Weak Measurements and Feedback

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Measurement and disturbance

In quantum mechanics, one of the fundamental principles is that any measurement which gives information about a quantum system must in general also disturb that system in some way. The canonical example of this is Heisenberg's Uncertainty principle:

$$\left(\Delta x^2\right)\left(\Delta p^2\right) \geq \frac{{}^{2}}{4}$$

A measurement of x disturbs the value of p, and vice versa. This relationship between complementary variables is general for all quantum measurements.

Projective measurements

The original formulation of measurement can be written in terms of orthogonal projectors:

$$\sum_j \hat{P}_j = \hat{I}, \qquad \hat{P}_j \hat{P}_k = \delta_{jk} \hat{P}_j,$$

$$|\psi\rangle \to \hat{P}_j |\psi\rangle / \sqrt{p_j}, \qquad p_j = \langle \psi | \hat{P}_j | \psi \rangle.$$

These projection operators project onto eigenspaces of an Hermitian operator (an observable):

$$\hat{O} = \sum_j o_j \hat{P}_j.$$

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Generalized Measurements

In QIP, a broader formulation of measurement is used: generalized measurements, which are not restricted to using projectors. Given any set of operators which obey the requirements

$$\sum_j \hat{M}_j^\dagger \hat{M}_j = \hat{I},$$

we can (in principle) carry out a measurement procedure:

$$|\psi\rangle \rightarrow \hat{M}_j|\psi\rangle/\sqrt{p_j}, \quad p_j = \langle\psi|\hat{M}_j^\dagger \hat{M}_j|\psi\rangle.$$

These generalized measurements are a very broad class of operations. They include projective measurements, as a special case, but they also include operations which cannot really be called measurements at all, since they provide no information about the state of the system. For instance, they include unitary transformations as a special case (when the set of operators has only one member), and operations like

$$\hat{M}_0 = \sqrt{1/2}\hat{I} = \hat{M}_1.$$

If a generalized measurement does give information, however, it must disturb the state.

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Weak Measurements

If every measurement that gives information about the system must disturb the state of the system, one might ask if there is a necessary relationship between the amount of information gained and the amount of disturbance? By gaining only a small amount of information about the system, can we disturb the system only very little?

It turns out that this is indeed the case. We can find measurements which disturb the system very little, but give only very little information. We call such measurements weak.

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Remember our example of a generalized measurement that gives no information– all the measurement operators were proportional to the identity:

$$\hat{M}_0 = \sqrt{1/2}\hat{I} = \hat{M}_1.$$

We can use the same idea to construct weak measurements. We simply require that all the measurement operators be close to proportional to the identity:

$$\hat{M}_j = \sqrt{q_j\left(I + \hat{\varepsilon}_j\right)}$$

$$q_j \geq 0, \quad \sum_j q_j = 1, \quad \left\|\hat{\varepsilon}_j\right\| \ll 1, \quad \sum_j \hat{\varepsilon}_j = 0.$$

We can decompose any projective measurement into a sequence of weak measurements. Consider a two-outcome projective measurement:

$$\hat{P}_0 + \hat{P}_1 = \hat{I},\ \hat{P}_0 \hat{P}_1 = 0.$$

We can define a weak generalized measurement with operators

$$\hat{M}_\pm = \sqrt{\frac{1 \pm \tanh(\varepsilon)}{2}}\hat{P}_0 + \sqrt{\frac{1 \mp \tanh(\varepsilon)}{2}}\hat{P}_1 \approx \sqrt{1/2}\left(\hat{I} \pm \frac{\varepsilon}{2}\left(\hat{P}_0 - \hat{P}_1\right)\right)$$

Doing this weak measurement repeatedly is equivalent at long times to doing the original projective measurement.

How can we see that this procedure gives the right outcomes?

Let's write down our initial state

$$|\psi\rangle = \sqrt{p_0}|\psi_0\rangle + \sqrt{p_1}|\psi_1\rangle,$$

where we have resolved the state into its orthogonal components

$$|\psi_0\rangle = \hat{P}_0|\psi\rangle / \sqrt{\langle\psi|\hat{P}_0|\psi\rangle},\ |\psi_1\rangle = \hat{P}_1|\psi\rangle / \sqrt{\langle\psi|\hat{P}_1|\psi\rangle}.$$

If we define a parameter

$$x = ln\sqrt{p_0/p_1},$$

we see that for small ε at each step the parameter changes by

$$x \rightarrow x \pm \varepsilon,$$

and that for very large/very negative x the state approaches

$$|\psi_0\rangle, |\psi_1\rangle.$$

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Generalized from Weak Measurements

If we can do any projective measurement as a sequence of weak measurements, can we do any generalized measurement as a sequence of weak measurements? Is it possible to find a procedure, similar to the one for projective measurements, which will work for any generalized measurement?

It turns out that it is indeed possible– but the choice of which measurement to do at each step will depend on the outcome of earlier measurements. In other words, our measurement procedure requires feedback. It is therefore a type of quantum control procedure.

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Consider a generalized measurement with two positive measurement operators. We would like to decompose this into a sequence of weak measurements, just as we did for projective measurements.

Define the operator:

$$\hat{M}(x) = \sqrt{\frac{\hat{I} + \tanh(x)\left(\hat{M}_2^2 - \hat{M}_1^2\right)}{2}}$$

In the limit of large |x|, this operator approaches

$$\hat{M}(x) \to \hat{M}_{1,2}, \quad x \to \infty.$$

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We can now define a weak measurement parametrized by x:

$$\hat{M}_{\pm}(x) = \sqrt{C_{\pm} \frac{\hat{I} + tanh(x \pm \varepsilon)\left(\hat{M}_2^2 - \hat{M}_1^2\right)}{\hat{I} + tanh(x)\left(\hat{M}_2^2 - \hat{M}_1^2\right)}}$$

where

$$C_{\pm} = \left(1 \pm tanh(\varepsilon)\, tanh(x)\right)/2.$$

- It is not hard to check that for small ε this is, indeed, a weak measurement; and moreover, that

$$\hat{M}_{\pm}(x)\hat{M}(x) \propto \hat{M}(x \pm \varepsilon).$$

We now see how the procedure works. At each step, we characterize the current state by the value of x. We do the weak measurement, and adjust x by $\pm\varepsilon$. The state at any given time is

$$\left|\psi(x)\right\rangle \propto \hat{M}(x)\left|\psi\right\rangle.$$

The state follows a random walk along a curve in state space, parametrized by x, whose end points represent the outcomes of the measurement. We continue this procedure until the parameter x exceeds some large threshold $\pm X$, and then halt. The probabilities of the two possible outcomes are

$$p_{\pm} = \left\langle\psi\right|\hat{M}_{2,1}^{2}\left|\psi\right\rangle.$$

What is it all good for?

I'm glad you asked that.

Entanglement Monotones

An entanglement monotone is a function f(ρ) of the state which is nonincreasing on average under any local operation. That is,

$$f(\rho) = f\left(\hat{U}\rho\hat{U}^{\dagger}\right) \quad f(\rho) \geq p_1 f\left(\hat{M}_1\rho\hat{M}_1^{\dagger}/p_1\right) + p_2 f\left(\hat{M}_2\rho\hat{M}_2^{\dagger}/p_2\right)$$

where both the unitary and the measurement are assumed to be local. Since both measurements and unitaries can now be done continuously, it is necessary and sufficient to show that a function is a monotone under weak local operations. This gives a set of differential conditions for an entanglement monotone.

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First, the function must be invariant under local unitary transformations. By expanding U $\approx$ I - iε we can derive a differential condition for local unitary invariance:

$$Tr\left\{\frac{\partial f}{\partial \rho}\left[\hat{\varepsilon},\rho\right]\right\}=0,\ \forall\hat{\varepsilon}=\hat{\varepsilon}^{\dagger},\ \forall\rho.$$

Similarly, we can find a (more complicated) condition for monotonicity under local measurements:

$$\frac{1}{8}Tr\left\{\frac{\partial f}{\partial \rho}\left[\left[\hat{\varepsilon},\rho\right],\hat{\varepsilon}\right]\right\}+Tr\left\{\frac{\partial^2 f}{\partial \rho^{\otimes 2}}\left(\langle\hat{\varepsilon}\rangle\rho-\frac{1}{2}\{\hat{\varepsilon},\rho\}\right)^{\otimes 2}\right\}\leq 0.$$

Note that ε is a local Hermitian operator, $\varepsilon = \varepsilon^{t} = \varepsilon \otimes I$.

These two conditions together are necessary and sufficient for f to be an entanglement monotone under local operations that preserve pure states (provided f is sufficiently differentiable). If we wish to include information-losing operations as well, we also need the convexity condition

$$f\left(\sum_j p_j \rho_j\right) \leq \sum_j p_j f(\rho_j).$$

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Continuous Error Correction

Error correction consists of several steps: first, encoding the system with an error correcting code; then periodically measuring the error syndromes S_j, and performing appropriate corrections U_j. This procedure can be written as a quantum operation:

$$\rho \rightarrow \sum_j U_j S_j \rho S_j U_j^t$$

Since both the measurements and the corrections can be done weakly, one can transform this into a continuous process, where in any one step dt the operation is

$$\rho \rightarrow (1-\varepsilon)\rho + \varepsilon \sum_j U_j S_j \rho S_j U_j^t$$

So long as the rate ε/dt is large compared to the rate of errors, this procedure will improve over no correction. (First proposed by Paz and Zurek, 1996!)

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Experimental realization

This decomposition into weak measurements proves that, in principle, weak measurements are universal. But can such a procedure be done in practice? It certainly can for projective measurements; what about generalized measurements?

One system that might allow such procedures is the electromagnetic field in a superconducting cavity, probed repeatedly by Rydberg atoms (Haroche and Raimond). This system contains a number of continuously adjustable parameters: the initial state of the incoming atoms, their velocity (and hence interaction time in the cavity), and the final measurement performed on them.

QuickTime™ and a
TIFF (Uncompressed) decompressor
are needed to see this picture.

Other experimental systems may also allow decompositions into weak measurements. For example, it may be possible to do balanced homodyne detection while continuously varying the parameters of the system, such as the phase of the local oscillator, or even the reflectivity of the cavity mirror.

An open question is the inverse problem: given an experimental setup which allows weak or continuous measurements, with some set of controllable parameters and feedback, what class of generalized measurements can be achieved?

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Conclusions

- It is possible to decompose any generalized measurement into a sequence of weak measurements, which have the form of a random walk along a curve in state space. Weak measurements are universal.
- Any entanglement monotone which is sufficiently differentiable must obey a set of differential conditions.
- It may be possible to carry out these procedures for some classes of measurements using well-known experimental systems. But it remains an open question what generalized measurements can be accomplished by a given measurement scheme with feedback.

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References

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There is an extensive literature on weak measurements from various points of view.

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It is possible to decompose measurements with n possible outcomes into a sequence of measurements with only two outcomes each. It is easy to see how to do this. Suppose we have three measurement operators,

$$\hat{M}_0, \hat{M}_1, \hat{M}_2.$$

We can then define two new measurements:

$$\hat{L}_0 = \hat{M}_0, \hat{L}_1 = \sqrt{\hat{I} - \hat{M}_0^\dagger \hat{M}_0}; \quad \hat{N}_0 = \hat{M}_1 \hat{L}_1^{-1}, \hat{N}_1 = \hat{M}_2 \hat{L}_1^{-1}.$$

Do the first measurement; if you get outcome 1, do the second measurement. The end result is the same as doing the original measurement. (If the operators are not invertible it can still be done, but is slightly more complicated.)

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Quantum State Diffusion Equation

For projective measurements, in the limit where the step size ε becomes infinitesimal, we can describe the evolution of the state by a Stochastic Schrödinger equation:

$$|d\psi\rangle = -\frac{\sigma^2}{2}\left(\hat{Z} - \langle\hat{Z}\rangle\right)^2|\psi\rangle dt + \sigma\left(\hat{Z} - \langle\hat{Z}\rangle\right)|\psi\rangle dW,$$

$$M[dW] = 0, M[dW^2] = dt. \quad \sigma = \sqrt{\varepsilon^2/dt}.$$

This equation is called the Quantum State Diffusion Equation with real noise, and can be considered a model of a continuous measurement process for the observable

$$\hat{Z} = \hat{P}_0 - \hat{P}_1.$$

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In the infinitesimal limit, we can find a stochastic Schrödinger equation for generalized measurements as well. But now, we must also keep track of the evolution of the parameter x:

$$|d\psi\rangle = -\frac{\sigma^2}{2}\left(\hat{Z}(x) - \langle \hat{Z}(x)\rangle\right)^2 |\psi\rangle dt + \sigma\left(\hat{Z}(x) - \langle \hat{Z}(x)\rangle\right)|\psi\rangle dW ,$$

$$dx = -2\sigma^2 \langle \hat{Z}(x)\rangle dt + \sigma\, dW ,$$

$$\hat{Z}(x) = \frac{1}{2}\frac{\left(\hat{M}_2^2 - \hat{M}_1^2\right) + \tanh(x)\hat{I}}{\hat{I} + \tanh(x)\left(\hat{M}_2^2 - \hat{M}_1^2\right)}.$$