Building new states of matter with polar molecules: a route toward topological order

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Motivation: Let's use ideas from QI to probe models of the natural world

Outline

- Surface codes as topological order
  - Protected quantum memory
  - Abelian anyonic excitations (realization of a $\mathbb{Z}_d$ lattice gauge theory)
- Proposed implementation with polar molecules
  - Structure of polar molecules
  - Engineering spin lattice models
  - Constructing a spin model on a honeycomb lattice having topologically protected ground states
  - Verification: Measuring anyonic statistics
- Extensions to spin one models
- Conclusions

I. Topologically protected q. memory

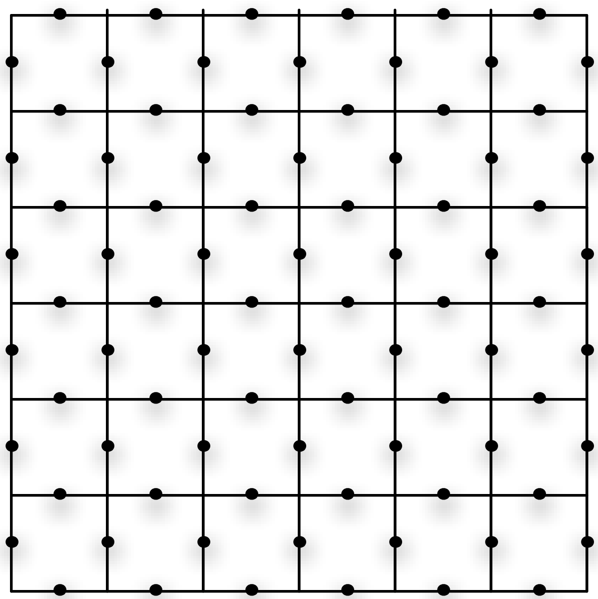
- **Idea: Isomorphism between spins and 1-chains (pieces of string) on a surface cellulation** $\Gamma = \Gamma(\mathcal{V}, \mathcal{E}, \mathcal{F})$
  - **e.g. n qubits on a square lattice**

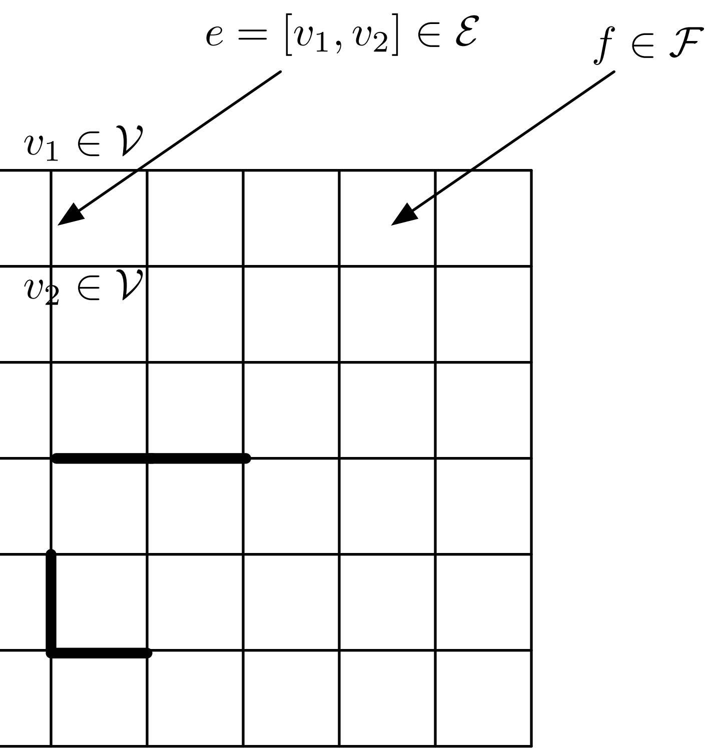
$$\mathcal{H} \cong (\mathbb{C}^2)^{\otimes n} \cong \mathbb{C}^{C_1(\Gamma, \mathbb{Z}_2)}$$

$$C_1(\Gamma, \mathbb{Z}_2) = \mathrm{span}_{\mathbb{Z}_2}(\mathcal{E})$$

no string (vacuum) $= 0$

string $= 1$





$$|\psi\rangle = \mathbf{1} \otimes \cdots \mathbf{1} \otimes X \otimes X \otimes \mathbf{1} \otimes X \otimes X \otimes \mathbf{1} \otimes \cdots \mathbf{1} |vac\rangle$$

  - **States invariant under loop operators in a given homology class are TO**

The stabilizer formulation

- **Want a Hamiltonian with vertex and face operators that commute**

$$H = -U\Big(\sum_{v\in\mathcal{V}} g_v + \sum_{f\in\mathcal{F}} g_f\Big)$$

$$g_v = \prod_{e\in\{[*,v],[v,*]\}} Z_e \qquad g_f = \prod_{e\in\partial f} X_e \qquad [g_v, g_{v'}] = [g_f, g_{f'}] = [g_v, g_f] = 0$$

  - **Generators of the stabilizer group G** $\quad G = \langle\{g_v, g_f\}\rangle$

- **Ground states of H are eigenstates of G with eigenvalue +1**

  - **dimension of this eigenspace**

$$\dim\,\mathcal{H}^G \;=\; \mathrm{Trace}\Big[\frac{1}{\#G}\sum_{g\in G} g\Big]$$

  - **Example**

    - For two qubits, define

$$G = \langle\{X_1X_2, Z_1Z_2\}\rangle = \{\mathbf{1}_4, X_1X_2, Z_1Z_2, (iY_1)(iY_3)\}$$

$$\mathrm{Trace}\Big[\frac{1}{\#G}\sum_{g\in G} g\Big] = \frac{4}{4} = 1 \qquad \mathcal{H}^G = |\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Qubits on a plane*

- **Ground state degeneracy for qubits on a p x q plane***

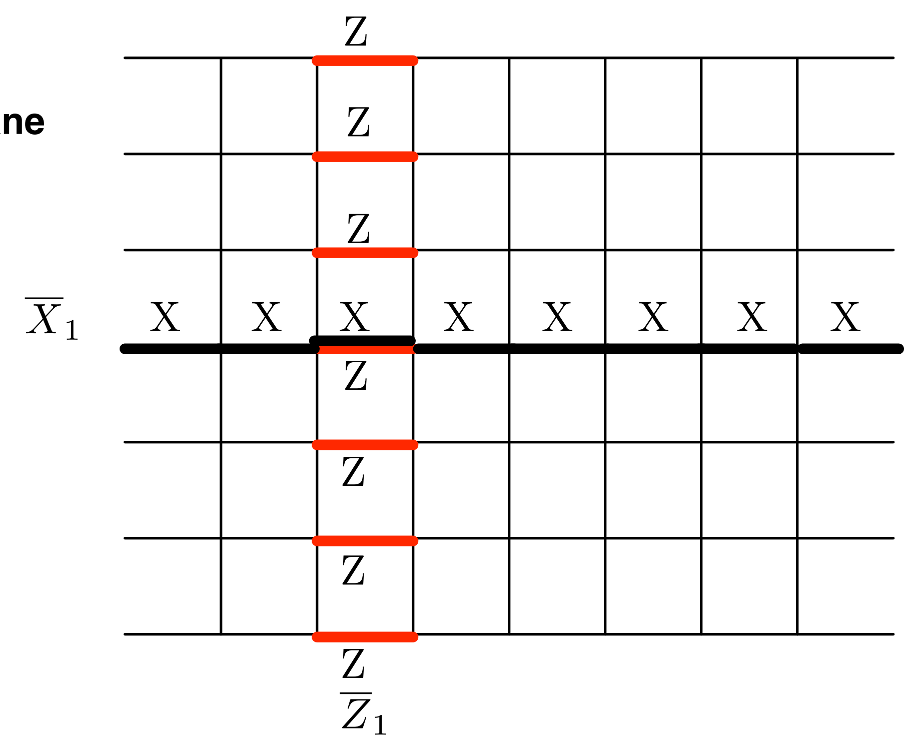
$$H = -U\Big(\sum_{+} Z_{e_1} Z_{e_2} Z_{e_3} Z_{e_4} + \sum_{\square} X_{e_1} X_{e_2} X_{e_3} X_{e_4}\Big)$$

$$\#\mathcal{V} = (p+1)q \qquad \#\mathcal{F} = p(q+1) \qquad \#\mathcal{E} = n = pq + (p+1)(q+1) = \#\mathcal{V} + \#\mathcal{F} + 1$$

$$\mathrm{dim}\mathcal{H}_{\mathrm{gr}} = \mathrm{Trace}\Big[\frac{1}{\#G}\sum_{g\in G} g\Big] = \mathrm{Trace}\Big[\frac{1}{2^{n-1}}\sum_{g\in G} g\Big] = 2$$

Can encode one qubit

6×7 **plane**



$$\{\overline{X}_1, \overline{Z}_1\} = 0$$

$$[H, \overline{X}_1] = [H, \overline{Z}_1] = 0$$

*A.Yu. Kitaev, Annals of Physics, **303**, 2 (2003); quant-ph/9707021

Slightly more generic construction using qudits*

- **Place a spin on each edge of lattice $\Gamma(\mathcal{V},\mathcal{E},\mathcal{V})$. Represent state space of each spin on a lattice by a qudit**

$$\mathcal{H}(1,d) = \mathbb{C}|0\rangle \oplus \cdots \oplus \mathbb{C}|d-1\rangle$$
$$\mathcal{H}(n,d) = \mathcal{H}(1,d)^{\otimes n} \qquad n = \#\mathcal{E}$$

Chain **Computational basis state**

$$\omega = \sum_{e\in\mathcal{E}} n_e e \ \leftrightarrow |\omega\rangle$$

- **Operator basis**

$$X\,|j\rangle = |j+1 \bmod d\rangle$$
$$Z\,|j\rangle = \xi^j\,|j\rangle\,, \qquad \text{for } \xi = \exp(2\pi i/d)$$
$$X^a Z^b = \xi^{a\cdot b} Z^b X^a$$

Pauli-group

$$\mathcal{P}(n,d) = \{\xi^c X^{\otimes \mathbf{a}} Z^{\otimes \mathbf{b}}, \mathbf{a},\mathbf{b} \in (\mathbb{Z}_d)^n\}$$

- **Vertex constraints**

$$g_v = \prod\nolimits_{e=[*,v]} Z_e \prod\nolimits_{e=[v,*]} Z_e^{-1}$$
$$H_v = -(g_v + g_v^\dagger)$$

For a lattice of valence k, this is of the form

$$Z^{\otimes k} + (Z^{-1})^{\otimes k}$$

- **Potential term** $H_\partial = U \sum_{v\in\mathcal{V}} H_v, \quad U>0$

  - **Claim.** $|\omega\rangle$ **is a ground state iff** $\partial\omega = 0$
  - **Check:** $g_v\,|\omega\rangle = \xi^c\,|\omega\rangle$ where $\partial\omega = cv + \sum_{w\neq v} c_w w$
  - **hence,** $|\omega\rangle$ **is in the stabilizer** $\langle\{g_v\}\rangle \subseteq \mathcal{P}(n,d)$ **iff** $|\omega\rangle$ **is an eigenstate of each** H_v
  - **with minimal eigenvalue iff** $|\omega\rangle$ **is a ground state of** H_∂

*SS Bullock and GKB in preparation

Construction of TO cont.

- **We're not there yet**
  - The ground states of H_∂ are superpositions of cycles, but they are not topologically ordered because the cycle space is not yet a topological invariant. Changing the cellulation changes the degeneracy.

- **Face constraints**

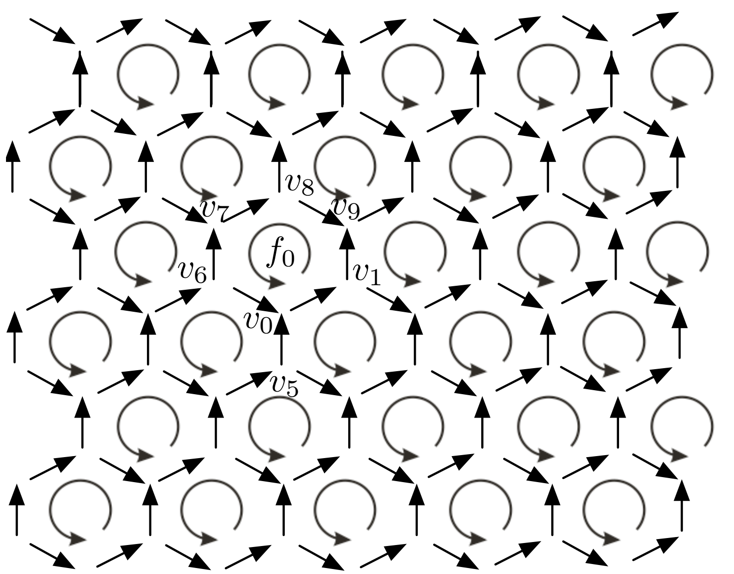
$$\begin{aligned} g_f &= X_{e_1}^{o_1} X_{e_2}^{o_2} X_{e_3}^{o_3} \dots X_{e_p}^{o_p} \\ H_f &= -(g_f + g_f^\dagger) \end{aligned}$$

$\partial f = \sum_{k=1}^{p} o_k e_k$ $\quad o_k \in \{1, d-1\}$ orientation (+/-)

Example:

$$g_{v_0} = Z_{[v_6,v_0]} Z_{[v_5,v_0]} Z_{[v_0,v_1]}^{-1}$$

$$g_{f_0} = X_{[v_0,v_1]} X_{[v_1,v_9]} X_{[v_8,v_9]}^{-1} X_{[v_7,v_8]}^{-1} X_{[v_6,v_7]}^{-1} X_{[v_6,v_0]}$$



- **Kinetic term**

$$H_{\mathrm{KE}} = g \sum_{f \in \mathcal{F}} H_f$$

- **Total Hamiltonian** $H = H_\partial + H_{\mathrm{KE}}$

$$\dim_{\mathbb{C}}(H_{\mathrm{gr}}) = \# H_1(\Gamma, \mathbb{Z}_d)$$

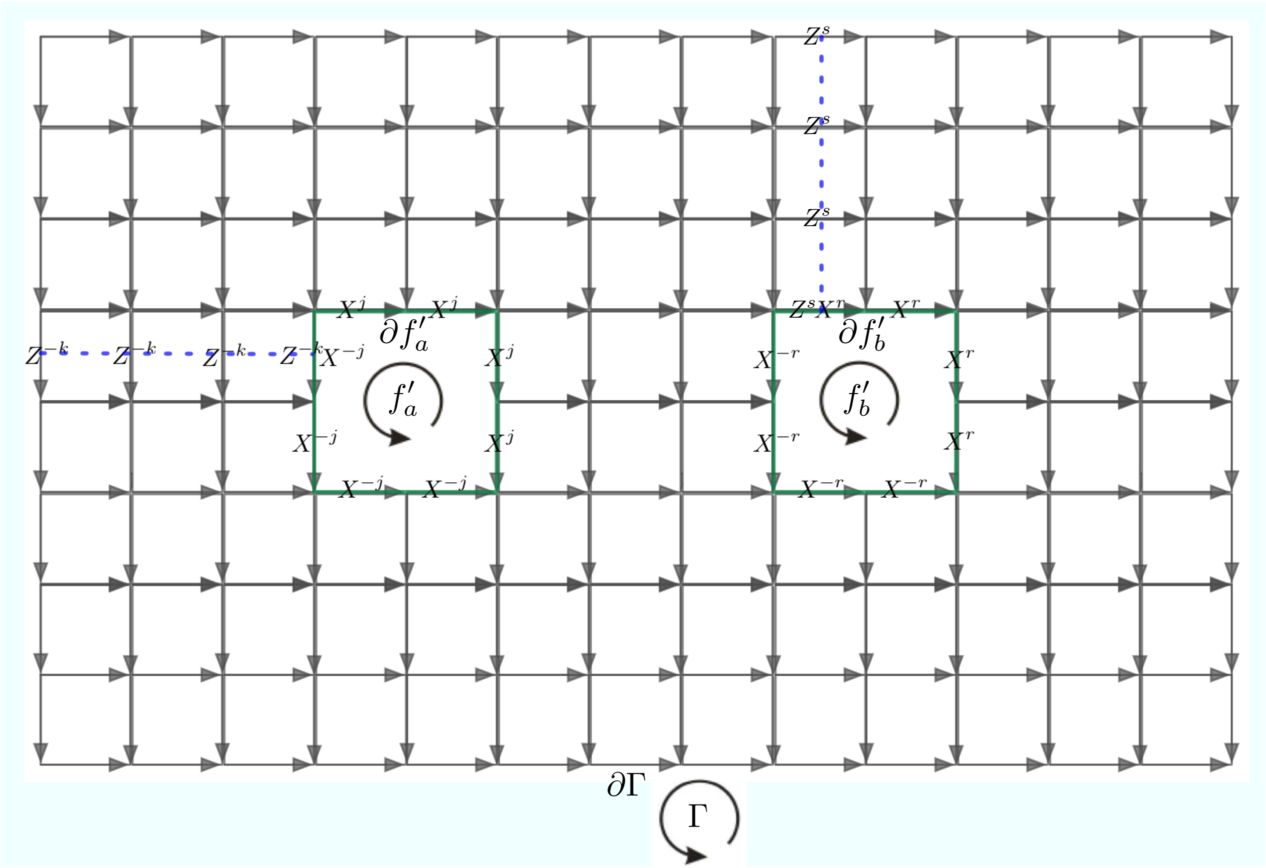
  - for a compact, connected, orientable surface of genus g,

$$H_1(\Gamma, \mathbb{Z}_d) = (\mathbb{Z}_d)^{2g}$$

  - ground subspace (code space)

$$\mathcal{H}_{\mathrm{gr}} \cong (\mathbb{C}^d)^{2g}$$

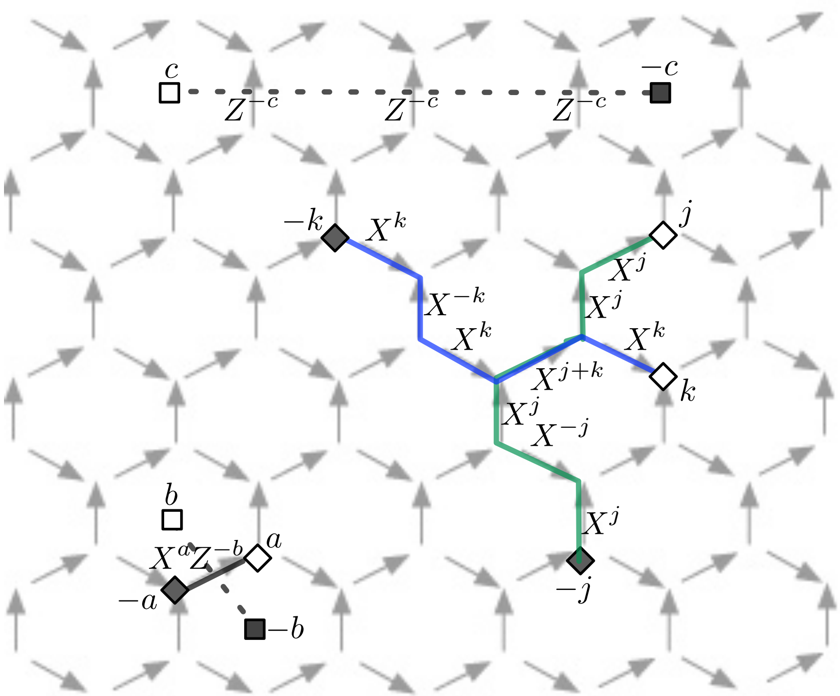
Example: 2 punctured plane encoding 2 qudits



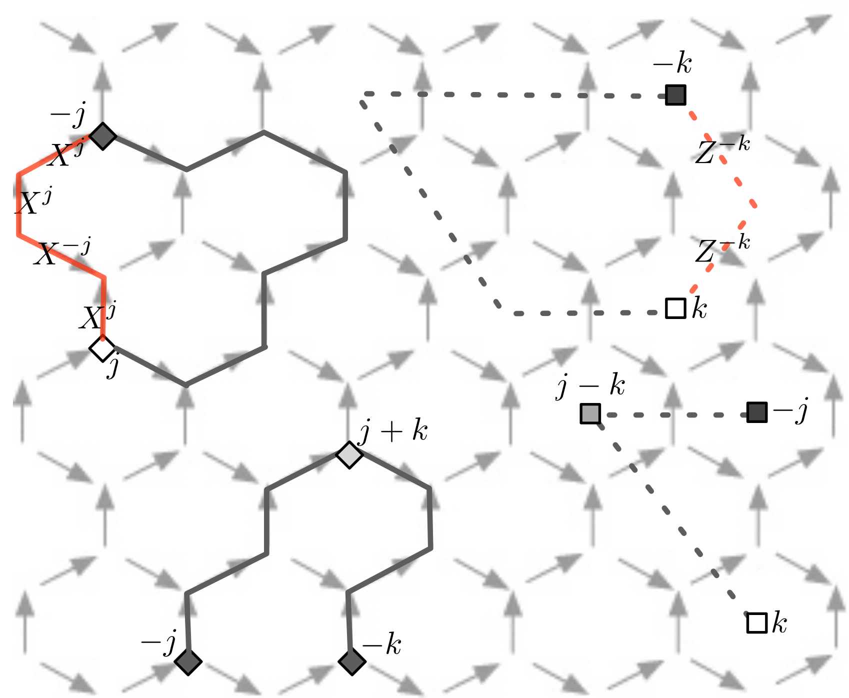
Excitations behave according to a $\mathbb{Z}_d$ gauge theory

Excitations come in particle anti particle pairs
Each particle a charge-flux dyonic combination $(a,b) \in \mathbb{Z}_d \times \mathbb{Z}_d$
Particle mass: $2U(1-\mathrm{Re}[\xi^a])+2h(1-\mathrm{Re}[\xi^b])$ $\xi = e^{i2\pi/d}$

Particle creation



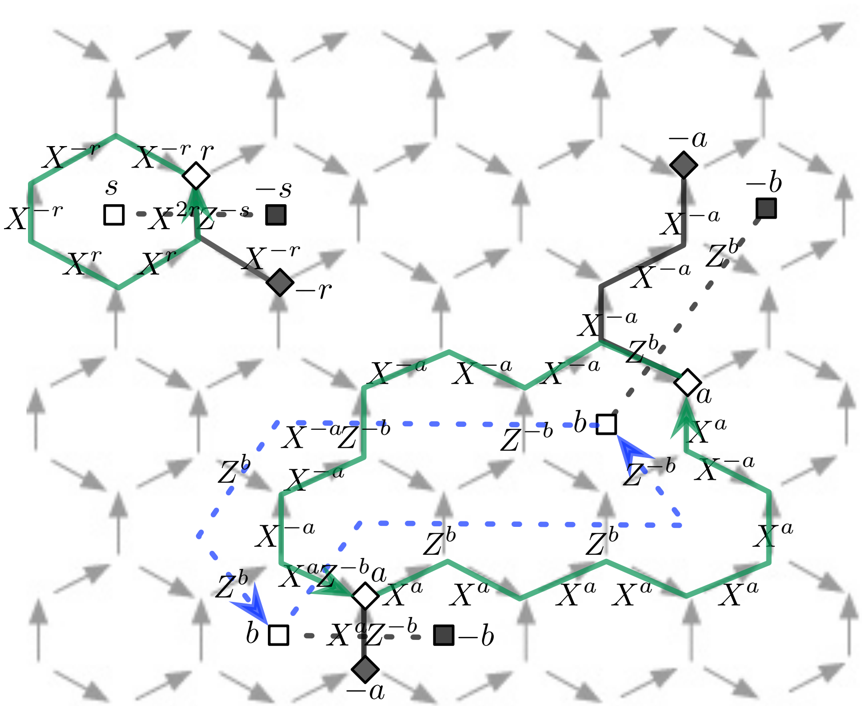
Fusion



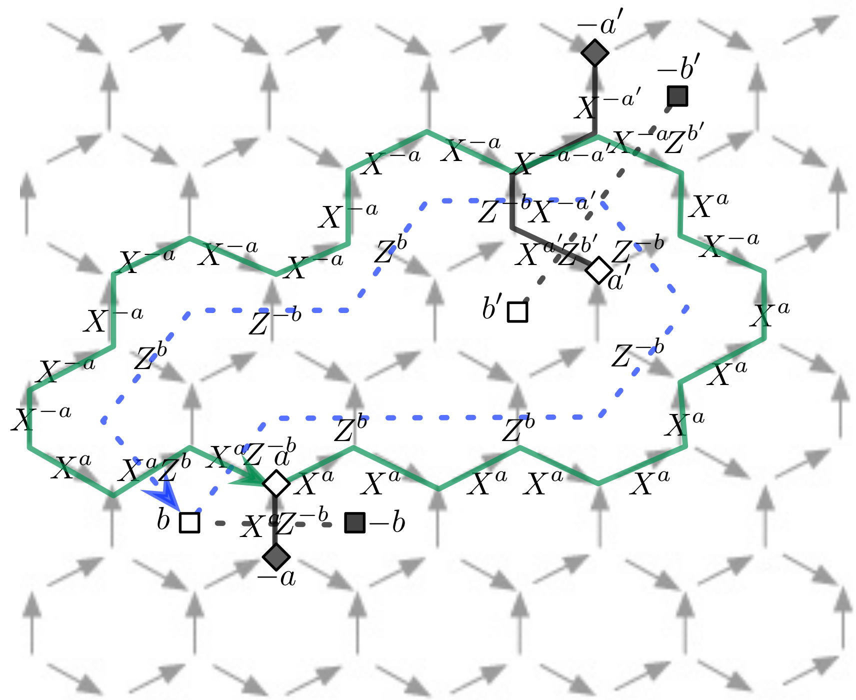
$$|(a,b;(v,f))\rangle \times |(a',b';(v,f))\rangle = |(a+a',b+b';(v,f))\rangle$$

Excitations cont.

Identical Particle exchange



Braiding

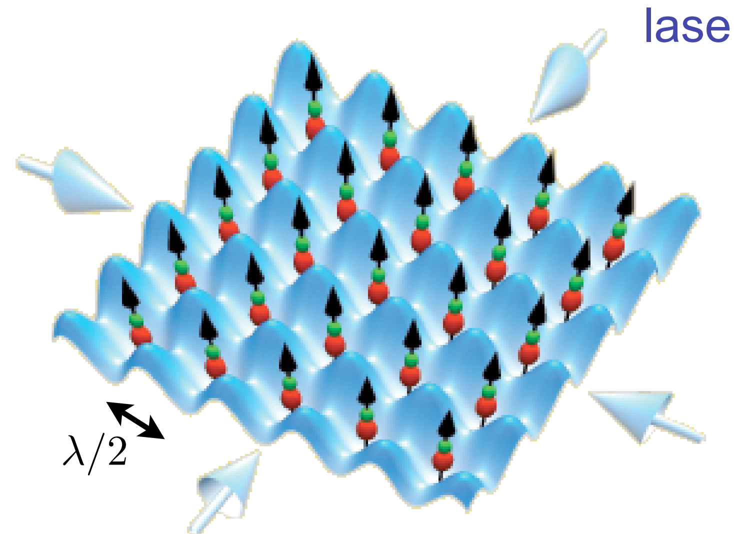


$$\mathcal{R}^2\,|(a,b)\rangle\,|(a',b')\rangle \;=\; \xi^{(a'b+b'a)}\,|(a,b)\rangle\,|(a',b')\rangle$$

$$\mathcal{R}\,|(a,b;(v,f))\rangle\,|(a,b;(v',f'))\rangle = \xi^{ab}\,|(a,b;(v,f))\rangle\,|(a,b;(v',f'))\rangle$$

II. Implementation of a qubit surface code

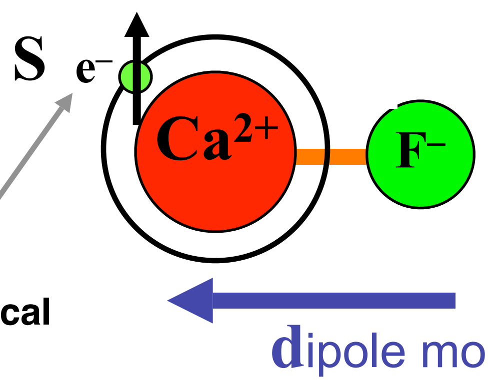
Platform: Polar molecules trapped in an optical lattice



- System: $^2\Sigma_{1/2}$ hetero-nuclear molecules in electronic-vibrational ground-states
  - Alkaline-earth monohalides (CaF,CaCl,MgCl...)
  - single electron in outer shell
- Electric dipole moment in superposition
- of rotational states

Energy scales:

$\gamma/\hbar \sim 100\ \mathrm{MHz}$	**Spin-rotational coupling**
$B/\hbar \sim 10\ \mathrm{GHz}$	**Rotational constant**
$\omega_{osc} \sim 100\ \mathrm{kHz} - 1\mathrm{MHz}$	**Lattice trap spacing**
$\Gamma/\hbar \sim 10^{-3}\ \mathrm{Hz}$	**Black-body scattering rate**
$\Gamma_{\mathrm{scat}}/\hbar \sim 10^{-1}\mathrm{Hz}$	**Spontaneous emission**



talks to microwave radiation ... as rotations on ~20 GHz

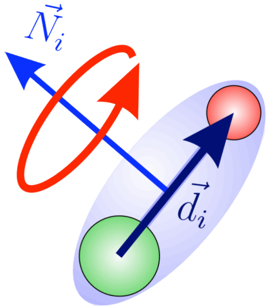
Rotational spectra of a single molecule

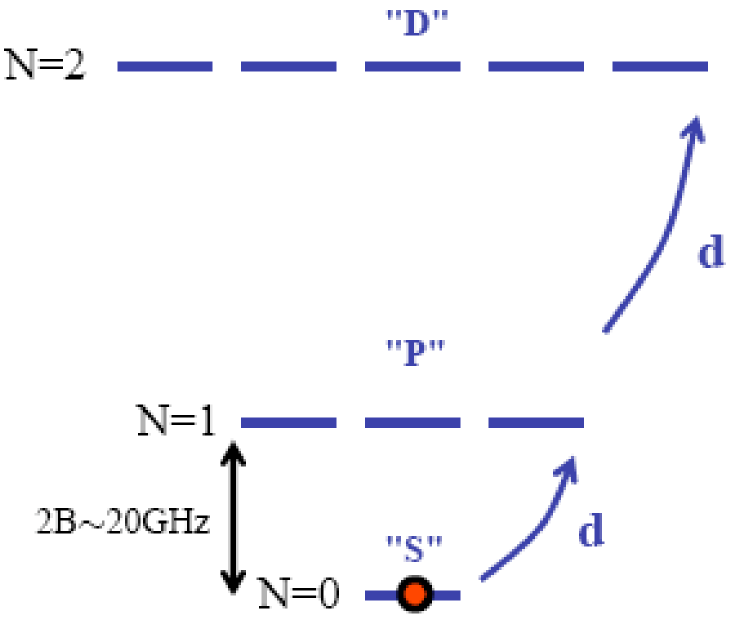
- rigid rotor

$$H = B\,\mathbf{N}^2$$

$$|N, M_N\rangle$$

$$E_N = B\,N(N+1)$$





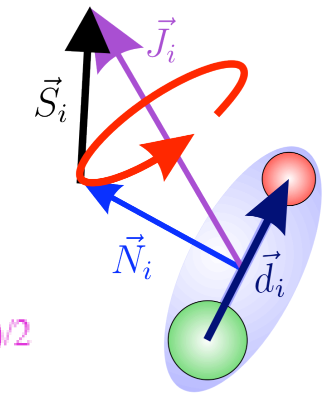
rotational ground state ...

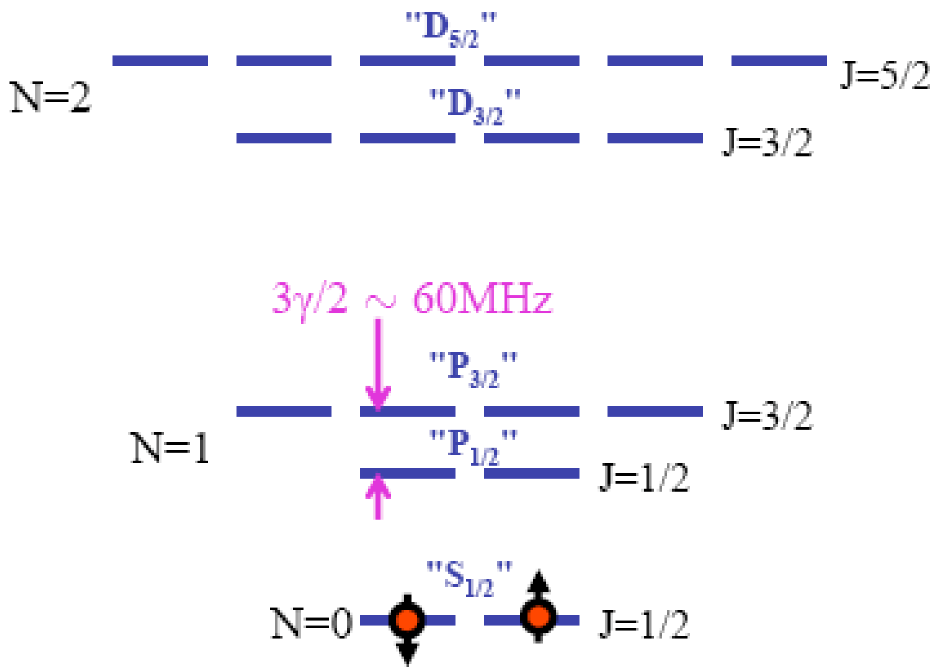
- add spin-rotation coupling

$$H = B\,\mathbf{N}^2 + \gamma\,\mathbf{N}\cdot\mathbf{S}$$

$$|N, J, M_J\rangle \quad (J=|N\pm 1/2|)$$

$$E_{N,J=N\pm 1/2} = B\,N(N+1) + \begin{cases} +\gamma N/2 \\ -\gamma(N+1)/2 \end{cases}$$

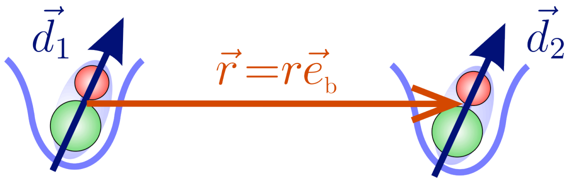




... as spin-1/2-system

Two polar molecules: dipole-dipole interactions

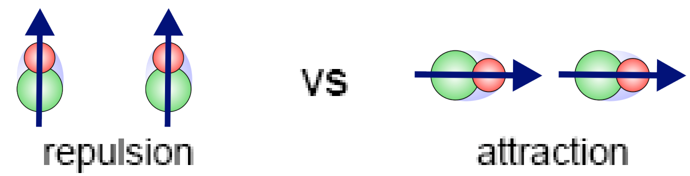
- interactions of two polar molecules



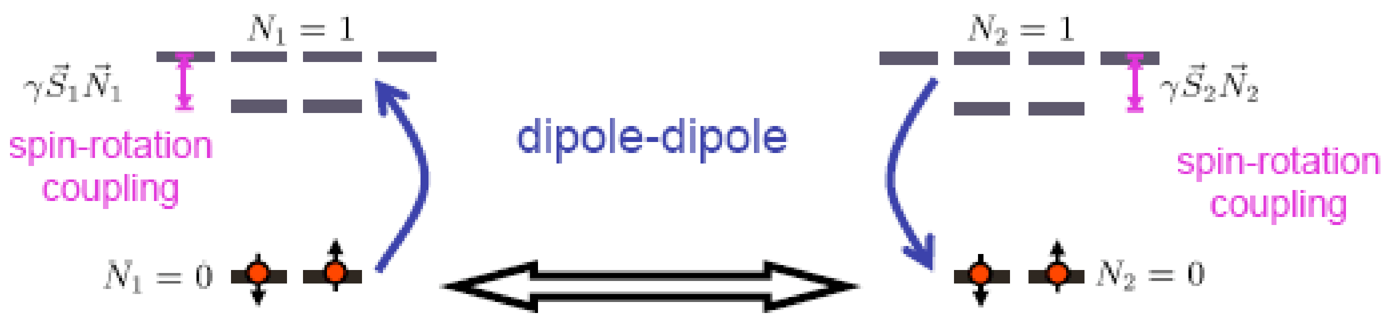
$$V_{\mathrm{dd}} = \frac{\vec{d}_1 \cdot \vec{d}_2 - 3\left(\vec{d}_1 \cdot \vec{e}_b\right)\left(\vec{e}_b \cdot \vec{d}_2\right)}{r^3}$$

features of dipole-dipole interaction:

- long range $\sim 1/r^3$
- angular dependence (anisotropic)



- include spin-rotation coupling in adiabatic potentials for molecular dimers

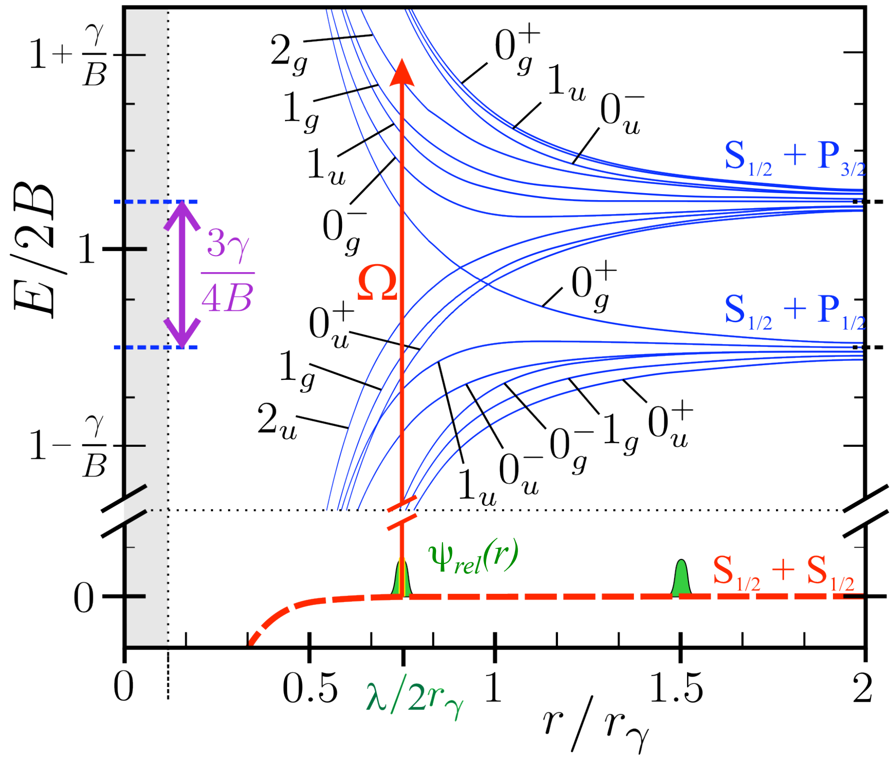


- At typical optical lattice spacing : $\lambda/2 \sim r_\gamma = (2d^2/\gamma)^{1/3}$
  - rotation of dimers strongly coupled to spins
  - Hunds case (c) excited states, $\{|Y|_{g,u}^{\pm}(r)\}$ $(Y = \Sigma_{i=1,2} M_{N,i} + M_{S,i})$
  - solvable in closed form due to symmetries

Microwave coupling with tunable spin patterns

$$H_{\mathrm{eff}}(r) = \sum_{i,f}\sum_{\lambda(r)} \frac{\langle g_f|H_{\mathrm{mf}}|\lambda(r)\rangle\langle\lambda(r)|H_{\mathrm{mf}}|g_i\rangle}{\hbar\omega_F - E(\lambda(r))}|g_f\rangle\langle g_i|$$

$$\begin{aligned} H_{\mathrm{spin}} &= \langle H_{\mathrm{eff}}(r)\rangle_{\mathrm{rel}} \\ &= \textstyle\sum_{\alpha,\beta} A_{\alpha,\beta}\sigma^{\alpha}\sigma^{\beta} \end{aligned}$$



- Feature 1: Tuning close to a resonance one select a *specific spin pattern, e.g.*

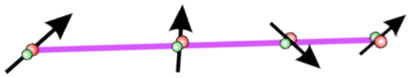
Polarization	Resonance	Spin pattern
$\hat{x}$	2_g	$\sigma^z\sigma^z$
$\hat{z}$	0_u^+	$\vec{\sigma}\cdot\vec{\sigma}$
$\hat{z}$	0_g^-	$\sigma^x\sigma^x+\sigma^y\sigma^y-\sigma^z\sigma^z$
$\hat{y}$	0_g^-	$\sigma^x\sigma^x-\sigma^y\sigma^y+\sigma^z\sigma^z$
$\hat{y}$	0_g^+	$-\sigma^x\sigma^x+\sigma^y\sigma^y+\sigma^z\sigma^z$
$(\hat{y}-\hat{x})/\sqrt{2}$	0_g^+	$-\sigma^x\sigma^y-\sigma^y\sigma^x+\sigma^z\sigma^z$

polarization rel. to body axis, here set $\vec{e}_b = \hat{z}$

Lattice Spin Models using multiple fields

- Feature 2: for a *multifrequency* field spin textures are *additive* => toolbox

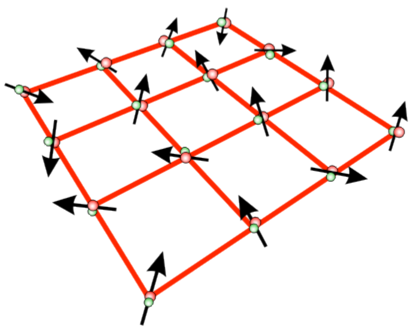
- 1D XYZ model



$$H = \sum_{<i,j>} J_x \sigma_i^x \sigma_j^x + J_y \sigma_i^y \sigma_j^y + J_z \sigma_i^z \sigma_j^z$$

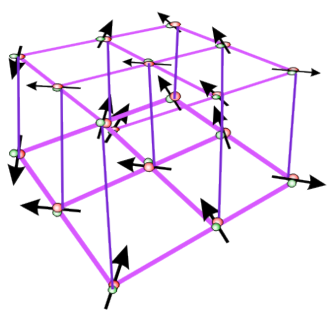
- 2D Ising model

$$H = \sum_{<i,j>} J \sigma_i^z \sigma_j^z$$



- 3D Heisenberg model

$$H = \sum_{<i,j>} J \vec{\sigma_i} \cdot \vec{\sigma_j}$$



Polarization	Resonance
$\hat{z}$	0_u^+
$\hat{y}$	0_g^-
$\hat{y}$	0_g^+
$\hat{x}$	2_g
$\hat{x}$	0_u^+
$\hat{z}$	0_g^-
$\hat{z}$	0_u^+
$\hat{x}$	1_u

- Typical coupling strengths: $|J| \sim 10$ - $100\mathrm{kHz}$

sign adjustable by tuning above or below given resonance

- Feature 3: Can choose the *range* of the interaction for a given spin texture

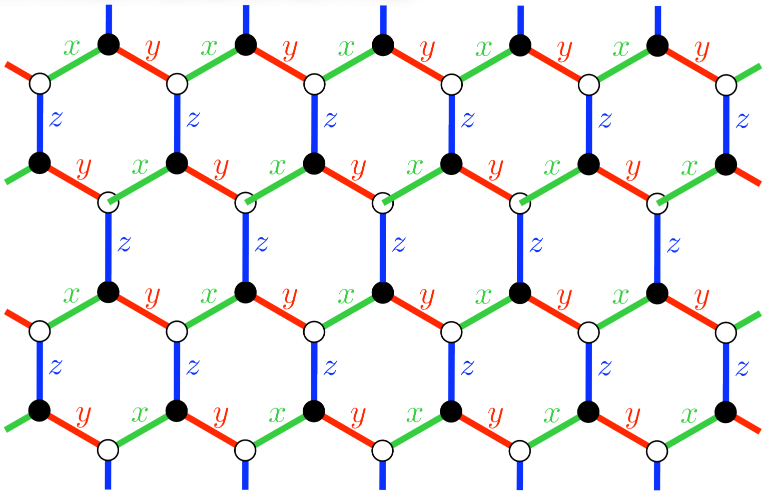
TO from 2-body interactions

- Spin-1/2 particles on a honeycomb lattice*

*A.Yu. Kitaev, Annals of Physics, 321,2 (2006)

$$H = J_\perp \sum_{x-\text{links}} \sigma_j^x \sigma_k^x + J_\perp \sum_{y-\text{links}} \sigma_j^y \sigma_k^y + J_z \sum_{z-\text{links}} \sigma_j^z \sigma_k^z.$$

  - **Exactly solvable**



- In the limit, $|J_z| \gg |J_\perp|$, pairs of spins along z-links are mapped to a qubit
  - **New spin operators on each z-link:**

$$\mathbf{1}_{2(1)} \otimes \sigma_2^z \to Z \qquad \sigma_1^y \otimes \sigma_2^x \to Y \quad \sigma_1^x \otimes \sigma_2^x \to X$$

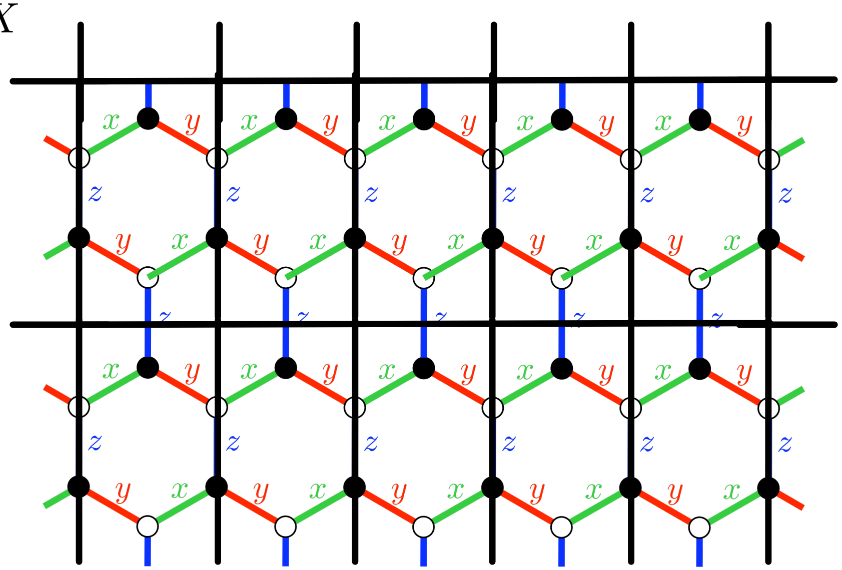
$$H_{\text{eff}} = -J_{\text{eff}} \sum_{\diamond} Y_{\text{left}} Z_{\text{up}} Y_{\text{right}} Z_{\text{down}}$$

Unitary transformation: $\prod_{j \ni \text{white}} e^{iX_j \pi/4}$

$$H_{\text{eff}} = -J_{\text{eff}} \left(\sum_{+} Z_{e_1} Z_{e_2} Z_{e_3} Z_{e_4} + \sum_{\square} X_{e_1} X_{e_2} X_{e_3} X_{e_4} \right)$$

$$J_{\text{eff}} = \frac{J_\perp^4 |J_z|}{16 J_z^4}$$

  - **Protected q. memory**

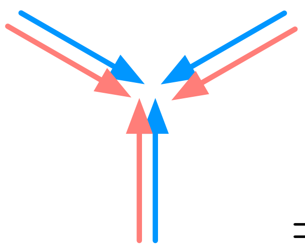


Construction in an optical lattice

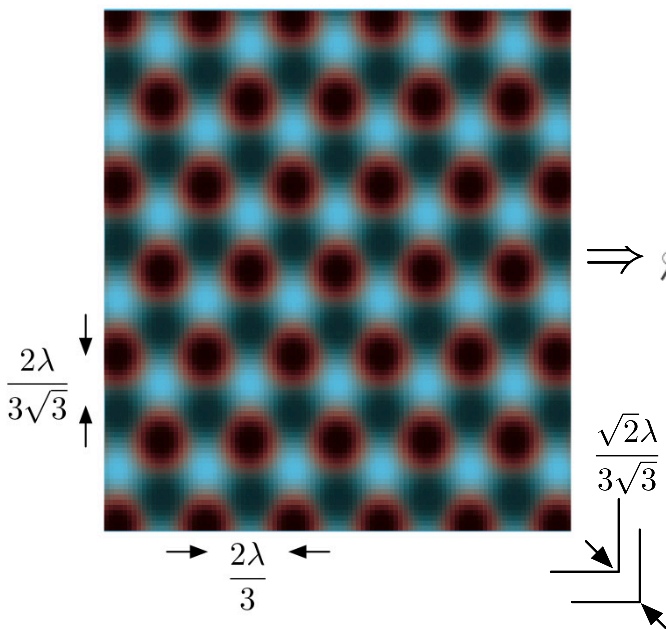
Bichromatic trapping beams in 2D, with relative phase shift

One triangular lattice staggered on top of another

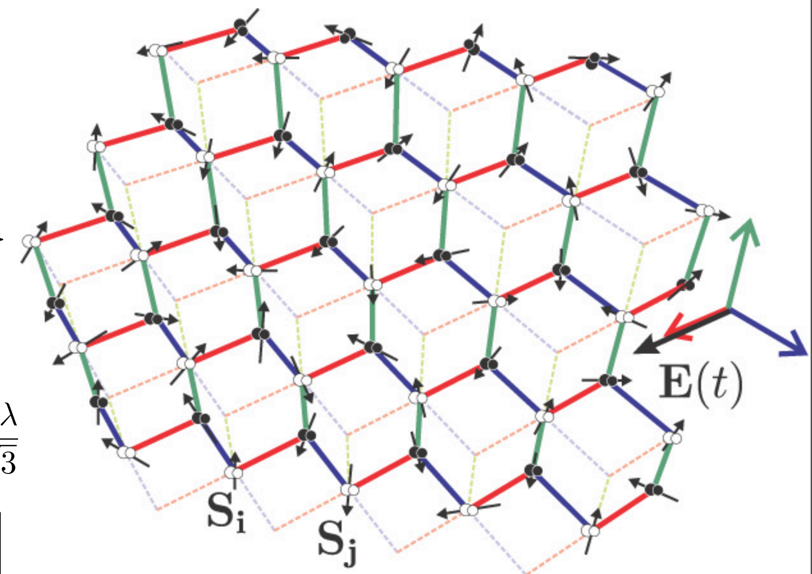
Q*bert lattice with nearest neighbor honeycomb graph. Edges connecting nearest neighbors form orthogonal triads



$\Rightarrow$

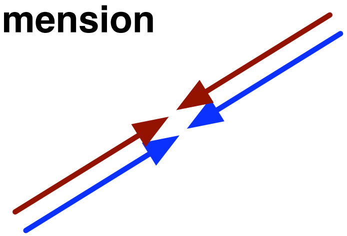


$\Rightarrow$



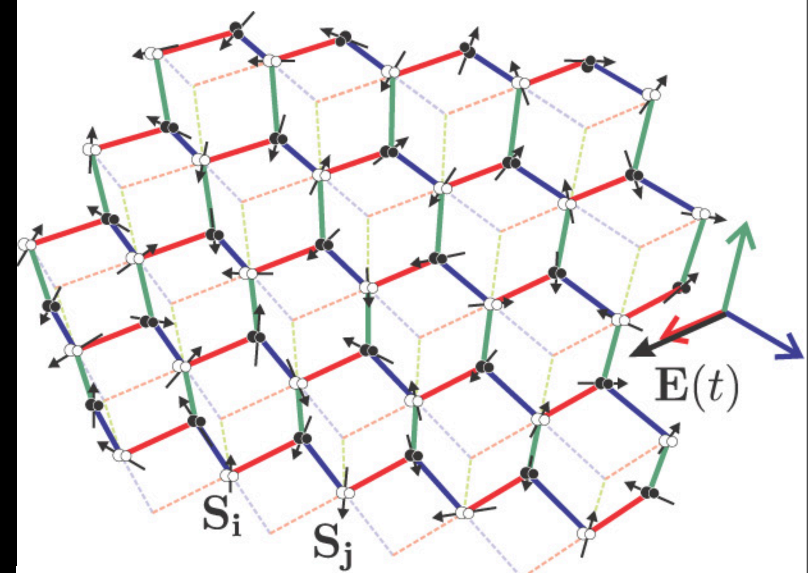
$+$

Biperiodic trap in third dimension



Construction in an optical lattice

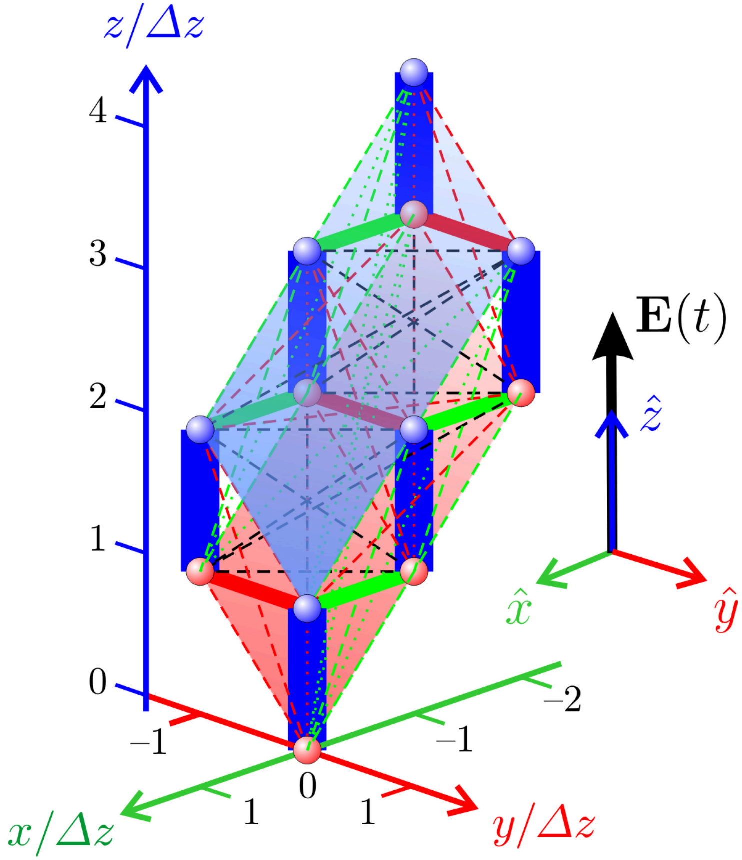
Q*bert lattice with nearest neighbor honeycomb graph. Edges connecting nearest neighbors form orthogonal triads

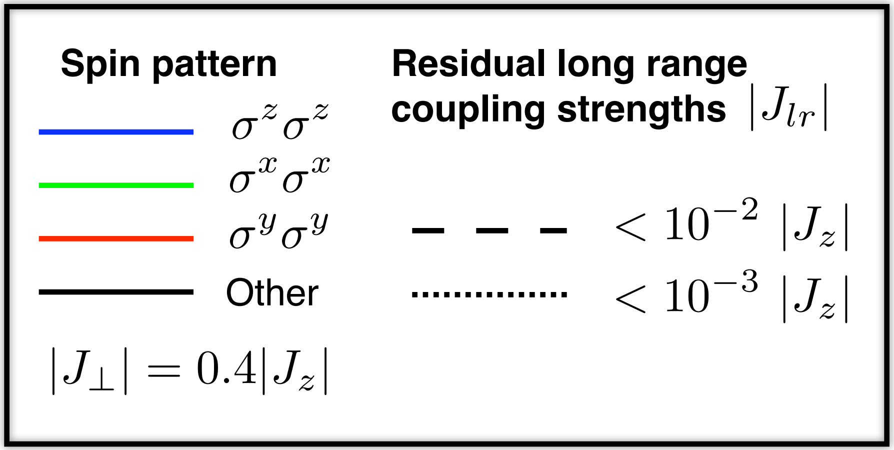


Results for system of 12 spins

- Realization with 3 fields. Several field choices possible, e.g. all polarized along $\hat{z}$ tuned to $1_g, 0_g^-, 2_g$

Coupling Graph





Operator fidelity (on a 4 spin configuration)

$$\sup\left[\| |H_{\text{spin}} - H_{\text{spin}}^{(\text{II})}|\psi\rangle||_2;\ \langle\psi|\psi\rangle = 1\right] = 10^{-4}\ |J_z|$$

For realistic parameters

$$|J_z| = 100\ \text{kHz} \Rightarrow\ J_{eff} \sim 167\ \text{Hz}$$

A. Micheli, GKB, P. Zoller, Nature Physics, **2**, 341 (2006)

Quasi-particle statistics

- **Excitations induced by single spin flips (along a z-link) represented by particle pairs**
  - **Consider translationally invariant 4-local interaction along diamonds with vertices on z-links**

$$H_{\rm eff} = -J_{\rm eff} \sum_{\diamond} Y_{\rm left} Z_{\rm up} Y_{\rm right} Z_{\rm down}$$

  - **Four superselection sectors: vacuum (no particles), Z particles ($\square$) on the left and right of a Z flipped spin, Y particles ($\diamond$) above and below a Y flipped spin, bound state of a Z particle and an Y particle ($\square\diamond$) flanking an X flipped spin.**
    - Fusion rules (as obtained from the action of the Pauli operators):

$$\square \times \square = 1 \qquad \diamond \times \diamond = 1 \qquad \square\diamond \times \square\diamond = 1$$
$$\square \times \diamond = \square\diamond \qquad \square \times \square\diamond = \diamond \qquad \diamond \times \square\diamond = \square$$

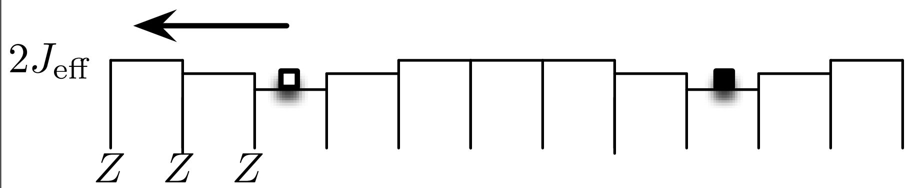
    - Relative statistics under braiding:

Particles	Statistical phase
$\square \ \square$	0
$\diamond \ \diamond$	0
$\square \ \diamond$	π
$\square\diamond \ \square\diamond$	0

Braiding

- $|\Psi(1)\rangle = S_A^Y S_B^Z |\lambda_g\rangle$
- $|\Psi(2)\rangle = e^{-iS_I^Z \pi/4}|\Psi(1)\rangle = \frac{1}{\sqrt{2}}(|\Psi(1)\rangle - iS_I^Z|\Psi(1)\rangle)$
- **Adiabatically drag □ left**

$$H'(t) = H + \sum_{e\in Path} \delta J_e(t)(\sigma_1^z\sigma_2^z)_e + \kappa(t) Z_e(t)$$



$$|\Psi(3)\rangle = \frac{1}{\sqrt{2}}(|\Psi(1)\rangle - iS^Z_{B'\cup I}|\Psi(1)\rangle)$$

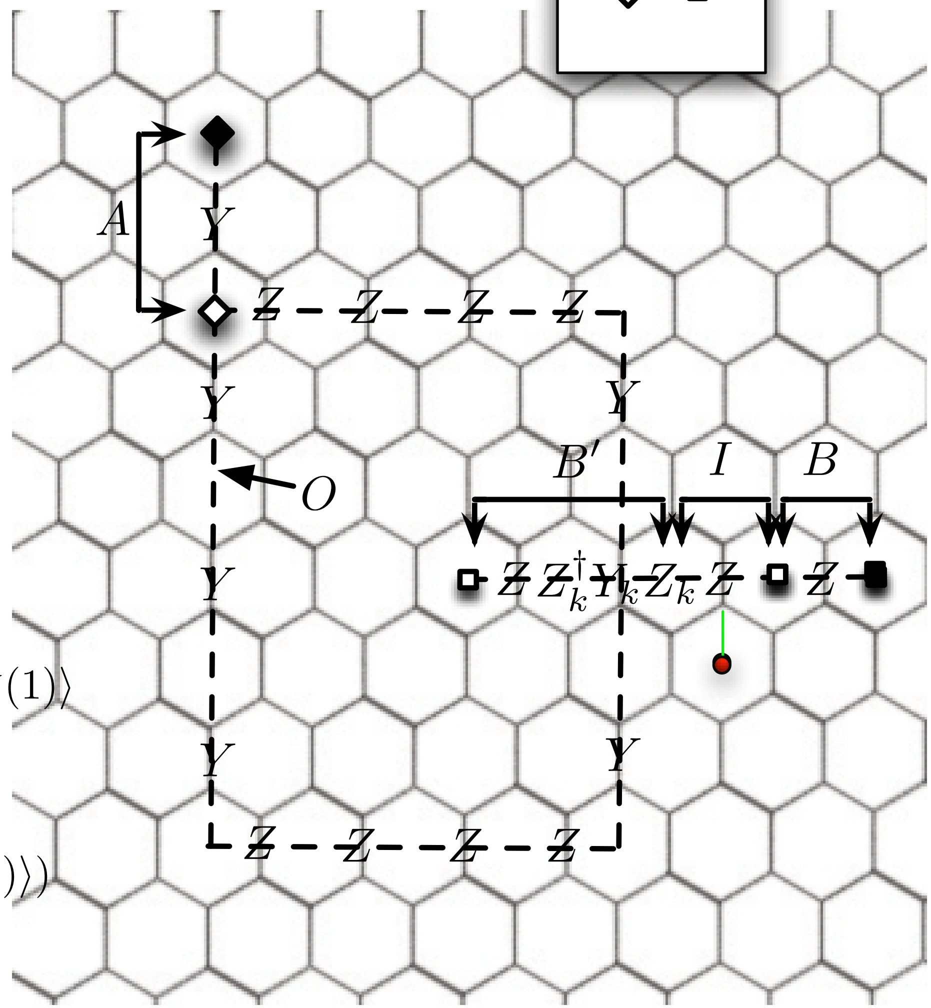
- **Adiabatically drag ◇ CCW around □**

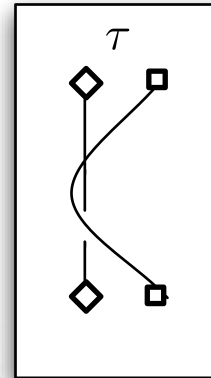
$$|\Psi(4)\rangle = \frac{1}{\sqrt{2}}(O|\Psi(1)\rangle - iOS^Z_{B'\cup I}|\Psi(1)\rangle)$$

- **Adiabatically drag □ right**

$$\begin{aligned} |\Psi(5)\rangle &= \tfrac{1}{\sqrt{2}}(O|\Psi(1)\rangle - ie^{i\beta}(Z_kY_kZ_k)Y_kOS_I^Z\Psi(1)\rangle) \\ &= \tfrac{1}{\sqrt{2}}(|\Psi(1)\rangle + ie^{i\beta}S_I^Z|\Psi(1)\rangle) \end{aligned}$$

- $$\begin{aligned} |\Psi(6)\rangle &= e^{iS_I^Z\pi/4}|\Psi(5)\rangle \\ &= \tfrac{1}{2}((1+e^{i\beta})iS_I^Z|\Psi(1)\rangle + (1-e^{i\beta})|\Psi(1)\rangle) \end{aligned}$$





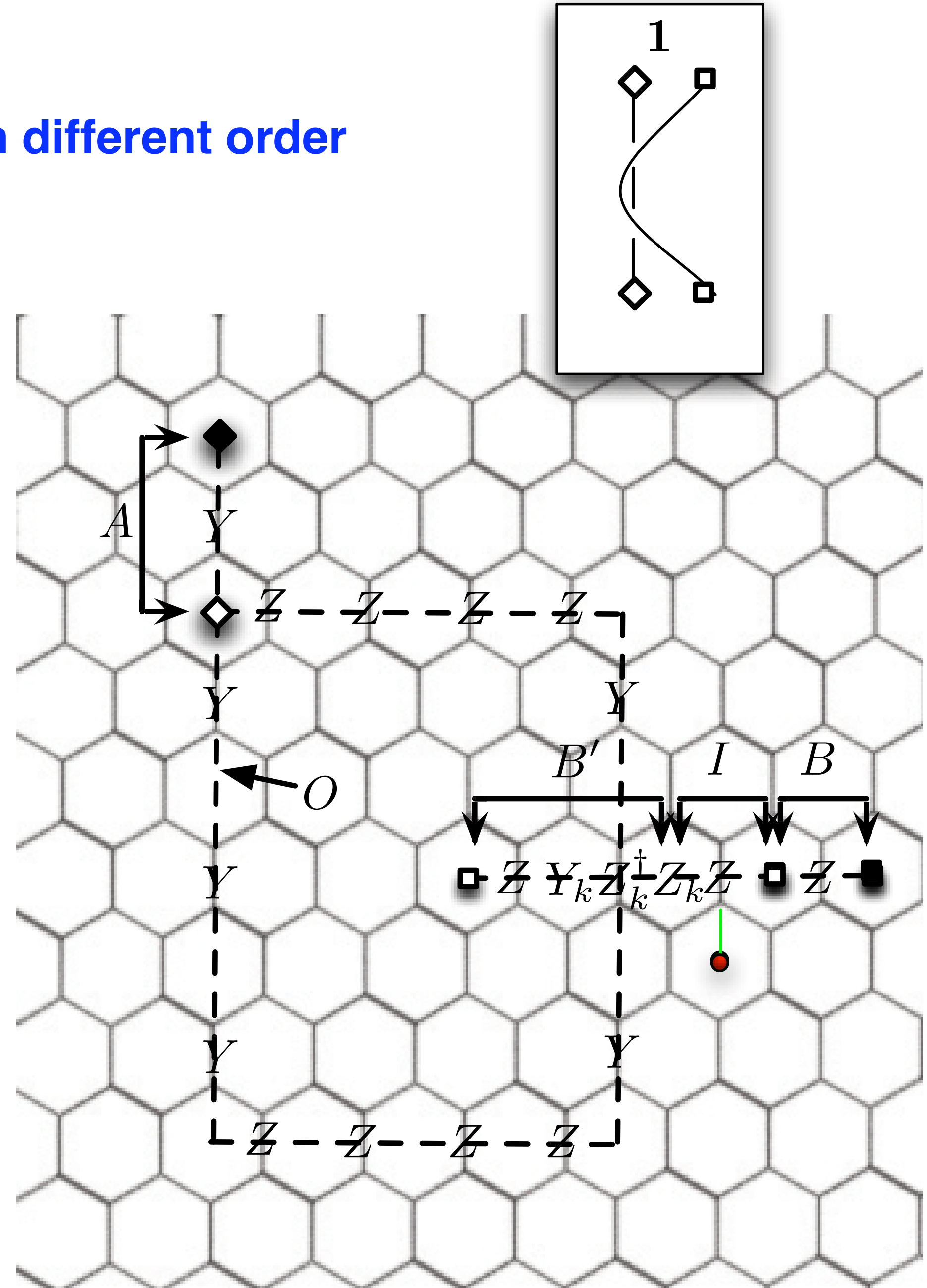
- **Measure location of □**

$$\langle S_I^Z\rangle = \sin(\beta + \pi)$$

Dynamical+Berry phases

Statistical phase

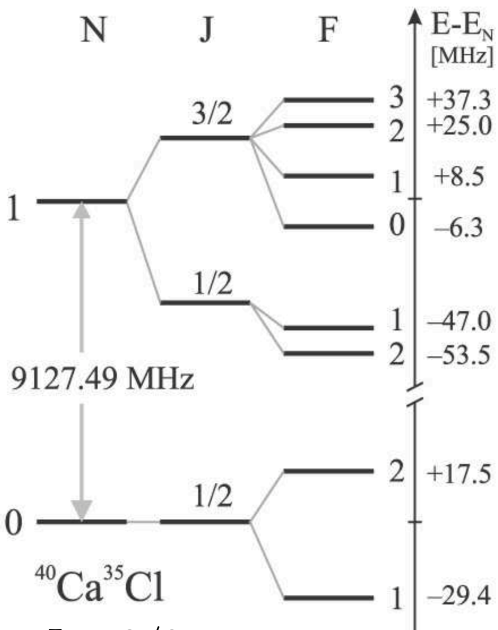
For trivial braid use same steps but in different order



- Adiabatically drag ◇ CCW around □
- Adiabatically drag □ left
- Adiabatically drag □ right
- Measure location of □

$$\langle S_I^Z \rangle = \sin(\beta)$$

III. Higher spin models using hyperfine levels



$I = 3/2$

$$H_{\mathrm{m}} = B\mathbf{N}^2 + \gamma \mathbf{N}\cdot\mathbf{S} + b\mathbf{I}\cdot\mathbf{S} + cI^z S^z + eQq\frac{3I^{z2} - I(I+1)}{4I(2I-1)}$$

Fermi contact **Dipolar** **Electric Quadrupole**

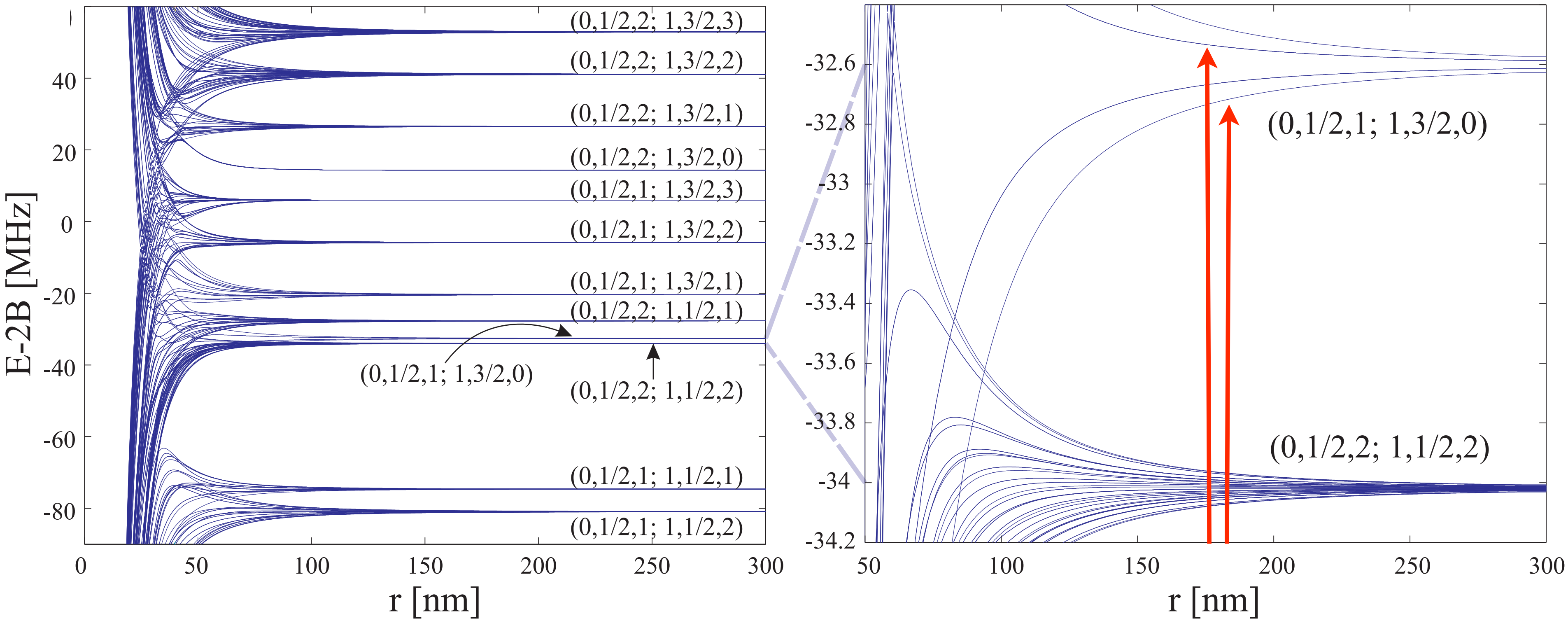
$N = 0$

$F = 2$

$F = 1$

Encode here

hyperfine cont.



Asymptotic couplings exactly solvable

Can't build generic two body Hamiltonians but can build a large class

Example Hamiltonian in terms of spin-1 rep of su(2), built with 8 microwave fields:

$$H_\beta = U(\mathbf{S}_1 \cdot \mathbf{S}_2 - \beta(\mathbf{S}_1 \cdot \mathbf{S}_2)^2)$$

For $\beta = -1/3$ system is gapped with valence bond ground state: spin 2 rep of $1 \otimes 1 = 0 \oplus 1 \oplus 2$

Summary & Outlook

- Recipe for building a class of Hamiltonians with topologically ordered ground states

- We can design spin-spin interactions with polar molecules

  - Tunable range and anisotropy
  - Large coherence to decoherence ratio   Q~800-10000 for reasonable trapping parameters

- Examples of Lattice Spin Model with TO

  - The Kitaev Model
    - Gapped system with abelian excitations
    - Feasible technique for measuring quasiparticle statistics

- Can we increase the effective coupling (increase the gap)? Possible with self assembled lattices---->closer lattice spacings

- Building three body interactions