Optimal Control of Josephson qubits

What can quantum control do for quantum computing?

F.K. Wilhelm[1 2] M.J. Storcz[2] J. Ferber[2] A. Spörl[3]
T. Schulte-Herbrüggen[3] S.J. Glaser[3] P. Rebentrost[1 2]

[1]Institute for Quantum Computing (IQC) and Physics Department
University of Waterloo, Canada

[2]Physics Department, Arnold Sommerfeld Center, and CeNS
Ludwig-Maximilians-Universität München, Germany [3]Chemistry Department
Munich University of Technology, Germany

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Outline

Basic problem setting

- Our physical system gives us a Hamiltonian

$$H(t) = H_\mathrm{d} + \sum_j u_j(t) H_j \tag{1}$$

with *static drift* H_d, controls u_j and *control Hamiltonians* H_j.

- Our goal: Build a *propagator*

$$U_\mathrm{gate} = U(t,0) = \mathcal{T} \exp\left(-\frac{i}{\hbar}\int_0^t dt' H(t')\right) \tag{2}$$

using physical $u_j(t)$.

Rotating wave and area theorem.

Spin in static *z* plus rotating *xy* field

$$H(t) = -\gamma \vec{B}(t) \cdot \vec{\sigma} = \frac{1}{2} \begin{pmatrix} E & \lambda(t) e^{i\omega t} \\ \lambda(t) e^{-i\omega t} & -E \end{pmatrix} \qquad (3)$$

in co-rotating frame

$$H'(t) = \frac{1}{2} \begin{pmatrix} E-\omega & \lambda(t) \\ \lambda(t) & -(E-\omega) \end{pmatrix} \qquad (4)$$

On resonance: $E - \omega = 0$ $[H'(t), H'(t')] = 0$, thus

$$\mathcal{T} \exp\left(-\frac{i}{\hbar}\int_0^t dt' H(t')\right) = \exp\left(-\frac{i}{\hbar}\int_0^t dt' H(t')\right) =$$

$$= \cos\phi(t) - i\sigma_x \sin\phi(t) \qquad \phi(t) = \frac{1}{\hbar}\int_0^t dt' \lambda(t') \qquad (5)$$

Area theorem

Beyond the area theorem

The area theorem does in general hold for $[H'(t), H'(t')] \neq 0$

- out of resonance
- for non-rotating wave Hamiltonians *and* strong driving (non-RWA) i.e. high pulses
- for multi-qubit systems

Power of quantum computing comes from the global non-validity of the area theorem!

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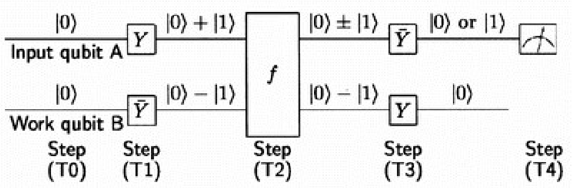
Complementing quantum circuits

Quantum circuit solution:

Discretize into RWA steps with full control.

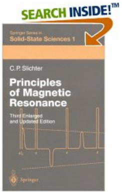


Complemented by control theory

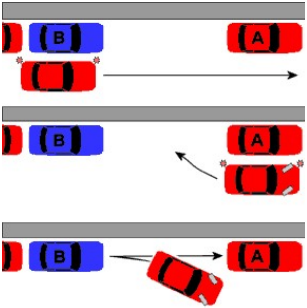
- even the single qubit gates may not be accessible by RWA
- decomposition into elementary gates may not be efficient

Complex control sequences

There are ingenious NMR solutions based on 50 years of quantum control
... do we have to do it again?



Analogous situation:
Steering / parallel parking



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Using control theory

- Established discipline in applied math / engineering
- Applied to quantum systems for state transfers e.g. in quantum chemistry
- Developed for NMR by N. Khaneja (Harvard), S.J. Glaser, T. Schulte-Herbrüggen . . . (TUM)

You do not need to know molecular biology in order to fry an egg.
(Donald E. Knuth)

Basic idea.

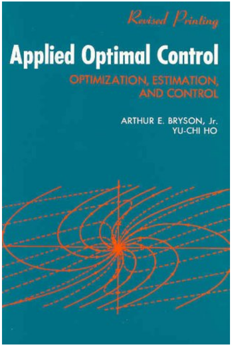
Take any *dynamical system* with variables x_i and controls u_j with EOM

$$\dot{x} = f(x,u,t) \tag{6}$$

Optimize a *performance index* at final time t_f, $F(x(t_f), u(t_f))$ using

$$J = F(x(t_f), u(t_f)) + \int_{t_i}^{t_f} dt \lambda^T(t)(\dot{x} - f(x,u,t)) \tag{7}$$

with initial conditions $x(t_i)$.



Solution of the control problem

Variation with constraints leads to initial value problem

$$\dot{x} = f(x, u, t) \quad x(t_i) = x_i \tag{8}$$

final value problem for influence function λ

$$\dot{\lambda} = -\left(\frac{\partial f}{\partial x}\right)^T \lambda \quad \lambda(t_f) = \left(\frac{\partial F}{\partial x}\right)^T \tag{9}$$

and equation for the controls

$$\left(\frac{\partial f}{\partial u}\right)^T \lambda = 0 \tag{10}$$

Solvable, typically hard (split conditions!)

From Rockets to Propagators

- Control problem for a quantum gate:

$$x \mapsto U(t) \quad U(t_i) = \hat{1} \tag{11}$$

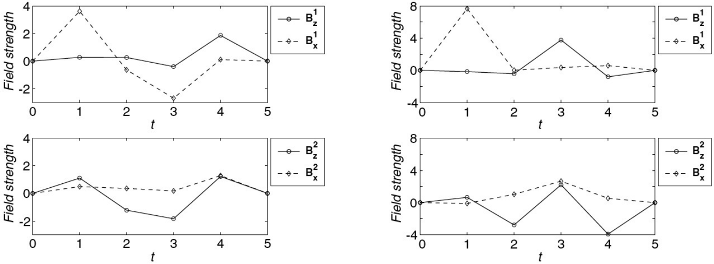
$$f \mapsto -i(H_d + \sum_i u_i(t) H_i) U \tag{12}$$

$$\phi = \left\| U_{\mathrm{gate}} - U(t_f) \right\|^2 = 2N - 2\mathrm{Re}\mathrm{Tr}(U_{\mathrm{gate}}^\dagger U(t_f)) \tag{13}$$

- So we need to *maximize* $\mathrm{Tr}(U_{\mathrm{gate}}^\dagger U(t_f))$.
- *Problem*: Fixes global phase, too
- *Solution*: Maximize $\Phi = |\mathrm{Tr}(U_{\mathrm{gate}}^\dagger U(t_f))|^2$ instead.

Numerical solution

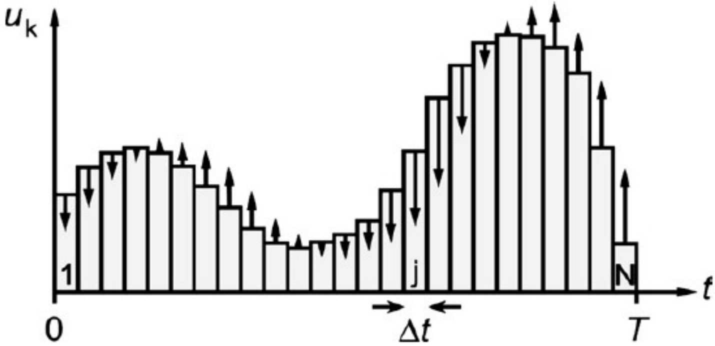
Numerical solution: Minimize J directly.
Problem: Computationally hard optimization, numerical gradients $\frac{\partial \phi}{\partial u_i}$ time-consuming ($\approx$ hours on supercomputer).



From A.O. Niskanen, J.J. Vartiainen and M.M. Salomaa, PRL **90**, 197901 (2003).

Challenge

In the discretized grid, how does Φ change when the control is changed in one point?



Gradient Ascent Pulse Engineering (GRAPE) I

Rewrite performance index

$$\begin{aligned}\Phi &= |\mathrm{Tr}(U_{\mathrm{gate}}^\dagger U(t_f))|^2 = \left|\mathrm{Tr}(U^\dagger(t_j,t_N)U_{\mathrm{gate}})^\dagger U(t_j,t_1)\right|^2 \\ &= \left|\mathrm{Tr}\left(U_{j+1}^\dagger \dots U_N^\dagger U_{\mathrm{gate}}\right)^\dagger U_j \dots U_1\right|^2\end{aligned}$$

Trotterized time-step propagators

$$U_i = \exp\left(-i\Delta t\left(H_d + \sum u_k(t_i)H_k\right)\right) \tag{14}$$

Using

$$\left.\frac{d}{dx}e^{A+Bx}\right|_{x=0} = e^A\int_0^1 d\tau e^{-A\tau}Be^{A\tau} \tag{15}$$

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Gradient Ascent Pulse Engineering (GRAPE) II

we can derive $\frac{\partial \Phi}{\partial u_k}$ analytically.

$$\frac{\partial \Phi}{\partial u_k(t_j)} = \delta t \mathrm{Re}\left[\left(\mathrm{Tr} U_{\mathrm{gate}}^\dagger U_N \dots U_{j+1} H_k U_j \dots U_1\right)\right.$$
$$\left.\left(\mathrm{Tr} U_{\mathrm{gate}}^\dagger U_N \dots U_{j+1} U_j \dots U_1\right)\right]$$

N. Khaneja, T. Reiss, C. Kehlet, T. Schulte-Herbrüggen, S.J. Glaser, JMR **172**, *296 (2005).*

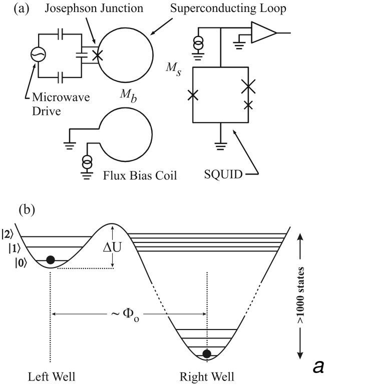
The physical problem.

Successful superconducting qubit with close leakage level

$$\delta\omega = \omega_{12} - \omega_{23} \simeq 0.1\omega_{12} \quad (16)$$

Drive resonantly on ω_{12}. RWA-Hamiltonian

$$H' = \begin{pmatrix} -\delta\omega & \sqrt{2}\lambda(t) & 0 \\ \sqrt{2}\lambda(t) & 0 & \lambda(t) \\ 0 & \lambda(t) & 0 \end{pmatrix} \quad (17)$$



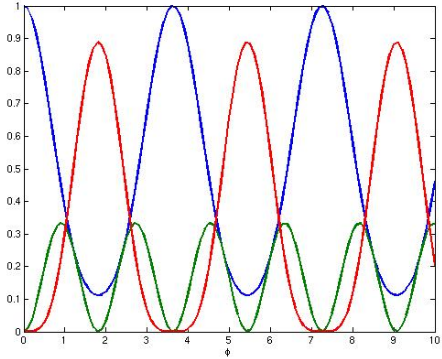
[a]

[a] Martinis and Simmonds groups, UCSB and NIST

How to avoid leakage to the higher level?

Properties of the problem.

- At low λ leakage is small $\propto \lambda/\delta\omega$, area theorem o.k. - slow pulse
- At extremely high $\lambda \gg \delta\omega$ area theorem again.
- Can we at least push the limits at intermediate $\lambda \simeq \delta\omega$?



Populations of $|0\rangle$, $|1\rangle$, $|2\rangle$

$\phi(t) = \int_0^t dt' \lambda(t')$

GRAPE for this problem.

We want an X gate on the two levels, i.e.

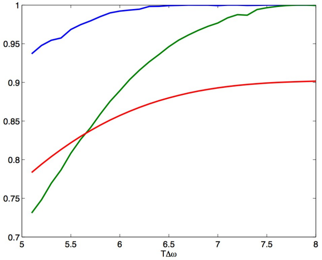
$$U_{\mathrm{gate}} = e^{i\phi_1} \begin{pmatrix} e^{i\phi_2} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{18}$$

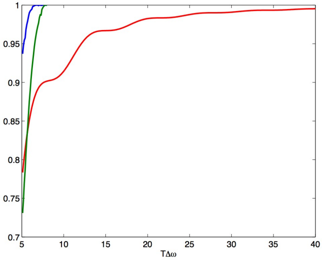
so we have *two* free phases.

Performance index

$$\Phi_d = \frac{1}{5}(|M_{22}|^2 + |M_{00} + M_{11}|^2) \qquad M = U_{\mathrm{gate}}^{\dagger} U(t_f). \tag{19}$$

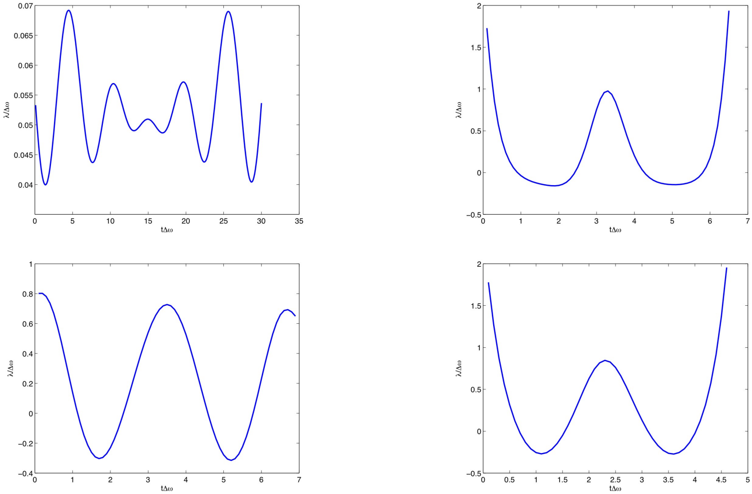
Overall performance



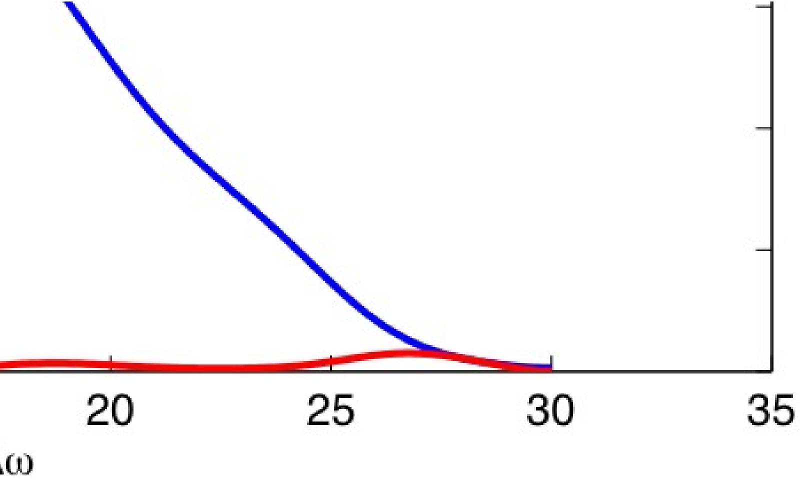


Rectangular Rabi pulse
GRAPE, fixed internal phase
GRAPE, free internal phase

Optimum pulse shapes

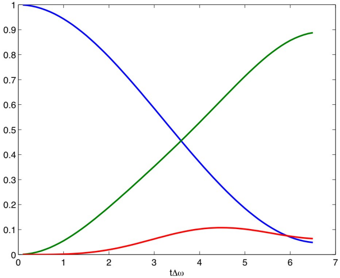


Populations in long pulses

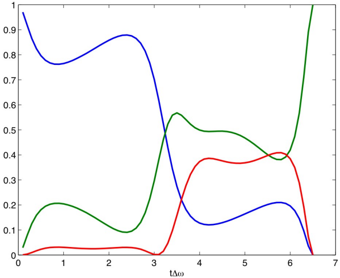


Populations in intermediate pulses

Rectangular pulse



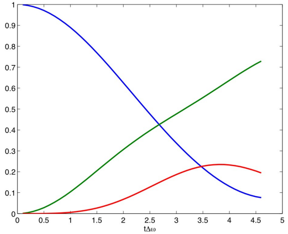
GRAPE pulse



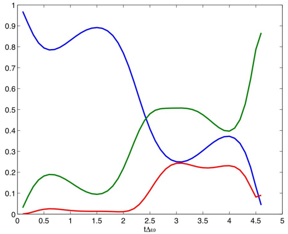
Aha! We do a $(2n+1)\pi$ pulse on the qubit transition and a $2n\pi$ pulse on the leakage transition

Populations in short pulses

Rectangular pulse

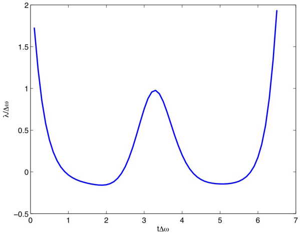


GRAPE pulse



GRAPE explores the physical limitations

Rise times and penalties



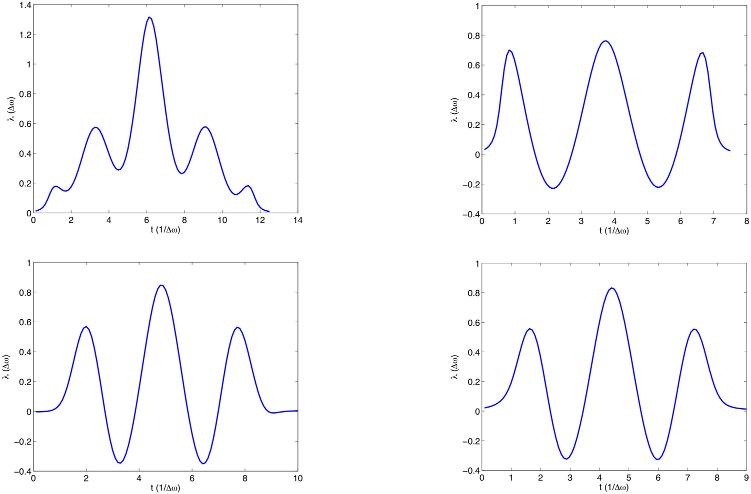
Problems:

- Does not start at zero
- Short rise time

Possible solutions:

- Additional Lagrange Multiplier: Not practical of inequalities
- Penalty in performance index:
  $F(x_f, u_f, t_f) + A \int_{t_i}^{t_f} dt \quad p^2(x(t), u(t), t)$.
  Here: $p = u\,[2 - \tanh(t/t_0) - \tanh((T - t)/t_0)]$

Easier pulse shapes



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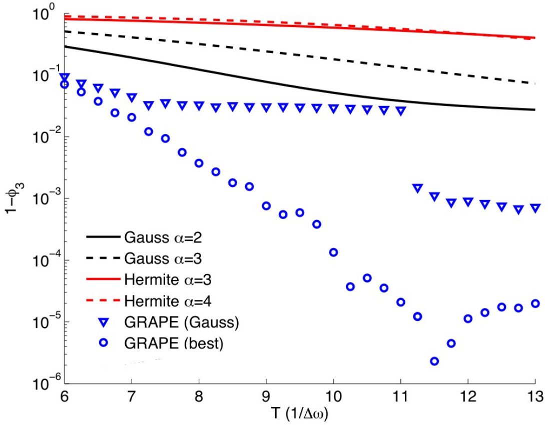
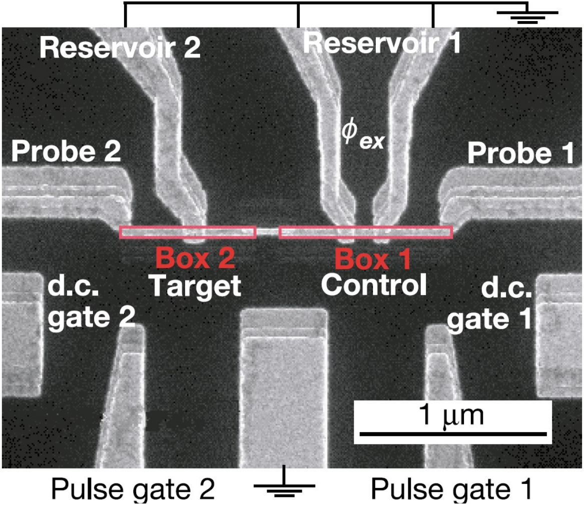
Performance



NEC coupled Cooper pair boxes.



Coupled boxes Hamitonian.

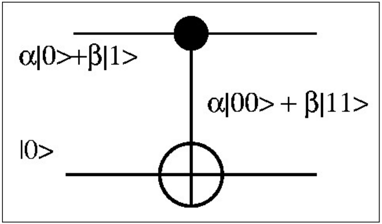
- Charge basis $|N_1, N_2\rangle$

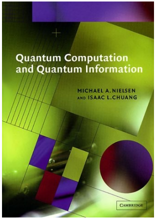
$$H = \sum_{N_1 N_2} E_{\mathrm{Ch},\mathrm{n_1 n_2}} |N_1, N_2\rangle \langle N_1, N_2| - \frac{E_J}{2}(Q_+^{(1)} + Q_-^{(1)}) \otimes \hat{1} - \frac{E_J}{2} \hat{1} \otimes (Q_+^{(2)} + Q_-^{(2)})$$

- Two-state approximation

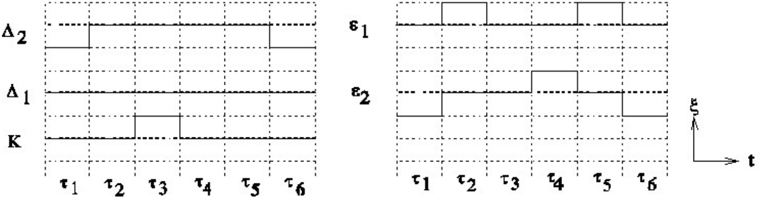
$$H = \frac{1}{4}\left[E_m(1 - 2n_{g2}(t)) + 2E_{c1}(1 - 2n_{g1}(t))\right]\sigma_z^{(1)} - \frac{E_{J1}}{2}\sigma_x^{(1)} \frac{1}{4}\left[E_m(1 - 2n_{g1}(t)) + 2E_{c2}(1 - 2n_{g2}(t))\right]\sigma_z^{(2)} - \frac{E_{J2}}{2}\sigma_x^{(2)} + \frac{E_m}{4}\sigma_z^{(1)} \otimes \sigma_z^{(2)}$$

Discretized CNOT quantum circuit.





For Ising interaction strength *K*



Needs more controls than available — also long pulse sequence.

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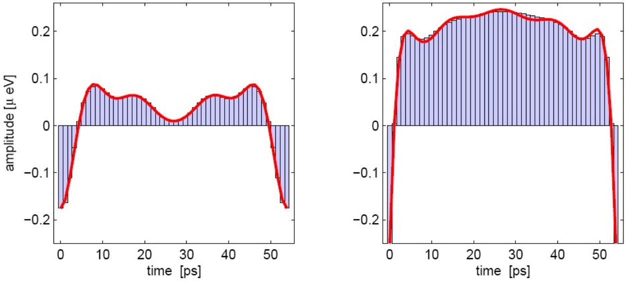
The GRAPE pulse



- 99.9999% precision (benchmark 70 %), short time
- *Palindrome pulse* $n_i(t) = n_i(T - t)$, as H is real and $U_{\mathrm{CNOT}} = U_{\mathrm{CNOT}}^{-1}$ [1]
- $T \simeq \pi / E_J = 55ps$: Local π pulses with phase gate: *Strong couling quantum control*

[1] see C. Griesinger, C. Gemperle, O. W. Sørensen, and R. R. Ernst, Molec. Phys. 62, **295** (1987).

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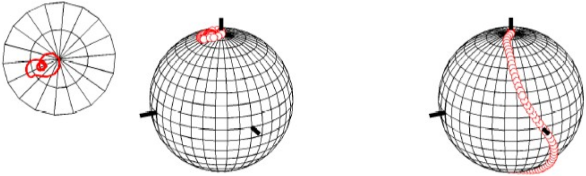
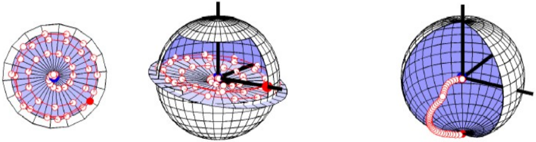
Dynamics under this pulse

Reduced Bloch spheres $\rho_i = \mathrm{Tr}\rho_{\neg i}$

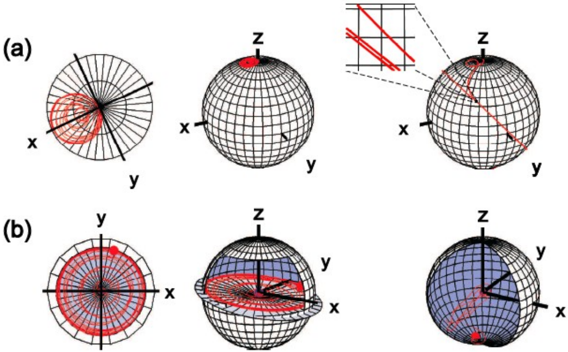


$|11\rangle \rightarrow |10\rangle$



$|00\rangle + |11\rangle \rightarrow (|0\rangle + |1\rangle) \otimes |0\rangle$

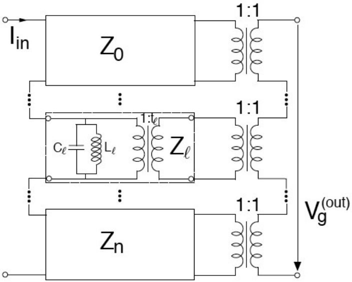
NECs evolution: Multiple loops



How to make such a pulse?

Time scale beyond current pulse generators.

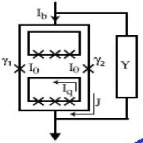
Broadband pulse (electro–optical) → Fourier Filter (Cauer) → Setup transfer function → Sample



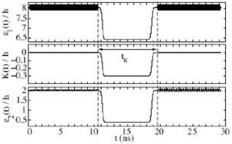
Input pulse $I(t) = f(t) - f(t - T)$ of arbitrary shape, rational approximation in Laplace space

Pulse optimization

Sample design

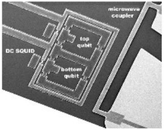


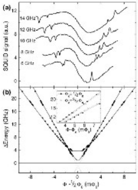
Pulse optimization

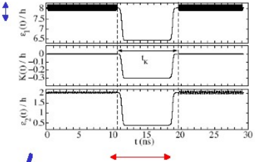


Control design

Fabrication → Spectroscopy → Pulse adjustment (30 s)

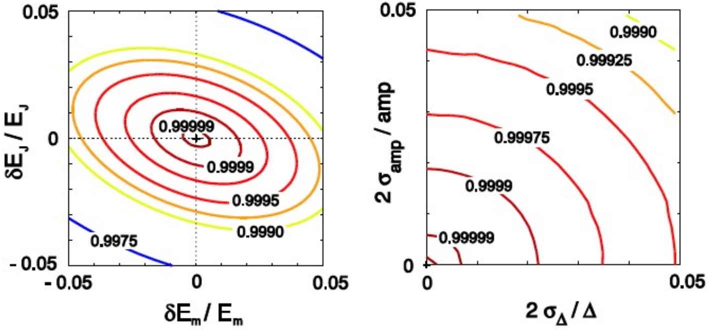






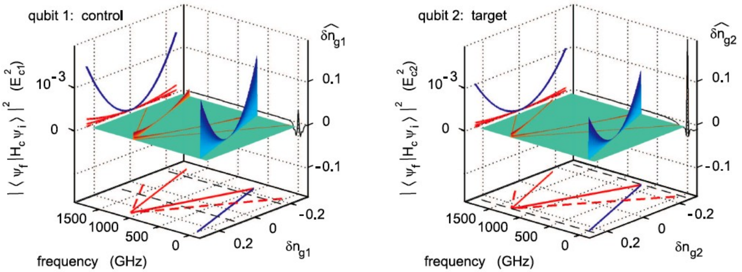
Quantum computing

Fault tolerance



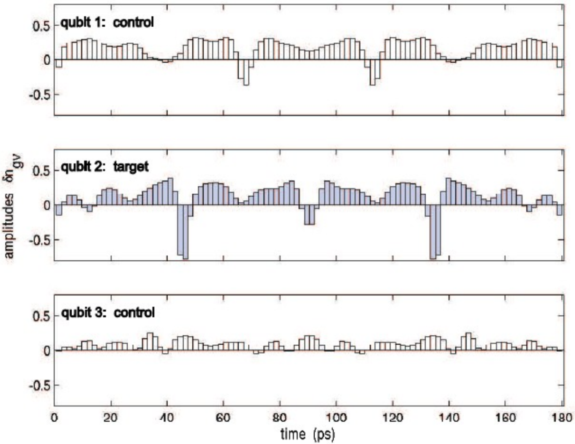
N.b.: Minimum makes errors $\propto (\delta u)^2$

Low leakage



High nonlinearity: With leakage $F = 99\%$

One-step Toffoli



Strong coupling leads to further acceleration

Summary

- Optimal control theory is a power tool for constructing pulses from Hamiltonians
- Leakage can be avoided in phase qubits
- Ultrafast CNOT in coupled Cooper pair boxes.

- Outlook
  - Help for experimental implementation
  - Optimization in the presence of decoherence

See also Poster by P. Rebentrost

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For Further Reading I

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