Continuity and compactness properties of pseudo-differential operators

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Weyl-Hörmander calculus

Given a positive definite quadratic form $G(X)$ on $\mathbb{R}^{2d}$, we define

- the *dual quadratic form:*

$$G^\sigma(X) = \sup_{G(Y)=1} \sigma(X,Y)^2$$

  with respect to the *standard symplectic form* in $\mathbb{R}^{2d}$:

$$\sigma(X,Y) = QX \cdot PY - PX \cdot QY,$$

  where $P : \mathbb{R}^{2d} \to \mathbb{R}^d \times \{0\}$ and $Q : \mathbb{R}^{2d} \to \{0\} \times \mathbb{R}^d$, are the orthogonal projections onto the first d coordinates and the last d coordinates.

- the *Plank constant:*

$$h_G = \sup_{X \in \mathbb{R}^{2d}} \sqrt{\frac{G(X)}{G^\sigma(X)}}.$$

An *admissible metric* is a measurable function $g : X \mapsto g_X$, of $\mathbb{R}^{2d}$ into the set of positive definite quadratic forms on $\mathbb{R}^{2d}$ and satisfying the following hypotheses:

- g is *slowly-varying:* there exists a positive constant r_g such that

$$g_X(T) \lesssim g_Y(T) \lesssim g_X(T),$$

  for all $T \in \mathbb{R}^{2d}$ and all $X, Y \in \mathbb{R}^{2d}$ such that

$$g_X(Y - X) < r_g^2.$$

- g is *σ-temperate:* there exists a positive constant M_g such that

$$g_X(T) \lesssim g_Y(T)\Big(1 + g_X^\sigma(Y - X)\Big)^{M_g},$$

  for all $X, Y, T \in \mathbb{R}^{2d}$.
- g satisfies the *uncertainty principle*:

$$h_g(X) = h_{g_X} \leqslant 1, \qquad \text{for all } X \in \mathbb{R}^{2d}.$$

A g-*weight* is a positive measurable function $\mu : \mathbb{R}^{2d} \to \mathbb{R}_+$ such that

- $\mu(X) \lesssim \mu(Y) \lesssim \mu(X)$, for all X, $Y \in \mathbb{R}^{2d}$ such that
$$g_X(Y - X) < r_g^2,$$

- there exists a constant $M_\mu > 0$ for which
$$\mu(X) \lesssim \mu(Y)\Big(1 + g_X^\sigma(X - Y)\Big)^{M_\mu},$$
for all X, $Y \in \mathbb{R}^{2d}$.

An example of g-weight is given by the Plank function h_g.

A smooth function $a : \mathbb{R}^{2d} \to \mathbb{C}$ is a *symbol* if there exists an admissible metric g and a g-weight μ such that

$$\|a\|^g_{\mu,k} = \sup_{j \leqslant k} \left\| |a|^g_j \, \mu^{-1} \right\|_{L^\infty} < \infty,$$

for all $k \in \mathbb{N}$, where

$$|a|^g_j \, (X) = \sup_{T_1,\ldots,T_j} \frac{\left| a^{(j)}(X; T_1, \ldots, T_j) \right|}{g_X(T_1)^{1/2} \cdots g_X(T_j)^{1/2}}.$$

We denote by $S(\mu, g)$ the class of all symbols of g-weight μ.

$S(\mu, g)$ is a Frechét space with respect to the norms $\|a\|^g_{\mu,k}$.

The *Weyl quantization* of a symbol $a \in S(\mu, g)$ is the pseudo-differential operator defined by

$$\langle a^{\mathrm{w}} u, \overline{v} \rangle = (2\pi)^{-d/2} \langle a, \mathcal{W}(u,v) \rangle, \qquad \text{for all } u,\, v \in \mathcal{S}(\mathbb{R}^d),$$

where $\mathcal{W}(u,v)$ is the *Wigner transform:*

$$\mathcal{W}(u,v)(X) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-iy \cdot QX} u\,(PX + y/2)\, \overline{v\,(PX - y/2)}\, dy.$$

a^{w} is a continuous operator on the Schwartz class $\mathcal{S}(\mathbb{R}^d)$, which extends to a continuous operator on the tempered distributions $\mathcal{S}'(\mathbb{R}^d)$.

The formal adjoint of a^{w} is $\overline{a}^{\mathrm{w}}$:

$$\langle a^{\mathrm{w}} u, \overline{v} \rangle = \langle u, \overline{\overline{a}^{\mathrm{w}} v} \rangle, \qquad \text{for all } u,\, v \in \mathcal{S}(\mathbb{R}^d).$$

In $S(\mu, g)$ we consider also the *weak topology*.

A sequence $a_j \in S(\mu, g)$ is *weakly convergent* to $a \in S(\mu, g)$ if

- a_j is bounded in the strong topology:

$$\sup_{j \in \mathbb{N}} \left\| a_j \right\|_{\mu,k}^{g} < \infty, \qquad \text{for all } k \in \mathbb{N},$$

- a_j converges point-wise to a.

The Weyl quantization is weakly continuous:

$$\lim_{j \to \infty} \langle a_j^{w} u, v \rangle = \langle a^{w} u, v \rangle, \qquad \text{for all } u \in \mathcal{S}'(\mathbb{R}^d) \text{ and } v \in \mathcal{S}(\mathbb{R}^d),$$

whenver $a_j \to a$ weakly in $S(\mu, g)$.

Theorem. *Let* $a \in S(\mu, g)$ *and assume* $\mu h_g^n \in L^\infty(\mathbb{R}^{2d})$ *for some* $n \in \mathbb{N}$. *Then* a^w *is bounded on* $L^2(\mathbb{R}^d)$ *if and only if* $a \in L^\infty(\mathbb{R}^{2d})$.

More precisely there exists $l \in \mathbb{N}$ *such that*

$$\|a^w\|_{\mathcal{B}(L^2)} \lesssim \|a\|_{L^\infty} + \|a\|^g_{\mu,l}\,,$$
$$\|a\|_{L^\infty} \lesssim \|a^w\|_{\mathcal{B}(L^2)} + \|a\|^g_{\mu,l}\,,$$

for all $a \in S(\mu, g)$.

Theorem. *Let* $a \in S(\mu, g)$ *and assume*

$$\mu(X) h_g(X)^n \to 0 \qquad \text{as } |X| \to \infty.$$

Then a^w *is compact on* $L^2(\mathbb{R}^d)$ *if and only if*

$$a(X) \to 0 \qquad \text{as } |X| \to \infty.$$

Theorem. *Given two symbols $a \in S(\mu, g)$ and $b \in S(\lambda, g)$, we have that $a^w b^w$ is a pseudo-differential operator with Weyl symbol*

$$a \# b \in S(\mu\lambda, g).$$

For each $n \in \mathbb{Z}_+$

$$a \# b - \sum_{j=0}^{n-1} \frac{1}{j!} \left\{a, b\right\}_j \in S(\mu\lambda h_g^n, g),$$

with

$$\left\{a, b\right\}_j (X) = (2i)^{-j} \Big[\Big(\sigma(\partial_X, \partial_Y) \Big)^j a(X) b(Y) \Big]_{Y=X}.$$

$a_j \# b \to a \# b$ *weakly in* $S(\mu\lambda, g)$ *whenever* $a_j \to a$ *weakly in* $S(\mu, g)$.

Hörmander's partition of unity.

There exists a sequence of points $Z_j \in \mathbb{R}^{2d}$ and an integer N_g such that the balls

$$B_j = \left\{ X \in \mathbb{R}^{2d} \ : \ g_{Z_j}(X - Z_j) < r_g^2/4 \right\}$$

cover $\mathbb{R}^{2d}$ and the intersection of more than N_g balls is always empty.

One can choose non-negative $\chi_j \in S(1, g) \cap \mathcal{C}_c^\infty(B_j)$ such that

$$\sum_{j=1}^{\infty} \chi_j = 1$$

and

$$\sup_{j \in \mathbb{Z}_+} \left\| \chi_j \right\|_{1,k}^{g} < \infty, \qquad \text{for all } k \in \mathbb{N}.$$

Local estimates.

Fix $n \in \mathbb{Z}_+$ and set

$$a_j = \chi_j a \quad \text{and} \quad b_j = \sum_{k=0}^{n-1} \frac{1}{k!} \left\{\chi_j, a\right\}_k .$$

Lemma. *There exists $l \in \mathbb{N}$ such that*

$$\left\| a_j^{\mathsf{w}} \right\|_{\mathcal{B}(L^2)} \lesssim \|a\|_{L^\infty} + \mu(Z_j) h_g(Z_j)^n \|a\|^g_{\mu,l} ,$$

and

$$\left\| b_j \right\|_{L^\infty} \lesssim \|a^{\mathsf{w}}\|_{\mathcal{B}(L^2)} + \mu(Z_j) h_g(Z_j)^n \|a\|^g_{\mu,l} ,$$

for all $j \in \mathbb{Z}_+$ and all $a \in S(\mu, g)$.

Almost orthogonal estimates.

A *system of almost orthogonal operators on a Hilbert space* H is a sequence A_j of bounded linear operators such that

$$\sup_{j\in\mathbb{Z}_+}\sum_{k=1}^{\infty}\left(\left\|A_j^*A_k\right\|_{\mathcal{B}(H)}^{1/2}+\left\|A_jA_k^*\right\|_{\mathcal{B}(H)}^{1/2}\right)<\infty.$$

Lemma. *There exists* $l\in\mathbb{N}$ *such that*

$$\sup_{j\in\mathbb{Z}_+}\sum_{k=1}^{\infty}\left(\left\|\overline{a}_j^{\mathsf{w}}a_k^{\mathsf{w}}\right\|_{\mathcal{B}(L^2)}^{1/2}+\left\|a_j^{\mathsf{w}}\overline{a}_k^{\mathsf{w}}\right\|_{\mathcal{B}(L^2)}^{1/2}\right)\lesssim\|a\|_{L^\infty}+\left\|\mu h_g^n\right\|_{L^\infty}\|a\|_{\mu,l}^{g},$$

and

$$\sup_{j\in\mathbb{Z}_+}\sum_{k=1}^{\infty}\left(\left\|\overline{b}_j^{\mathsf{w}}b_k^{\mathsf{w}}\right\|_{\mathcal{B}(L^2)}^{1/2}+\left\|b_j^{\mathsf{w}}\overline{b}_k^{\mathsf{w}}\right\|_{\mathcal{B}(L^2)}^{1/2}\right)\lesssim\|a^{\mathsf{w}}\|_{\mathcal{B}(L^2)}+\left\|\mu h_g^n\right\|_{L^\infty}\|a\|_{\mu,l}^{g},$$

for all $a\in S(\mu,g)$.

Assume that $a \in S(\mu, g) \cap L^\infty(\mathbb{R}^{2d})$ and $\mu h_g^n \in L^\infty(\mathbb{R}^{2d})$.
From previous lemmas $a_j^w = \chi_j^w a^w$ is a system of almost orthogonal operators on $L^2(\mathbb{R}^d)$.
From Cotlar's lemma we have that $a^w = \sum_{j=1}^\infty a_j^w$ is bounded:

$$\|a^w\|_{\mathcal{B}(L^2)} \lesssim \|a\|_{L^\infty} + \|a\|_{\mu,l}^g .$$

Assume that $a \in S(\mu, g)$, $a^w \in \mathcal{B}(L^2)$ and $\mu h_g^n \in L^\infty(\mathbb{R}^{2d})$.
Let J be the set of the indexes j such that B_j contains a given point X. J has at most N_g elements.
$\sum_{j \in J} \chi_j = 1$, on the open set $\bigcap_{j \in J} B_j$; so all derivatives of $\sum_{j \in J} \chi_j$ vanish at X.
Then $a(X) = \sum_{j \in J} b_j(X)$ and we obtain

$$|a(X)| \leqslant \Big\| \sum_{j \in J} b_j \Big\|_{L^\infty} \leqslant N_g \sup_{j \in J} \big\| b_j \big\|_{L^\infty} \lesssim \|a^w\|_{\mathcal{B}(L^2)} + \|a\|_{\mu,l}^g .$$

Assume that

$$\lim_{|X|\to\infty} a(X) = 0 \quad \text{and} \quad \lim_{|X|\to\infty} \mu(X) h_g(X)^n = 0.$$

Then for all $\epsilon > 0$ there exists $m_\epsilon \in \mathbb{Z}_+$ such that

$$|a(X)| \leqslant \epsilon \quad \text{and} \quad \mu(X) h_g(X)^n \leqslant \epsilon, \qquad \text{for all } X \in \bigcup_{j > m_\epsilon} B_j.$$

Then the quantization of $(1 - \sum_{j \leqslant m_\epsilon} \chi_j) a/\epsilon \in S(\mu/\epsilon, g)$ is bounded:

$$\frac{1}{\epsilon} \Big\| a^w - \sum_{j \leqslant m_\epsilon} a_j^w \Big\|_{\mathcal{B}(L^2)} \lesssim \|a/\epsilon\|_{L^\infty} + \Big\| (\mu/\epsilon) h_g^n \Big\|_{L^\infty} \|a/\epsilon\|^g_{\mu/\epsilon, l} \leqslant 1 + \|a\|^g_{\mu, l},$$

for all $\epsilon > 0$.

This implies that a^w is a compact operator on $L^2(\mathbb{R}^d)$, because every a_j^w is regularizing.

Assume that a^w is compact and that $\mu h_g^n(X) \to 0$ as $|X| \to \infty$, and let us show that $a(X) \to 0$ as $|X| \to \infty$.

We have already proven that b_j^w, with

$$b_j = \sum_{k=1}^{n} \frac{1}{k!}\{\chi_j, a\}_k,$$

is an almost orthogonal system on $L^2(\mathbb{R}^d)$.

$\sum_{k=1}^{\infty} \chi_{j_k}^w a^w = (\sum_{k=1}^{\infty} \chi_{j_k}^w) a^w$ is compact for all increasing sequence of integers j_k, because $\sum_{k=1}^{\infty} \chi_{j_k}$ is weakly convergent in $S(1, g)$.

The Weyl quantization of $\sum_{k=1}^{\infty}\Big(\chi_{j_k} \# a - b_{j_k}\Big) \in S(\mu h_g^n, g)$ is compact because $\mu h_g(X)^n \to 0$ as $|X| \to \infty$.

Then $\sum_{k=1}^{\infty} b_{j_k}^w$ is compact for all increasing sequence of integers j_k.

Apply the following lemma due to Hörmander:

Lemma. *Let* A_j *be a system of almost orthogonal operators on a Hilbert space* H *such that* $\sum_{k=1}^{\infty} A_{j_k}$ *is compact for all increasing sequence of integers* j_k.

Then

$$\lim_{j\to\infty} \left\| A_j \right\|_{\mathcal{B}(H)} = 0.$$

We obtain

$$\lim_{j\to\infty} \left\| b_j^w \right\|_{\mathcal{B}(L^2)} = 0.$$

Assume now that there exists an increasing sequence on integers j_k such that

$$\lim_{k\to\infty} \|a\|_{L^\infty(B_{j_k})} > 0,$$

that is a sequence of points $X_k \in B_{j_k}$ such that

$$\lim_{k\to\infty} |a(X_k)| > 0.$$

For each k let J_k be the set of indexes j such that B_j contains X_k.

Each J_k has at most N_g elements and $\sum_{j\in J_k} \chi_j = 1$ on the open set $\bigcap_{j\in J_k} B_j$. So

$$a(X_k) = \sum_{j\in J_k} b_j(X_k).$$

Consider the translations

$$c_k = T_{-X_k} \sum_{j \in J_k} b_j,$$

then

$$c_k(0) = a(X_k)$$

and

$$\|c_k\|^g_{\mu,m} \leqslant N_g \, \|a\|^g_{\mu,n+m} \, .$$

Since $\mathcal{C}^\infty(\mathbb{R}^{2d})$ is a Montel space, bounded subsets of $S(\mu, g)$ are compact with respect to the weak topology.

So there exists a sub-sequence c_{k_m} converging weakly to a symbol $c \in S(\mu, g)$. Then

$$c^{\mathsf{W}}_{k_m} \to c^{\mathsf{W}}, \qquad \text{weakly in } \mathcal{B}(\mathcal{S}').$$

Now

$$\left\|c_{k_m}^{\mathsf{w}}\right\|_{\mathcal{B}(L^2)} \leqslant \sum_{j \in J_{k_m}} \left\|b_j^{\mathsf{w}}\right\|_{\mathcal{B}(L^2)} \leqslant N_g \sup_{j \in J_{k_m}} \left\|b_j^{\mathsf{w}}\right\|_{\mathcal{B}(L^2)}.$$

Since $\left\|b_j^{\mathsf{w}}\right\|_{\mathcal{B}(L^2)} \to 0$, we obtain that $\left\|c_{k_m}^{\mathsf{w}}\right\|_{\mathcal{B}(L^2)} \to 0$.

This implies $c^{\mathsf{w}} = 0$, that is $c = 0$.

Then

$$\lim_{k\to\infty} a(X_k) = \lim_{m\to\infty} a(X_{k_m}) = \lim_{m\to\infty} c_{k_m}(0) = 0,$$

in contradiction with the assumption

$$\lim_{k\to\infty} |a(X_k)| > 0.$$

Proof of the local estimate

$$\left\| a_j^{\mathsf{w}} \right\|_{\mathcal{B}(L^2)} \lesssim \|a\|_{L^\infty} + \mu(Z_j) h_g(Z_j)^n \|a\|_{\mu,l}^g, \qquad \text{for all } j \in \mathbb{Z}_+.$$

We show that there exists $l \in \mathbb{N}$ such that

$$\|a^{\mathsf{w}}\|_{\mathcal{B}(L^2)} \lesssim \|a\|_{L^\infty} + \mu(Z) h_g(Z)^n \|a\|_{\mu,l}^g,$$

for all $a \in S(\mu, g) \cap \mathcal{C}_{\mathsf{c}}^\infty(B_Z)$, where

$$B_Z = \left\{ Y \in \mathbb{R}^{2d} \ : \ g_Z(Y - Z) < r_g^2/4 \right\}.$$

Diagonalize the the quadratic form g_Z by a linear symplectic map Σ_Z:

$$g_Z \circ \Sigma_Z = \sum_{j=1}^{d} \theta_j^2 (T_j^2 + T_{d+j}^2).$$

Since the Plank function is symplectically invariant, we have

$$h_g(Z)^{1/2} = \sup_{1 \leqslant j \leqslant d} \theta_j.$$

Let

$$a_Z = (T_{-Z} a) \circ \Sigma_Z,$$

then for all $|\alpha| = k$ we have

$$|\partial^\alpha a_Z(0)| \lesssim |a|_k^g (Z) \Theta^\alpha,$$

where

$$\Theta = (\theta, \theta) \in \mathbb{R}^{2d}.$$

Now a_Z is supported in $\Sigma_Z^{-1} T_Z B_Z$, so from the slowly-varying property we have for all $|\alpha| = k$:

$$|\partial^\alpha a_Z(X)| \lesssim |a|_k^g (Z) \Theta^\alpha, \qquad \text{for all } X \in \mathbb{R}^{2d}.$$

Let

$$p_1 = p_0 - p_0 * \Psi$$

with

$$p_0 = a_Z \quad \text{and} \quad \Psi(X) = \pi^{-d} e^{-|X|^2}.$$

Then we can write

$$p_1 = p_1' + p_1'',$$

where p_1' is supported in $\Sigma_Z^{-1} T_Z B_Z$ and for all $|\alpha| = k$

$$\left|\partial^\alpha p_1'(X)\right| \lesssim |a|_k^g (Z) h_g(Z) \Theta^\alpha, \qquad \text{for all } X \in \mathbb{R}^{2d}.$$

$$\left\|\partial^\alpha p_1''\right\|_{L^\infty} \lesssim \mu(Z) h_g(Z)^n \, \|a\|_{\mu, 2n+k}^g \, .$$

Now we iterate

$$p_j = p_{j-1} - p_{j-1} * \Psi, \qquad \text{for } 1 \leqslant j \leqslant n$$

and write

$$a_Z = \sum_{j=0}^{n-1} p_j * \Psi + p_n = p_0' * \Psi + \sum_{j=1}^{n-1} (p_j' + p_j'') * \Psi + p_n = q * \Psi + q',$$

where $q = \sum_{j=0}^{n-1} p_j' * \Psi$ is supported in $\Sigma_Z^{-1} T_Z B_Z$ and for all $|\alpha| = k$

$$|\partial^\alpha q(X)| \lesssim |a|_k^g (Z) \Theta^\alpha, \qquad \text{for all } X \in \mathbb{R}^{2d},$$
$$\left\|\partial^\alpha q'\right\|_{L^\infty} \lesssim \mu(Z) h_g(Z)^n \|a\|_{\mu,2n+k}^g .$$

Moreover $\left\|p_{j+1}'\right\|_{L^\infty} \leqslant 2 \left\|p_j'\right\|_{L^\infty}$ for $1 \leqslant j \leqslant n-1$, so

$$\|q\|_{L^\infty} \leqslant (1 + 2^n) \|p_0\|_{L^\infty} = (1 + 2^n) \|a\|_{L^\infty} .$$

$a_Z^w - (q')^w$ has anti-Wick symbol q, and therefore

$$\left\| a_Z^w - (q')^w \right\|_{\mathcal{B}(L^2)} \leqslant \|q\|_{L^\infty} \leqslant (1 + 2^n) \|a\|_{L^\infty} .$$

On the other hand $(q')^w$ is bounded, since by the Theorem of Calderon-Vaillancourt there exists k such that:

$$\left\| (q')^w \right\|_{\mathcal{B}(L^2)} \lesssim \sup_{|\alpha| \leqslant k} \left\| \partial^\alpha q' \right\|_{L^\infty} \lesssim \mu(Z) h_g(Z)^n \|a\|^g_{\mu,2n+k} .$$

By Segal's Theorem there exists a unitary operator U_Z such that

$$U_Z^{-1} a_Z^w U_Z = (T_{-Z} a)^w,$$

and therefore

$$\|a^w\|_{\mathcal{B}(L^2)} = \|a_Z^w\|_{\mathcal{B}(L^2)} \lesssim \|a\|_{L^\infty} + \mu(Z) h_g(Z)^n \|a\|^g_{\mu,2n+k} .$$

Proof of second local estimate:

$$\left\|b_j\right\|_{L^\infty} \lesssim \|a^w\|_{\mathcal{B}(L^2)} + \mu(Z_j)h_g(Z_j)^n \|a\|^g_{\mu,l}\,, \qquad \text{for all } j \in \mathbb{Z}_+,$$

where

$$b_j = \sum_{k=0}^{n-1} \tfrac{1}{k!} \left\{\chi_j, a\right\}_k.$$

The Weyl quantization of

$$b_j - \chi_j \# a \in S(\mu h_g^n, g) \subset S(1, g)$$

is bounded on $L^2(\mathbb{R}^d)$:

$$\left\|b_j - \chi_j \# a\right\|^g_{1,k} \leqslant \mu(Z_j)h_g(Z_j)^n \left\|b_j - \chi_j \# a\right\|^g_{\mu h_g^n,k} \lesssim \mu(Z_j)h_g(Z_j)^n \|a\|^g_{\mu,l}\,,$$

for some $l \geqslant k$.

Moreover

$$\left\|\chi_j^w a^w\right\|_{\mathcal{B}(L^2)} \leqslant \left\|\chi_j^w\right\|_{\mathcal{B}(L^2)} \|a^w\|_{\mathcal{B}(L^2)} \lesssim \|a^w\|_{\mathcal{B}(L^2)}.$$

Then also b_j^w is bounded on $L^2(\mathbb{R}^d)$:

$$\left\|b_j^w\right\|_{\mathcal{B}(L^2)} \lesssim \|a^w\|_{\mathcal{B}(L^2)} + \left\|\mu h_g^n\right\|_{L^\infty} \|a\|_{\mu,l}^g.$$

Then we have to estimate $\left\|b_j\right\|_{L^\infty}$ in terms of $\left\|b_j^w\right\|_{\mathcal{B}(L^2)}$. Since b_j is supported in $B_j = B_{Z_j}$, and $\left\|b_j\right\|_{\mu,l}^g \lesssim \|a\|_{\mu,l}^g$ we need to prove the following estimate

$$\|b\|_{L^\infty} \lesssim \|b^w\|_{\mathcal{B}(L^2)} + \mu(Z)h_g(Z)^n \|b\|_{\mu,l}^g.$$

for all $b \in S(\mu, g) \cap \mathcal{C}_c^\infty(B_Z)$.

Recall that Ψ is the Wigner transform of the Gaussian

$$\Phi(x) = \pi^{-d/4} e^{-|x|^2/2},$$

so the *Wick symbol* $p = b * \Psi$ of the operator b^w can be written as

$$p(X) = \langle b^w \mathcal{U}_X \Phi, \overline{\mathcal{U}_X \Phi} \rangle,$$

where $\mathcal{U}_X$ is the unitary operator of *time-frequency shift*:

$$\mathcal{U}_X \Phi(y) = e^{iy \cdot QX} \Phi(y - PX).$$

In particular $p \in L^\infty(\mathbb{R}^{2d})$ whenever b^w is bounded on $L^2(\mathbb{R}^d)$:

$$\|p\|_{L^\infty} \leqslant \|b^w\|_{\mathcal{B}(L^2)} \, .$$

Set $p_0 = b$. Then we have

$$\|p_0 * \Psi\|_{L^\infty} \leqslant \|p_0^w\|_{\mathcal{B}(L^2)} = \|b^w\|_{\mathcal{B}(L^2)}\,.$$

On the other hand

$$\|(p_0 * \Psi)^w\|_{\mathcal{B}(L^2)} \leqslant \|p_0^w\|_{\mathcal{B}(L^2)}\,\|\Psi^w\|_{\mathcal{B}(L^2)} \leqslant \|b^w\|_{\mathcal{B}(L^2)}\,\|\Psi^w\|_{\mathcal{B}(L^2)}\,,$$

and consequently also $p_1^w = p_0^w - (p_0 * \Psi)^w$ is bounded.

Moreover, $p_0 = b$ is supported in B_Z, so we can write

$$p_1 = p_1' + p_1''$$

with p_1' supported in B_Z and $p_1'' \in S(\mu h_g^n, g) \subset S(1, g)$.

Then we can iterate:

$$p_j = p_{j-1} - p_{j-1} * \Psi, \qquad \text{with } 1 \leqslant j \leqslant n.$$

All p_j^w and $p_j * \Psi$ are bounded:

$$\left\| p_j * \Psi \right\|_{L^\infty} \leqslant \left\| p_j^w \right\|_{\mathcal{B}(L^2)} \lesssim \| b^w \|_{\mathcal{B}(L^2)} + \mu(Z) h_g(Z)^n \, \|b\|_{\mu,l}^g ,$$

and $p_n \in S(\mu h_g^n, g)$.

This implies that

$$\|b\|_{L^\infty} \leqslant \sum_{j=0}^{n-1} \left\| p_j * \Psi \right\|_{L^\infty} + \|p_n\|_{L^\infty} \lesssim \|b^w\|_{\mathcal{B}(L^2)} + \mu(Z) h_g(Z)^n \, \|b\|_{\mu,l}^g .$$

Almost orthogonal estimates

The following estimates are estracted from the third volume of Hörmander's book.

Let

$$U_Z = \Big\{X \in \mathbb{R}^{2d} \ : \ g_Z(X-Z) \leqslant r_g^2/2\Big\},$$
$$U'_Z = \Big\{X \in \mathbb{R}^{2d} \ : \ g_Z(X-Z) \leqslant r_g^2/\sqrt{2}\Big\},$$
$$U''_Z = \Big\{X \in \mathbb{R}^{2d} \ : \ g_Z(X-Z) \leqslant r_g^2\Big\},$$

and

$$d_{ZW} = \min_{\substack{X \in U_Z \\ Y \in U_W}} \Big\{g_Z^\sigma(X-Y)\Big\}.$$

For all $k \in \mathbb{N}$ and all $\nu > 0$ there exists $l \in \mathbb{N}$ such that

$$|p\#q|_k^g(X) \lesssim \mu(X)^2 h_g(X)^\nu (1 + d_{ZW})^{-\nu} \|p\|_{\mu,l}^g \|q\|_{\mu,l}^g$$

for all X, Z, $W \in \mathbb{R}^{2d}$ and all p, $q \in S(\mu, g)$ such that

$$X \notin U'_Z \cap U'_W, \quad \operatorname{supp} p \subset B_Z, \quad \operatorname{supp} q \subset B_W.$$

There exists $\nu > 0$ such that

$$\sup_{j \in \mathbb{Z}_+} \sum_{k \in \mathbb{Z}_+} (1 + d_{jk})^{-\nu} < \infty.$$

where

$$d_{jk} = d_{Z_j Z_k},$$

and Z_j is the sequence of points in Hörmander's partition of unity.

Now we estimate

$$\left\|\overline{a}_j^{w} a_k^{w}\right\|_{\mathcal{B}(L^2)} \quad \text{and} \quad \left\|a_j^{w} \overline{a}_k^{w}\right\|_{\mathcal{B}(L^2)}.$$

The estimate is the same, so we discuss the first one only. Set

$$U_j' = U_{Z_j}', \qquad U_j'' = U_{Z_j}''.$$

For all $m \in \mathbb{N}$ and $\nu > 0$ there exists l such that

$$\left|\overline{a}_j \# a_k\right|_m^g (X) \lesssim \mu(X)^2 h_g(X)^{2n} (1 + d_{jk})^{-\nu} \left\|a_j\right\|_{\mu,l}^g \|a_k\|_{\mu,l}^g \lesssim$$
$$\lesssim \mu(X)^2 h_g(X)^{2n} (1 + d_{jk})^{-\nu} \Big(\|a\|_{\mu,l}^g\Big)^2,$$

for all j, $k \in \mathbb{Z}_+$ and X such that

$$X \notin U_j' \cap U_k'.$$

Let $\omega_j \in S(1, g)$ be a non-negative symbol such that

$$\omega_j(X) = \begin{cases} 1, & \text{for } X \in U'_j, \\ 0, & \text{for } X \notin U''_j. \end{cases}$$

For all m we have

$$\left\| (1 - \omega_j \omega_k) \overline{a}_j \# a_k \right\|^g_{1,m} \lesssim (1 + d_{jk})^{-\nu} \left\| \mu h^n_g \right\|^2_{L^\infty} \left(\|a\|^g_{\mu,l} \right)^2$$

for all j, $k \in \mathbb{Z}_+$.

But then, for a large ν, we have

$$\sum_{k \in \mathbb{Z}_+} \left\| \left((1 - \omega_j \omega_k) \overline{a}_j \# a_k \right)^w \right\|_{\mathcal{B}(L^2)} \lesssim \left\| \mu h^n_g \right\|^2_{L^\infty} \left(\|a\|^g_{\mu,l} \right)^2,$$

for all $j \in \mathbb{Z}_+$.

Consider now $\omega_j\omega_k\overline{a}_j\#a_k$, which is supported in $U'_j \cap U'_k$.

We have

$$\begin{aligned}
&\left|\omega_j\omega_k\overline{a}_j\#a_k(X)\right| \lesssim \\
&\qquad \lesssim \left\|(\omega_j\omega_k\overline{a}_j\#a_k)^w\right\|_{\mathcal{B}(L^2)} + \mu(Z_j)^2 h_g(Z_j)^{2n}\Big(\|a\|^g_{\mu,l}\Big)^2,
\end{aligned}$$

for all $X \in U'_j \cap U'_k$.

On the other hand we have

$$\begin{aligned}
&\left\|(\omega_j\omega_k\overline{a}_j\#a_k)^w\right\|_{\mathcal{B}(L^2)} \leqslant \\
&\qquad \leqslant \left\|\Big((1-\omega_j\omega_k)\overline{a}_j\#a_k\Big)^w\right\|_{\mathcal{B}(L^2)} + \left\|(\overline{a}_j\#a_k)^w\right\|_{\mathcal{B}(L^2)} \\
&\qquad \lesssim \|a\|^2_{L^\infty} + \left\|\mu h_g^n\right\|^2_{L^\infty}\Big(\|a\|^g_{\mu,l}\Big)^2.
\end{aligned}$$

Since there are only N_g indices k such that $U'_j \cap U'_k$ contains X, we have

$$\sum_{k \in \mathbb{Z}_+} \left| \omega_j \omega_k \overline{a}_j \# a_k(X) \right| \lesssim \|a\|^2_{L^\infty} + \left\| \mu h^n_g \right\|^2_{L^\infty} \left(\|a\|^g_{\mu,l} \right)^2 .$$

It follows that

$$\sum_{k \in \mathbb{Z}_+} \left\| \omega_j \omega_k \overline{a}_j \# a_k \right\|_{L^\infty} \lesssim \|a\|^2_{L^\infty} + \left\| \mu h^n_g \right\|^2_{L^\infty} \left(\|a\|^g_{\mu,l} \right)^2 .$$

and therefore

$$\sum_{k \in \mathbb{Z}_+} \left\| (\omega_j \omega_k \overline{a}_j \# a_k)^w \right\|_{\mathcal{B}(L^2)} \lesssim \|a\|^2_{L^\infty} + \left\| \mu h^n_g \right\|^2_{L^\infty} \left(\|a\|^g_{\mu,l} \right)^2 .$$

for all $j \in \mathbb{Z}_+$, because $\omega_j \omega_k \overline{a}_j \# a_k$ is supported in $U'_j \cap U'_k$.

The estimate of $\left\| \overline{b}^w_j b^w_k \right\|_{\mathcal{B}(L^2)}$ and $\left\| b^w_j \overline{b}^w_k \right\|_{\mathcal{B}(L^2)}$ is similar.