Wiener's Lemma for the Heisenberg Group and a Class of Pseudodifferential Operators

Brendan Farrell*
University of California, Davis
farrell@math.ucdavis.edu

In collaboration with Thomas Strohmer

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1. Motivation

   - Harmonic Analysis

   - Mobile Communications

2. Integral Operator Result

3. Wiener's Lemma for $L^1_v(\mathbb{H})$

4. Twisted convolution, Pseudodifferential Operators and Mobile Communications

Wiener's lemma (1932) states that if a periodic function $f : \mathbb{R} \to \mathbb{R}$ has an absolutely summable Fourier series

$$f(t) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n t}$$

and is nowhere zero, then $1/f$ also has an absolutely convergent Fourier series.

Our Harmonic Analysis question: for what non-abelian groups does Wiener's lemma hold?

Wiener's lemma in terms of Banach algebras

$\mathcal{A}$: Banach algebra of 1-periodic functions $f : \mathbb{R} \to \mathbb{R}$, $f(t) \neq 0 \; \forall t \in \mathbb{R}$.

$\mathcal{B} : \{f \in \mathcal{A} : f(t) = \sum_n a_n e^{2\pi i n t}, \sum_n |a_n| < \infty\}$

If $f \neq 0$, $1/f \in \mathcal{A}$.

If $f \in \mathcal{B}$ and $f \neq 0$, is $1/f \in \mathcal{B}$?

Wiener's lemma: YES.

A Wiener-type Theorem: the subalgebra $\mathcal{B}$ is inverse-closed in $\mathcal{A}$. That is, $\mathbf{b} \in \mathcal{B}$ and $\mathbf{b}$ invertible $\Rightarrow \mathbf{b}^{-1} \in \mathcal{B}$.

$\widetilde{\mathcal{A}}$: $\mathcal{A}$ with adjoined identity.

Bochner and Phillips (1942) contributed the first essential step towards a general operator version of Wiener's lemma. They showed that the a_n in

$$f(t) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n t}$$

may belong to a noncommutative Banach algebra.

Gohberg, Kaashoek and Woerdeman (1989) and Baskakov (1992): Let X_n be subspaces of X, indexed by a discrete abelian group $\mathbb{I}$, satisfying $X_i \cap X_j = \emptyset$ for $i \neq j$ and $X = \bigcup_{i\in\mathbb{I}} X_i$. Set P_i to be the projection onto X_i. For the linear operator $T: X \to X$ they set

$$a_n = \sum_{i-j=n} P_i T P_j,$$

and consider the operator-valued Fourier series

$$f(t) = \sum_{n\in\mathbb{I}} a_n e^{2\pi i n t} \tag{1}$$

satisfying $\sum_{n\in\mathbb{Z}} \|a_n\| < \infty$. They use Bochner and Phillips's work to establish that operators of the form (1) satisfying $\sum_{n\in\mathbb{Z}} \|a_n\| = \sum_{n\in\mathbb{Z}} \sup_{i-j=n} \|P_i T P_j\| < \infty$ form an inverse-closed Banach algebra in $\mathcal{B}(X)$.

In the commutative setting, Gelfand, Raikov and Shilov (1964) addressed the important question: what rates of decay of an element are preserved in its inverse?

Definition: *A weight function* $v : \mathbb{G} \to \mathbb{R}$ *is* admissible *if:*

1. *v is continuous, even in each coordinate, and normalized so that $v(0) = 1$.*

2. *v is submultiplicative, i.e. $v(x+y) \leq v(x)v(y)$ for all x, y.*

3. *v satisfies the Gelfand-Raikov-Shilov (GRS) condition:*

$$\lim_{n \to \infty} v(nx)^{1/n} = 1 \quad \text{for all } x.$$

Typical examples: $v(x) = (1 + |x|^2)^{k/2}$
$v(x) = e^{-x^\alpha}$, $0 \leq \alpha < 1$.
GRS showed that $l^1_v(\mathbb{Z})$ is inverse-closed in $(l^1(\mathbb{Z}), *)$.

Baskakov incorporated the GRS condition and proved the following operator version of Wiener's lemma: let v be an admissible weight; if the linear operator T satisfies

$$\sum_{n\in\mathbb{Z}} \sup_{i-j=n} \|P_i T P_j\| v(n) < \infty \tag{2}$$

and is invertible, then

$$\sum_{n\in\mathbb{Z}} \sup_{i-j=n} \|P_i T^{-1} P_j\| v(n) < \infty. \tag{3}$$

Kurbatov (1999)considered a class of operators satisfying

$$(Tf)(t) \leq \int \beta(t-s)|f(s)|ds \qquad (4)$$

for some $\beta \in L^1$. If $\alpha_1 I + T$ is invertible and T satisfies (4) for $\beta_1 \in L^1$, then $(\alpha_1 I + T)^{-1} = \alpha_2 I + T_2$ and T_2 satisfies (4) for $\beta_2 \in L^1$.

This theorem, as stated for integral operators, is our point of departure.

Other recent Wiener-type results:

- Sjöstrand (1995): Pseudodifferential operators with symbol in $M^{\infty 1}(\mathbb{R}^{2d})$.
- Gröchenig (2004): Generalization of Sjöstrand using time-frequency techniques
- Gröchenig and Leinert (2004, 2006): $(l_v^1(\mathbb{Z}^d \times \mathbb{Z}^d), \natural_\theta)$.
- Gröchenig and Strohmer (2006): Pseudodifferential operators with symbol in $M^{\infty 1}(\mathbb{G} \times \widehat{\mathbb{G}})$.
- Balan (2006): Summable time-frequency shifts on a discrete subset of $\mathbb{R}^{2d}$.

All of these recent results feature locally compact abelian groups, their dual groups, and various forms of twisted convolution.

Fundamental property:

$$(T_x M_\omega)(T_{x'} M_{\omega'}) = e^{2\pi i x' \cdot \omega} T_{x+x'} M_{\omega+\omega'}$$

It is, therefore, natural to look at the Heisenberg group:

$$\mathbb{H} = \mathbb{G} \times \widehat{\mathbb{G}} \times \mathbb{T}$$

$$\begin{aligned} \mathbf{h}\mathbf{h}' &= (x, \omega, e^{2\pi i \tau})(x', \omega', e^{2\pi i \tau'}) \\ &= (x + x', \omega + \omega', e^{2\pi i (\tau+\tau')} e^{\pi i (x' \cdot \omega - x \cdot \omega')}) \end{aligned}$$

$$(F_1 \star F_2)(\mathbf{h}_0) = \int_{\mathbb{H}} F_1(\mathbf{h}) F_2(\mathbf{h}^{-1}\mathbf{h}_0) d\mathbf{h}$$

Our question: Is $\widetilde{L^1_v}(\mathbb{H})$ inverse-closed with respect to convolution $\star$?

Our Second Motivation: Propagation Channel of a Mobile Communication System

View $f(t)$ as a signal transmitted by a single source.

- Reflections result in various paths with different travel times

- Movement causes Doppler effect, a frequency shift

- Collection of time-shifted and frequency-shifted (modulated) copies of transmitted signal is received

$$f_{rec}(t) = \int_{\mathbb{R}} \int_{\mathbb{R}+} \widehat{\sigma}(x,\omega) T_x M_\omega f_{trans}(t) dx d\omega.$$

$$T_x f(t) = f(t-x) \qquad M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t)$$

Mobile communication channel:

$$f_{rec}(t) = \int_{\mathbb{R}} \int_{\mathbb{R}+} \widehat{\sigma}(x, \omega) T_x M_\omega f_{trans}(t) dx d\omega.$$

Weyl pseudodifferential operator L_σ:

$$L_\sigma f(t) = \int_{\mathbb{G}} \int_{\widehat{\mathbb{G}}} \widehat{\sigma}(\omega, x) e^{-\pi i \omega \cdot x} T_{-x} M_\omega f(t) dx d\omega.$$

In practice, we must numerically "invert" L_σ.

$$L_\sigma \cong \mathbf{A}, \quad \mathbf{A}_{k,l} = \langle L_\sigma \phi_l, \phi_k \rangle$$

If $\widehat{\sigma} \in L^1_v(\mathbb{G})$, $\mathbf{A}$ will decay off the diagonal.

$$L_\sigma^{-1} \cong \mathbf{B}, \quad \mathbf{B}_{k,l} = \langle L_\sigma^{-1} \phi_l, \phi_k \rangle$$

Want to know if L_σ^{-1}, and hence $\mathbf{B}$, will have the same off-diagonal decay.

Then we can truncate the $\mathbf{A}$ to $\mathbf{A}_{trunc}$, and $\mathbf{A}_{trunc}^{-1} \approx \mathbf{B}$.

This truncation is essential for real world computation.

$$L_\sigma f(t) = \int_{\mathbb{G}} \int_{\widehat{\mathbb{G}}} \widehat{\sigma}(\omega, x) e^{-\pi i \omega \cdot x} T_{-x} M_\omega f(t) dx d\omega.$$

Composition rule for Weyl symbols:

$$L_\sigma L_\tau = L_{\mathcal{F}^{-1}(\widehat{\sigma} \natural \widehat{\tau})}.$$

Thus, we are also (in fact, initially) interested in twisted convolution on $L^1_v(\mathbb{G} \times \widehat{\mathbb{G}})$.

If $(\widetilde{L^1_v}(\mathbb{G} \times \widehat{\mathbb{G}}), \natural)$ is inverse closed, then the class of Weyl pseudodifferential operators is also inverse-closed.

We will see that $L^1_v(\mathbb{G} \times \widehat{\mathbb{G}})$ can be treated by our more general approach for $L^1_v(\mathbb{H})$.

Theorem 1 (B.F & T.S) *Let* $\mathcal{N}_v^1(\mathbb{H})$ *denote those bounded integral operators* N *of the form*

$$(Nf)(\mathbf{h}_0) = \int_{\mathbb{H}} N(\mathbf{h}_0, \mathbf{h}) f(\mathbf{h}) d\mathbf{h},$$

for which there exists $\beta \in L_v^1(\mathbb{H})$ *satisfying*

$$|N(\mathbf{h}_0, \mathbf{h})| \leq \beta(\mathbf{h}_0^{-1}\mathbf{h})$$

for all $\mathbf{h}_0, \mathbf{h} \in \mathbb{H}$*. Let* $\widetilde{\mathcal{N}_v^1(\mathbb{H})}$ *denote* $\mathcal{N}_v^1(\mathbb{H})$ *with adjoined unit. Then* $\widetilde{\mathcal{N}_v^1(\mathbb{H})}$ *is an inverse-closed Banach algebra in* $\mathcal{B}(L^p(\mathbb{H}))$ $1 \leq p \leq \infty$*..*

We begin with

- $\mathcal{N}_v^1(\mathbb{H})$: integral operators with kernel majorized by $\beta \in L_v^1(\mathbb{H})$.

- $\mathcal{N}_v^\infty(\mathbb{H})$: not necessarily integral operators, defined by the norm:

$$\|N\|_{\mathcal{N}_v^\infty(\mathbb{H})} =$$

$$\sum_{i \in \mathbb{I}} \sup_{j-k=i} \|N : L^1(Q_j) \to L^\infty(Q_k)\| v(i) < \infty$$

- $\mathcal{N}_v^\infty(\mathbb{H})$ is a dense, two-side ideal in $\mathcal{N}_v^1(\mathbb{H})$

- We can apply Baskakov to $\mathcal{N}_v^\infty(\mathbb{H})$

Proof, Part 1: Structure for $L^1_v(\mathbb{H})$

GKW & B:

- $\mathbb{I}$: discrete abelian group
- $X = \bigcup_{i\in\mathbb{I}} X_i$
- P_i : projection onto X_i

$$\sum_{i\in\mathbb{I}} \sup_{j-k=i} \|P_i T P_j\| < \infty, \text{ and } T \text{ invertible}$$

$$\Rightarrow \sum_{i\in\mathbb{I}} \sup_{j-k=i} \|P_i T^{-1} P_j\| < \infty.$$

By the structure theorem:

$$\mathbb{G} \cong \mathbb{R}^d \times \mathbb{G}_0 \cong \bigcup_{(i,d)\in\mathbb{Z}^d\times\mathbb{D}} (i,d) + [0,1)^d \times \mathbb{K}, \quad \mathbb{D} = \mathbb{G}_0/\mathbb{K}$$

$$\begin{aligned}
\mathbb{G}\times\widehat{\mathbb{G}} &\cong \mathbb{R}^{2d}\times\mathbb{G}_0\times\widehat{\mathbb{G}}_0 \\
&\cong \bigcup_{(i,d)\in\mathbb{Z}^{2d}\times\mathbb{D}} (i,d) + [0,1)^{2d}\times\mathbb{K}\times\mathbb{K}^{\perp} \\
\mathbb{D} &= (\mathbb{G}_0\times\widehat{\mathbb{G}}_0)/(\mathbb{K}\times\mathbb{K}^{\perp})
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{H} &= \mathbb{G} \times \widehat{\mathbb{G}} \times \mathbb{T} \\
&\cong \bigcup_{(i,d)\in\mathbb{Z}^{2d}\times\mathbb{D}} (i,d,0) + [0,1)^{2d} \times \mathbb{K} \times \mathbb{K}^{\perp} \times \mathbb{T}
\end{aligned}$$

We can set up subspaces while avoiding the noncommutativity of $\mathbb{H}$ and apply Baskakov's result to $\mathcal{N}_v^{\infty}(\mathbb{H})$.

Proof, Part 2

Show that $\mathcal{N}_v^\infty(\mathbb{H})$ is a two-sided ideal in $\mathcal{N}_v^1(\mathbb{H})$.

The noncommutativity of $\mathbb{H}$ makes this lemma a little technical and delicate.

Definition: $(\mathcal{I}, \mathbb{U})$ is a *partition* of $\mathbb{G}$ if $\mathcal{I}$ is a discrete subset of $\mathbb{G}$, $\mathbb{U}$ is a compact subgroup of $\mathbb{G}$, and $\mathbb{G} = \bigcup_{i\in\mathcal{I}}(i+\mathbb{U})$.

Definition: Let $\mathbb{G}$ be a group with partition $(\mathcal{I}, \mathbb{U})$ and v an admissible weight function, and set $Q_i = i+\mathbb{U}$. The *amalgam space* $W(L_v^p(\mathbb{G}), l_v^q)$ is the space of functions finite in the local L^p norm and the global l_v^q norm as follows:

$$\|f\|_{W(L^p(\mathbb{G}),l_v^q)} = \left(\sum_{i\in\mathcal{I}} \|f\|_{L^p(Q_i)}^q v(i)^q\right)^{1/q}.$$

Proposition $\mathcal{N}_v^\infty(\mathbb{H})$ is equivalent to the class of integral operators with kernels majorized by functions in $W(L^\infty(\mathbb{H}), l_v^1)$.

Proof, Part 2

By previous propostion, we may equivalently prove that $W(L^1(\mathbb{H}), l^1_v)$ is a two-sided ideal in $W(L^\infty(\mathbb{H}), l^1_v)$.

Suppose $F \in W(L^1(\mathbb{H}), l^1_v)$ and $G \in W(L^\infty(\mathbb{H}), l^1_v)$ for v an admissible weight.

Show:

$$\|F \star G\|_{W(L^\infty(\mathbb{H}), l^1_v)} \leq C \|F\|_{W(L^1(\mathbb{H}), l^1_v)} \|G\|_{W(L^\infty(\mathbb{H}), l^1_v)}$$

$$\|G \star F\|_{W(L^\infty(\mathbb{H}), l^1_v)} \leq C \|F\|_{W(L^1(\mathbb{H}), l^1_v)} \|G\|_{W(L^\infty(\mathbb{H}), l^1_v)}$$

- Noncommutative!
- Prove for any partition of $\mathbb{H}$
- Use $\mathbb{H}$ is unimodular, $\mathbb{U}$ is a group

Idea of Proof: Assume $\alpha I + N$ is invertible in $\widetilde{\mathcal{N}_v^1}(\mathbb{H})$.

1. Invertible implies $\alpha \neq 0$.

2. For $\alpha I + N$, $N \in \mathcal{N}_v^1(\mathbb{H})$, there exists $\overline{N} \in \mathcal{N}_v^\infty(\mathbb{H})$ s.t. $\|\overline{N} - \mathcal{N}\| < \alpha/2$.

$$\begin{aligned} K &= (\alpha I + (N - \overline{N}))^{-1}(\alpha I + N) \\ &= (\alpha I + (N - \overline{N}))^{-1}(\alpha I + (N - \overline{N}) + \overline{N}) \\ &= I + (\alpha I + (N - \overline{N}))^{-1}\overline{N}. \end{aligned}$$

$\mathbf{1}$. By the ideal property, $(\alpha I + (N - \overline{N}))^{-1}\overline{N} \in \mathcal{N}_v^\infty(\mathbb{H})$.

$\mathbf{2}$. $\Rightarrow \quad K \in \widetilde{\mathcal{N}_v^\infty}(\mathbb{H})$

$\mathbf{3}$. K invertible $\Rightarrow \quad K^{-1} \in \widetilde{\mathcal{N}_v^\infty}(\mathbb{H})$

$\mathbf{4}$. $K^{-1} \in \widetilde{\mathcal{N}_v^\infty}(\mathbb{H})$ & $(\alpha I + (N - \overline{N}))^{-1} \in \widetilde{\mathcal{N}_v^1}(\mathbb{H})$

$\Rightarrow K^{-1}(\alpha I + (N - \overline{N}))^{-1} \in \widetilde{\mathcal{N}_v^1}(\mathbb{H})$

$K^{-1}(\alpha I + (N - \overline{N}))^{-1} = (\alpha I + N)^{-1} \in \widetilde{\mathcal{N}_v^1}(\mathbb{H})$.

Theorem 2 (B.F, T.S) *Let $\mathbb{H}$ be the general, reduced Heisenberg group as defined above, and v an admissible weight function. If $\alpha_1\delta+f$, $f \in L^1_v(\mathbb{H})$, is invertible with respect to convolution over $\mathbb{H}$, then $(\alpha_1\delta + f)^{-1} = \alpha_2\delta + g$, $g \in L^1_v(\mathbb{H})$.*

Idea of proof: Set $S_{\alpha\delta+F}f = (\alpha\delta + F) \star f$ and assume the $S_{\alpha\delta+F}$ is invertible. By the previous theorem, $(S_{\alpha\delta+F})^{-1} = \alpha_2\delta + A$, where A is majorized by $\beta \in L^1_v(\mathbb{H})$. We use an approximate identity $\{\psi_n\}_{n\geq 0}$:

$$\theta = \alpha\delta + \lim_{n\to\infty} A\psi_n = \alpha\delta + G.$$

For $\phi \in \mathbb{C}_0(\mathbb{H})$,

$$\begin{aligned} S_{\alpha_1\delta+F} & (S_\theta - (\alpha_2 I + A))\phi \\ &= S_{\alpha_2\delta+F}S_{\alpha_2\delta+G}\phi - S_{\alpha_1\delta+F}S^{-1}_{\alpha_1\delta+F}\phi \\ &= (\alpha_1\delta + F) \star (\alpha_2\delta + G) - \phi \\ &= \delta \star \phi - \phi \\ &= 0 \end{aligned}$$

Theorem 3 (Kurbatov (without weights)**)** *Let* $\mathbb{G}$ *be a locally compact abelian group. Then* $\widetilde{\mathcal{N}_v^1}(\mathbb{G})$ *is an inverse-closed Banach algebra in* $\mathcal{B}(L^p(\mathbb{G}))$*.*

Twisted convolution is defined by

$$F \natural G(x_0, \omega_0) = \int_{\mathbb{G}} \int_{\widehat{\mathbb{G}}} F(x, \omega) G(x_0 - x, \omega_0 - \omega) e^{\pi i (x \omega_0 - \omega x_0)}$$

Theorem 4 (B.F. & T.S.) *Let* $\mathbb{G}$ *be a locally compact abelian group and* $\widehat{\mathbb{G}}$ *its dual group. If* $\alpha_1 \delta + f$*,* $f \in L_v^1(\mathbb{G} \times \widehat{\mathbb{G}})$*, is invertible with respect to twisted convolution, then* $(\alpha_1 \delta + f)^{-1} = \alpha_2 \delta + g$*,* $g \in L_v^1(\mathbb{G} \times \widehat{\mathbb{G}})$*.*

Proof: The proof for the analogous theorem for $L_v^1(\mathbb{H})$ holds with the substitution of $\mathbb{G} \times \widehat{\mathbb{G}}$ for $\mathbb{H}$.

Our mobile communication channel:

$$f_{rec}(t) = \int_{\mathbb{R}} \int_{\mathbb{R}+} \widehat{\sigma}(x,\omega) T_x M_\omega f_{trans}(t) dx d\omega$$

The assumption that $\widehat{\sigma} \in L^1_v(\mathbb{R}^2)$ is appropriate: Why is $L^1_v(\mathbb{G} \times \widehat{\mathbb{G}})$ the appropriate spreading function space for modile communications?

- In practice, strength of delayed copies fades quickly

- Doppler effect depends on relative speeds and angles of bodies and signal

- These are bounded, so Doppler effect is bounded, say to $[-D, D]$

- If signal is band-limited to $[-W, W]$, then the support of $\widehat{\sigma}(x, \cdot) \in [-W - D, W + D]$ for all x

Theorem 5 (B.F. & T.S.) *Let* $\mathrm{OP}(\mathcal{F}^{-1}L^1_v(\widehat{\mathbb{G}}\times\mathbb{G}))$ *denote the space of pseudodifferential operators with Weyl symbol* σ *satisfying* $\widehat{\sigma} \in L^1(\widehat{\mathbb{G}}\times\mathbb{G})$.

Then $\mathrm{OP}(\mathcal{F}^{-1}\widetilde{L^1_v}(\widehat{\mathbb{G}}\times\mathbb{G}))$ *is an inverse-closed subalgebra of* $\mathcal{B}(L^p(\mathbb{G}))$. *That is*

(i) $\alpha I + L_\sigma$ *is bounded on all* $L^p(\mathbb{G})$.

(ii) If $\widehat{\sigma}, \widehat{\tau} \in L^1(\widehat{\mathbb{G}}\times\mathbb{G})$, *then* $(\alpha_1 I + L_\sigma)(\alpha_2 I + L_\tau) = (\alpha_3 I + L_\gamma)$, *where* $\widehat{\gamma} \in L^1_v(\widehat{\mathbb{G}}\times\mathbb{G})$.

(iii) If $\alpha_1 I + L_\sigma$ *is invertible on* $L^p(\mathbb{G})$, *then* $(\alpha_1 I + L_\sigma)^{-1} = (\alpha_2 I + L_\tau)$ *where* $\widehat{\tau} \in L^1_v(\widehat{\mathbb{G}}\times\mathbb{G})$.

iii. means that the matrices for $\alpha_1 I + L_\sigma$ and $(\alpha_1 I + L_\sigma)^{-1}$ will have the same off-diagonal decay and can be truncated to a small number of diagonals.

Proof: (i).

$$
\begin{aligned}
\|L_\sigma f\|_{L^p}^p &\le \int \left| \int\int \widehat{\sigma}(\omega,x) e^{-\pi i \xi\cdot x} T_{-x} M_\omega dx d\omega \right|^p dt \\
&\le \int \left(\int\int |\widehat{\sigma}(\omega,x)||f(t+x)| d\omega dx \right)^p dt \\
&= \int \left(\int \|\widehat{\sigma}(\cdot,-x)\|_{L^1} |f(t+x)| dx \right)^p dt \\
&= \|\widehat{\sigma}(\cdot,u) * |f|\|_{L^p}^p \\
&\le \|\widehat{\sigma}\|_{L^1}^p \|f\|_{L^p}^p \\
&\le \|\widehat{\sigma}\|_{L^1_v}^p \|f\|_{L^p}^p
\end{aligned}
$$

Therefore, $\|(\alpha I + L_\sigma) f\|_{Lp} \le (|\alpha| + \|\widehat{\sigma}\|_{L^1}) \|f\|_{L^p}$.

(ii). $(\alpha_1 I + L_\sigma)(\alpha_2 I + L_\tau) = L_{\mathcal{F}^{-1}((\alpha_1\delta+\widehat{\sigma})(\alpha_2\delta+\widehat{\tau}))}$. Therefore, by Theorem 4, if $\widehat{\sigma}, \widehat{\tau} \in L^1(\widehat{\mathbb{G}} \times \mathbb{G})$, $(\alpha_1\delta + \widehat{\sigma})\natural(\alpha_2\delta + \widehat{\tau}) = (\alpha_3\delta + \widehat{\gamma})$, where $\widehat{\gamma} \in L^1(\widehat{\mathbb{G}} \times \mathbb{G})$. Then $\mathcal{F}^{-1}(\alpha\delta + \widehat{\gamma}) = \alpha + \gamma$, and $(\alpha_1 I + L_\sigma)(\alpha_2 I + L_\tau) = (\alpha_3 I + L_\gamma)$.

(iii). Follows immediately from Theorem 4.