $L^2(\mathbb{R}^n)$ from its Fourier spectrum $\{\hat{f}(\xi) : \xi \in \mathbb{R}^n\}$ by means of the following resolution of the identity formula.
- **Theorem** For all $f \in L^2(\mathbb{R}^n)$,

$$f = (2\pi)^{-n} \int_{\mathbb{R}^n} (f, e_\xi) e_\xi d\xi.$$

Modern Perspective

- Lie Group: \mathbb{R}^n , a group with respect to addition
- Haar Measure: $d\xi$ =the Lebesgue measure on \mathbb{R}^n
- Unitary Representation: $\pi : \mathbb{R}^n \rightarrow U(L^2(\mathbb{R}^n))$,
where $U(L^2(\mathbb{R}^n))$ = group of unitary operators on $L^2(\mathbb{R}^n)$,

$$(\pi(\xi)f)(x) = e^{ix \cdot \xi} f(x), \quad x, \xi \in \mathbb{R}^n; f \in L^2(\mathbb{R}^n).$$

- Admissible Wavelet: $\varphi(x) = 1, \quad x \in \mathbb{R}^n.$
- $e_\xi = \pi(\xi)\varphi, \xi \in \mathbb{R}^n$

The Plancherel Formula in Modern Perspective

- **Theorem** For all f and g in $L^2(\mathbb{R}^n)$,

$$(f, g) = (2\pi)^{-n} \int_{\mathbb{R}^n} (f, \pi(\xi)\varphi)(\pi(\xi)\varphi, g) d\xi.$$

- Ingredients: Lie group, Haar measure, unitary representation on $L^2(\mathbb{R}^n)$, admissible wavelet
- **Remark** The group \mathbb{R}^n is commutative.

Defects of the Fourier Transform

- To compute the spectrum $\hat{f}(\xi)$ of f localized at a single frequency ξ , information about f for all x is required.
- The Fourier transform gives the spectrum with precise information about frequency, but no information about time or position.

The Gabor Transform

- Idea: Look at the signal f through a window $\varphi \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ at time b and compute the Fourier transform. For simplicity, we let $n = 1$ in \mathbb{R}^n .
- Gabor Transform: For all $b, \xi \in \mathbb{R}$,

$$(G_\varphi f)(b, \xi) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-ix\xi} f(x) \overline{\varphi(x-b)} dx.$$

- Windowed Fourier Transform or Short-Time Fourier Transform

Reformulation of the Gabor Transform

■ Gabor Transform Reformulated:

$$(G_\varphi f)(b, \xi) = (2\pi)^{-1/2} (f, M_\xi T_{-b} \varphi), \quad b, \xi \in \mathbb{R},$$

where

$$(M_\xi h)(x) = e^{ix\xi} h(x), \quad x \in \mathbb{R},$$

$$(T_{-b} h)(x) = h(x - b), \quad x \in \mathbb{R},$$

for all signals h .

■ M_ξ =Modulation; T_{-b} = Translation

The Plancherel Formula for the Gabor Transform

■ **Theorem** Suppose that $\|\varphi\|_2 = 1$. Then for all $f, g \in L^2(\mathbb{R})$,

$$(f, g) = (2\pi)^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (f, M_{\xi} T_{-b} \varphi)(M_{\xi} T_{-b} \varphi, g) db d\xi.$$

The Resolution of the Identity Formula for the Gabor Transform

- **Theorem** Suppose $\|\varphi\|_2 = 1$. Then for all f in $L^2(\mathbb{R})$,

$$f = (2\pi)^{-1} \int_{-\infty}^{\infty} (f, M_{\xi} T_{-b}) M_{\xi} T_{-b} \varphi \, db \, d\xi.$$

- This says that every signal can be reconstructed from its Gabor spectrum

$$\{(G_{\varphi} f)(b, \xi) : b, \xi \in \mathbb{R}\}.$$

The Weyl–Heisenberg Group

- Set: $\text{WH} = \mathbb{R} \times \mathbb{R} \times [0, 2\pi]$
- Group Law:

$$(b_1, \xi_1, t_1) \cdot (b_2, \xi_2, t_2) = (b_1 + b_2, \xi_1 + \xi_2, t_1 + t_2 + b_1 \xi_2),$$

where $t_1 + t_2 + b_1 \xi_2$ is understood to be addition modulo 2π .

- WH is a noncommutative Lie group with Haar measure $db d\xi dt$

Representations

- Irreducible and Unitary Representation:

$$\pi : \mathbb{W}\mathbb{H} \rightarrow U(L^2(\mathbb{R}))$$

- Action:

$$(\pi(b, \xi, t)f)(x) = e^{i(x\xi - b\xi + t)} f(x - b)$$

for $x \in \mathbb{R}$, $(b, \xi, t) \in \mathbb{W}\mathbb{H}$ and $f \in L^2(\mathbb{R})$.

Square Integrability

- Square-Integrability (Admissibility): For all $\varphi \in L^2(\mathbb{R})$ with $\|\varphi\|_2 = 1$,

$$\int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |(\varphi, \pi(b, \xi, t)\varphi)|^2 db d\xi dt = 4\pi^2.$$

- Every φ in $L^2(\mathbb{R})$ with $\|\varphi\|_2 = 1$ is an admissible wavelet and has the same wavelet constant $4\pi^2$.

The Plancherel Formula for the Weyl–Heisenberg Group

- **Theorem** Let $\varphi \in L^2(\mathbb{R})$ be such that $\|\varphi\|_2 = 1$. Then for all $f, g \in L^2(\mathbb{R})$,

$$(f, g) = \frac{1}{4\pi^2} \int_{\mathbb{WH}} (f, \pi(z, t)\varphi)(\pi(z, t)\varphi, g) dz dt,$$

where $z = (b, \xi)$ and $dz = db d\xi$.

- This formula is exactly the same as the Plancherel formula for the Gabor transform.

Defects of the Gabor Transform

The window has fixed size. We want an adaptive window in the following sense:

- THE WINDOW IS WIDE FOR REGIONS WITH LOW FREQUENCY
- THE WINDOW IS NARROW FOR REGIONS WITH HIGH FREQUENCY

How? The answer comes from wavelets.

Wavelets

- Let $\varphi \in L^2(\mathbb{R})$ be such that $\|\varphi\|_2 = 1$ and

$$\int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi)|^2}{|\xi|} d\xi < \infty.$$

Then φ is admissible and is called a *mother wavelet*.

- Wavelets: For all $b \in \mathbb{R}$ and $a \in \mathbb{R} \setminus \{0\}$, the wavelet $\varphi_{b,a}$ is defined by

$$\varphi_{b,a}(x) = \frac{1}{\sqrt{|a|}} \varphi \left(\frac{x - b}{a} \right), \quad x \in \mathbb{R}.$$

Wavelets Reformulated

■ Note:

$$\varphi_{b,a} = T_{-b} D_{1/a} \varphi,$$

where

$$(D_{1/a} h)(x) = \frac{1}{\sqrt{|a|}} h\left(\frac{x}{a}\right), \quad x \in \mathbb{R},$$

for all signals h .

The Wavelet Transform

- Let φ be a mother wavelet. Then the wavelet transform $\Omega_\varphi f$ of a signal f is the function on $\mathbb{R} \times \mathbb{R} \setminus \{0\}$ defined by

$$(\Omega_\varphi f)(b, a) = (f, \varphi_{b,a})$$

for all $b \in \mathbb{R}$ and $a \in \mathbb{R} \setminus \{0\}$.

The Plancherel Formula for the Wavelet Transform

■ **Theorem** Let φ be a mother wavelet. Then for all $f, g \in L^2(\mathbb{R})$,

$$(f, g) = \frac{1}{c_\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (f, \varphi_{b,a})(\varphi_{b,a}, g) \frac{db da}{a^2},$$

where

$$c_\varphi = 2\pi \int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi)|^2}{|\xi|} d\xi.$$

The Resolution of the Identity Formula for the Wavelet Transform

- **Theorem** Let φ be a mother wavelet. Then for all $f \in L^2(\mathbb{R})$,

$$f = \frac{1}{c_\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (f, \varphi_{b,a}) \varphi_{b,a} \frac{db da}{a^2}.$$

- This says that every signal f can be reconstructed from its wavelet spectrum

$$\{(f, \varphi_{b,a}) : b \in \mathbb{R}, a \in \mathbb{R} \setminus \{0\}\}.$$

The Affine Group

- Set: $\mathbb{A} = \mathbb{R} \times \mathbb{R} \setminus \{0\}$
- Group Law:

$$(b_1, a_1) \cdot (b_2, a_2) = (b_1 + a_1 b_2, a_1 a_2)$$

- \mathbb{A} is a noncommutative Lie group with left Haar measure $\frac{db da}{a^2}$.

Representations

- Irreducible and Unitary Representation:

$$\pi : \mathbb{A} \rightarrow U(L^2(\mathbb{R}))$$

- Action:

$$(\pi(b, a)f)(x) = \frac{1}{\sqrt{|a|}} f\left(\frac{x - b}{a}\right)$$

for $x \in \mathbb{R}$, $(b, a) \in \mathbb{A}$ and $f \in L^2(\mathbb{R})$.

Square-Integrability

- Square-Integrability (Admissibility): For all $\varphi \in L^2(\mathbb{R})$ with $\|\varphi\|_2 = 1$,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |(\varphi, \pi(b, a)\varphi)|^2 \frac{db da}{a^2} = 2\pi \int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi)|^2}{|\xi|} d\xi.$$

- So, $\text{LHS} < \infty \Leftrightarrow \text{RHS} < \infty$.

The Plancherel Formula for the Affine Group

■ **Theorem** Let φ be a mother wavelet. Then for all $f, g \in L^2(\mathbb{R})$,

$$(f, g) = \frac{1}{c_\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (f, \pi(b, a)\varphi)(\pi(b, a)\varphi, g) \frac{db da}{a^2}.$$

The Stockwell Transform

- Let $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then the Stockwell transform $S_\varphi f$ of a signal f is the function on $\mathbb{R} \times \mathbb{R}$ defined by

$$(S_\varphi f)(b, \xi) = (2\pi)^{-1/2} |\xi| \int_{-\infty}^{\infty} e^{-ix\xi} f(x) \overline{\varphi(\xi(x - b))} dx$$

for all $b, \xi \in \mathbb{R}$.

- R. G. Stockwell, L. Mansinha and R. P. Lowe, *IEEE Trans. Signal Processing* 44 1996, 998–1001

A Transparent Expression for the Stockwell Transform

- For all $b, \xi \in \mathbb{R}$,

$$(S_\varphi f)(b, \xi) = (f, \varphi^{b, \xi}),$$

where

$$\varphi^{b, \xi} = (2\pi)^{-1/2} M_\xi T_{-b} \tilde{D}_\xi \varphi.$$

Here,

$$(\tilde{D}_\xi h)(x) = |\xi| h(\xi x), \quad x \in \mathbb{R},$$

for all signals h .

The Plancherel Formula for the Stockwell Transform

■ **Theorem** Let $\varphi \in L^2(\mathbb{R})$ be such that $\|\varphi\|_2 = 1$ and

$$c_\varphi = \int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi - 1)|^2}{|\xi|} d\xi < \infty.$$

Then for all $f, g \in L^2(\mathbb{R})$,

$$(f, g) = \frac{1}{c_\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (f, \varphi^{b,\xi})(\varphi^{b,\xi}, g) \frac{db d\xi}{|\xi|}.$$

The Resolution of the Identity Formula for the Stockwell Transform

- **Theorem** Let $\varphi \in L^2(\mathbb{R})$ be such that $\|\varphi\|_2 = 1$ and

$$c_\varphi = \int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi - 1)|^2}{|\xi|} d\xi < \infty.$$

Then for all $f \in L^2(\mathbb{R})$,

$$f = \frac{1}{c_\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (f, \varphi^{b,\xi}) \varphi^{b,\xi} \frac{db d\xi}{|\xi|}.$$

Remarks on the Stockwell Transform

■ The admissibility condition

$$\int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi - 1)|^2}{|\xi|} d\xi < \infty$$

means that $\hat{\varphi}(-1) = 0$ whenever $\hat{\varphi}$ is continuous at -1 .

■ The Gaussian window φ used exclusively for the Stockwell transform in the literature is not “admissible” because $\hat{\varphi}(-1) \neq 0$.

More Remarks on the Stockwell Transform

- The Gabor, wavelet and Stockwell reconstruction formulas in this talk can be discretized using frames.
- The Stockwell and wavelet transforms are related by

$$\varphi^{b,\xi} = (2\pi)^{-1/2} |\xi|^{1/2} M_\xi \varphi_{b,1/\xi}$$

for all $b \in \mathbb{R}$ and $\xi \in \mathbb{R} \setminus \{0\}$, but the Stockwell transform is not a special case of a wavelet transform.

Further Remarks on the Stockwell Transform

- Jingde Du, M. W. Wong and Hongmei Zhu,
Continuous and Discrete Reconstruction Formulas
for the Stockwell Transform, to appear in *Integral
Equations and Special Functions*
- Yu Liu: 2-D Polar Stockwell Transforms

Absolutely Referenced Phase Information

The modulation in the Stockwell transform gives the following result:

- **Theorem** Suppose that $\|\varphi\|_1 = 1$. Then

$$\int_{-\infty}^{\infty} (S_\varphi f)(b, \xi) db = \hat{f}(\xi), \quad \xi \in \mathbb{R}.$$

The Time-Time Transform

- If we take the inverse Fourier transform of the Stockwell transform with respect to frequency, then we get a new integral transform known as the time-time transform or TT -transform. The TT -transform is an integral operator whose kernel is given in terms of Dawson's integral.
- Dawson's Integral: $D(x) = e^{-x^2} \int_0^x e^{t^2} dt$, $x \in \mathbb{R}$.
- C. R. Pinnegar, M. W. Wong and Hongmei Zhu,
Integral Representations of the TT -Transform,
Applicable Analysis 85 2006, 933–940

Wigner Transforms

- Wigner Transform: Let $f, g \in L^2(\mathbb{R})$. The Wigner transform $W(f, g)$ of f and g is defined by

$$W(f, g)(x, \xi) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-i\xi p} f\left(x + \frac{p}{2}\right) \overline{g\left(x - \frac{p}{2}\right)} dp$$

for all $x, \xi \in \mathbb{R}$.

- Moyal Identity: For all f_1, g_1, f_2 and g_2 in $L^2(\mathbb{R})$,

$$(W(f_1, g_1), W(f_2, g_2)) = (f_1, f_2) \overline{(g_1, g_2)}.$$

Weyl Transforms

- Weyl Transform: Let $\sigma \in L^2(\mathbb{R} \times \mathbb{R})$. Then the Weyl transform $W_\sigma : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is defined by

$$(W_\sigma f, g) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma(x, \xi) W(f, g)(x, \xi) dx d\xi$$

for all $f, g \in L^2(\mathbb{R})$.

- M. W. Wong, Weyl Transforms, *Springer-Verlag*, 1998.

Localization Operators Associated to the Stockwell Transform

- Let $\varphi \in L^2(\mathbb{R})$ be admissible with respect to the Stockwell transform. Let $F \in L^2(\mathbb{R} \times \mathbb{R})$. Then we define the Stockwell localization operator $S_{F,\varphi} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ by

$$(S_{F,\varphi} f, g) = \frac{1}{c_\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(b, \xi) (f, \varphi^{b,\xi})(\varphi^{b,\xi}, g) \frac{db d\xi}{|\xi|}$$

for all f and g in $L^2(\mathbb{R})$.

- M. W. Wong, Wavelet Transforms and Localization Operators, *Birkhäuser*, 2002.

Stockwell Localization Operators and Weyl Transforms

Theorem Let $\varphi \in L^2(\mathbb{R})$ be admissible with respect to the Stockwell transform. Then for all $F \in L^2(\mathbb{R} \times \mathbb{R})$, $S_{F,\varphi} = W_\sigma$, where for all $q, p \in \mathbb{R}$,

$$\begin{aligned} & \sigma(q, p) \\ &= \frac{(2\pi)^{1/2}}{c_\varphi} \int_{\mathbb{R}^2} F(b, \xi) \overline{W(\varphi, \varphi)(\xi(q - p), (p - \xi)/\xi)} \frac{db d\xi}{|\xi|} \end{aligned}$$

Proof

The proof is based on a key computation.

■ **Lemma** Let $\varphi \in L^2(\mathbb{R})$. Then for all q, p, b and ξ in \mathbb{R} with $\xi \neq 0$,

$$W(\varphi^{b,\xi}, \varphi^{b,\xi}) = W(\varphi, \varphi)(\xi(q - b), (p - \xi)/\xi).$$

Proof Continued

Let $u, v \in L^2(\mathbb{R})$. Then

$$\begin{aligned} & (S_{F,\varphi} u, v) \\ &= \frac{1}{\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(b, \xi) (u, \varphi^{b,\xi})(\varphi^{b,\xi}, v) \frac{db d\xi}{|\xi|} \\ &= \frac{1}{\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(b, \xi) (W(u, v), W(\varphi^{b,\xi}, \varphi^{b,\xi})) \frac{db d\xi}{|\xi|} \end{aligned}$$

Proof Continued

Using the lemma on the Wigner transform of $\varphi^{b,\xi}$, we get

$$\begin{aligned} & (W(u, v), W(\varphi^{b,\xi}, \varphi^{b,\xi})) \\ = & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(u, v) \overline{W(\varphi, \varphi)(\xi(q - p), (p - \xi)/\xi)} dq dp. \end{aligned}$$

Proof Continued

So, putting the formulas in the previous two slides together, we get

$$(S_\varphi u, v) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma(q, p) W(u, v)(q, p) dq dp.$$

Therefore

$$S_{F,\varphi} = W_\sigma.$$