Non completely positive maps

in physics

UNIWERSYTET
JAGIELLOŃSKI
W KRAKOWIE

Daniel R Terno
Karol Życzkowski
Hilary Carteret

History, references, etc

H. Carteret, D. R. Terno, K. Życzkowski, quant-ph/0512167

P. Pechukas, Phys. Rev. Lett. **73**, 1060 (1994)
R. Alicki, Phys. Rev. Lett. **75**, 3020 (1995)

P. Štelmachovič and V. Bužek, Phys. Rev. A **64**, 062106 (2001)

T. F. Jordan, A. Shaji, and E. C. G. Sudarshan, Phys. Rev. A **70**, 052110 (2004).

K. Życzkowski and I. Bengtsson, Open. Sys. Info. Dyn. **11**, 3 (2004).

K. M. Fonseca Romero, P. Talkner, and P. Hänggi, Phys. Rev. A **69**, 052109 (2004)
Y. S. Weinstein, T. F. Havel, J. Emerson, N. Boulant, M. Saraceno, S. Lloyd, and D. G. Cory, J. Chem. Phys. **121**, 6117 (2004).

Maps, extensions & non-linearity

Explicit form of a non-unitary evolution: extensions

$$E(\rho_A) = \rho_A \otimes \rho_B^0$$

$$\& \; U$$

CP map

$$E(\rho_A) = \rho_{AB}$$

$$\mathrm{tr}_B \rho_{AB} = \rho_B$$

$$E_{\mathbf{V}}(\rho_A) = \rho_{AB} \;\; \forall \rho_A \in \mathbf{V}$$

$$\& \; U$$

Non- CP map

$$\rho_{AB} = \frac{1}{d_A d_B}\left(1_A \otimes 1_B + \alpha_i \sigma_i^A \otimes 1_{B\backslash} + \beta_j 1_A \otimes \sigma_j^B + \gamma_{ij} \sigma_i^A \otimes \sigma_j^B\right)$$

$$\rho_A \otimes \rho_B \;\; \Leftrightarrow \;\; \gamma_{ij} = \alpha_i \beta_j$$

Non-CP maps: definitions

Defs: $\Xi(\rho)$ Non-CP map with domain of positivity W

- assignment map $E_V(\rho_A) \quad \forall \rho \in V \subset W$

  $E_V(\rho_A) = \rho_{AB}$
- unitary transformation U

$\Xi(\rho)$ is physically accessible

Without restrictions it is trivial

$\Xi(\rho) = \rho^T$: $E(\rho_A) = \rho_A \otimes \rho_A^T$ & U=SWAP

A warning & three conclusions

Assignments: definitions

Linear assignments: motivated by (uncontrolled) interactions

$$\beta_j = \beta_j^0 + \vec{B}_j \cdot \vec{\alpha}, \quad \gamma_{ij} = \gamma_{ij}^0 + \vec{C}_{ij} \cdot \vec{\alpha}$$

Non-linear assignments: motivated by quantum comp hardware

$$\beta_j = \beta_j^0 + \varepsilon \beta_j^1(\vec{\alpha}), \quad \gamma_{ij} = \alpha_i \beta_j^0 + \varepsilon \gamma_{ij}^0(\vec{\alpha}) \qquad \varepsilon \ll 1$$

Example $\quad E(\rho_A) = \dfrac{1}{4_B}\left(1_A \otimes 1_B + \alpha_i \sigma_i^A \otimes 1_{B\backslash} + a \sigma_i^A \otimes \sigma_i^B\right)$

$$\rho_A = \tfrac{1}{2}(1_A + \alpha_i \sigma_i^A) \qquad 0 \le |a| \le a_{\max} = \left(\sqrt{4 - 3|\alpha|^2} - 1\right)\Big/3$$

Maps & dynamical matrices

Dynamical (Choi) matrix $\rho' = \Phi(\rho)$ $\quad \rho_{mn} = D(\Phi)_{ms,nt}\,\rho_{st}$

$$\rho_{mn} \to r_b$$

$$D^R_{ms;nt} \equiv D_{mn;st} \to \bar{D}_{ab}$$

Properties

- Hermitian $\quad \left(D^\dagger\right)_{ms;nt} = D^*_{nt;ms}$
- If trace-preserving $\quad \sum_m D_{ms;mt} = \delta_{st} \quad \Rightarrow \sum_a \lambda_a = d$
- If unital $\quad \Phi(1) = 1 \quad \Rightarrow \sum_s D_{ms;ns} = \delta_{mn}$
- Choi theorem $\quad D \geq 0, \text{ iff } \Phi \text{ is CP}$

$$(\Phi_A \otimes 1_B)\rho_{AB} \geq 0$$

Maps & dynamical matrices

Eigendecomposition $$D_{ms;nt} = \sum_a \lambda_a (M_a)_{ms} (M_a^\dagger)_{nt}$$

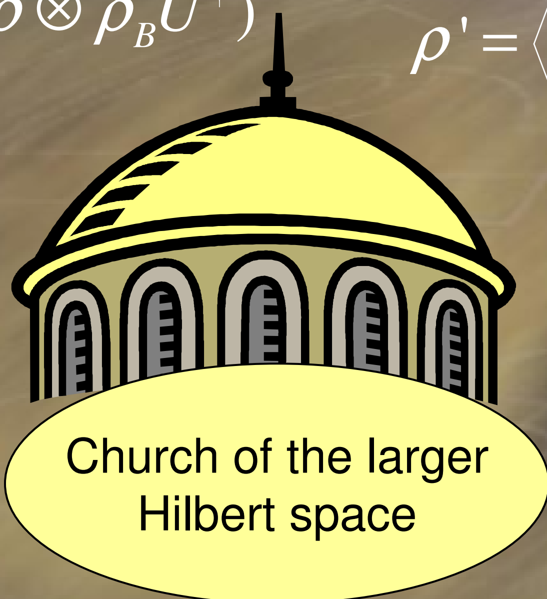
Kraus form of a CP map $$\rho' = \Lambda(\rho) = \sum_a M_a \rho M_a^\dagger$$

Difference form $$\rho' = \Lambda_1(\rho) - \Lambda_2(\rho)$$

$$\Lambda(\rho) = \mathrm{tr}_B(U \rho \otimes \rho_B U^\dagger)$$

$$\rho' = \langle \mu | \sqrt{p_\nu} U | \nu \rangle \rho_A \langle \nu | \sqrt{p_\nu} U^\dagger | \mu \rangle$$

$$M_{\mu\nu} = \langle \mu | \sqrt{p_\nu} U | \nu \rangle$$



Non-CP maps: physics

$$\rho_{AB} = \frac{1}{d_A d_B}\left(1_A \otimes 1_B + \alpha_i \sigma_i^A \otimes 1_{B\backslash} + \beta_j 1_A \otimes \sigma_j^B + \gamma_{ij}\sigma_i^A \otimes \sigma_j^B\right)$$

Unitary evolution & partial trace

$$\Xi(\rho_A) = \sum_{\mu,\nu} M_{\mu\nu}\rho_A M_{\mu\nu}^\dagger + \sum_{\mu}\langle\mu|\Gamma_{ij} U \sigma_i^A \otimes \sigma_j^B U^\dagger|\mu\rangle$$

$$\Gamma_{ij} = \left(\gamma_{ij} - \alpha_i\beta_j\right)/d_A d_B$$

$$\boxed{\rho'_{mn} = G_{ms;nt}\rho_{st} + \boldsymbol{\xi}\cdot\boldsymbol{\sigma}_{mn}}$$

$$D_{ms;nt} = G_{ms;nt} + \boldsymbol{\xi}\cdot\boldsymbol{\sigma}_{mn}\delta_{st}$$

$$G_{ms;nt} = \sum_{\mu,\nu}\left(M_{\mu\nu}\right)_{ms}\left(M_{\mu\nu}^\dagger\right)_{mt}$$

$$\xi_i\sigma_i^A = \sum_{\mu}\langle\mu|\Gamma_{ij} U \sigma_i^A \otimes \sigma_j^B U^\dagger|\mu\rangle$$

Non-CP maps: examples

$$E(\rho_A) = \frac{1}{4_B}\left(1_A \otimes 1_B + \alpha_i \sigma_i^A \otimes 1_{B\backslash} + a\sigma_i^A \otimes \sigma_i^B\right)$$

$$U_1 = \begin{pmatrix} e^{i\theta}\cos\phi_1 & 0 & 0 & ie^{i\theta}\sin\phi_1 \\ 0 & \cos\phi_2 & i\sin\phi_2 & 0 \\ 0 & i\sin\phi_2 & \cos\phi_2 & 0 \\ ie^{i\theta}\sin\phi_1 & 0 & 0 & e^{i\theta}\cos\phi_1 \end{pmatrix}$$

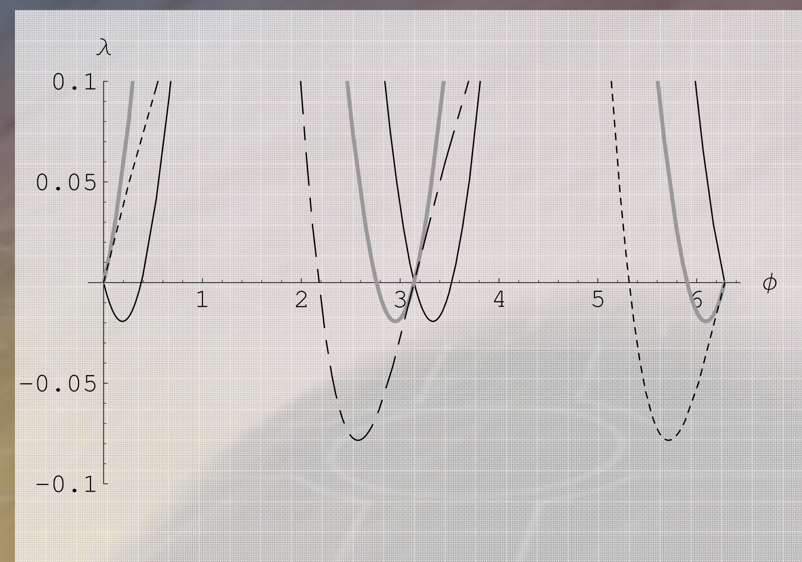
$$\left[U, \sigma_i^A \otimes \sigma_i^B\right] = 0$$

CP map

$$U_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xi \cdot \boldsymbol{\sigma} = \tfrac{1}{2}\begin{pmatrix} a\sin 2\phi & 0 \\ 0 & -a\sin 2\phi \end{pmatrix}$$

so what?



Entanglement

PPT test:

$$\forall \rho: \quad a_{\text{ent}} > a_{\max}$$

No entanglement
both before & after

Necessary condition 1

$\Xi(\rho)$ is physically accessible through linear extension, then

$$\Xi(\rho) = \sum_a M_a \rho M_a^\dagger + \xi_i \sigma_i$$

- Finite-dim extension: trivial
- Infinite-dim extension: no decomposition à la Bloch

$$\rho_{AB} = [\rho_A \otimes \rho_B] + [\rho_{AB} - \rho_A \otimes \rho_B]$$

$$\rho'_{AB} = U[\rho_A \otimes \rho_B]U^\dagger + U[\rho_{AB} - \rho_A \otimes \rho_B]U^\dagger$$

$$\rho'_A = \Lambda(\rho_A) + \text{traceless part}$$

Example: transposition

$$D_T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$D_T(\rho) = G\rho + \boldsymbol{\xi} \cdot \boldsymbol{\sigma}$$

$$G(\boldsymbol{\xi}) > 0$$

Difference form:

$$T(\rho) = -(i\sigma_y)\rho(i\sigma_y)^\dagger / 2 + (\sigma_+\rho\sigma_+ + \sigma_-\rho\sigma_- + \sigma_x\rho\sigma_x)$$

Negative eigenvalue:

$$\lambda_- = -\sqrt{1+\boldsymbol{\xi}^2}$$

Necessary condition 2

$\Xi(\rho)$ is physically accessible with state-independent affine form & unital, then it is CP

$$\Xi(\rho) = \Lambda(\rho) + \boldsymbol{\xi} \cdot \boldsymbol{\sigma}$$

$$\Lambda(\rho) = \Lambda_0(\rho) + \boldsymbol{\varsigma} \cdot \boldsymbol{\sigma}$$

$$\Xi(\rho) = \Lambda_0(\rho)$$

Non-linear assignment...

$$\beta_j = \beta_j^0 + \varepsilon\beta_j^1(\vec{\alpha}), \quad \gamma_{ij} = \alpha_i\beta_j^0 + \varepsilon\gamma_{ij}^0(\vec{\alpha}) \qquad \varepsilon \ll 1$$

$$H_{AB} = H_A \otimes 1 + \eta H_{\text{int}},$$

At the first order of perturbation theory the reduced dynamics is linear and CP

Proof:

$$U_{AB} = U \otimes 1 - i\eta t\left(\sum_{ij} h_{ij} U_A \sigma_i^A \otimes \sigma_j^B + \sum_{ijkl} h_i h_{jk} C_{lij} U_A \sigma_l^A \otimes \sigma_j^B / 2\right) +$$

+affine form + perturbation theory

Summary & questions

- Accessibility= (interesting) extension & unitary
- Necessary conditions for linear extension
- No entanglement is required
- May not occur
- Good gates=> linear & CP
- Variety of systems…

Sufficient conditions
Non-linear & non-Markovian
Channels