The Generalized Complex Geometry of Supersymmetry

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Mainly based on
hep-th/0505212 [Graña, Minasian, Petrini, AT]
also:
hep-th/0502148 [AT]
hep-th/0311122 [Fidanza, Minasian, AT]

Introduction

Use pairs (Φ_+, Φ_-) that determine metric g

each $\Phi = \sum_k \mathrm{form}_k$

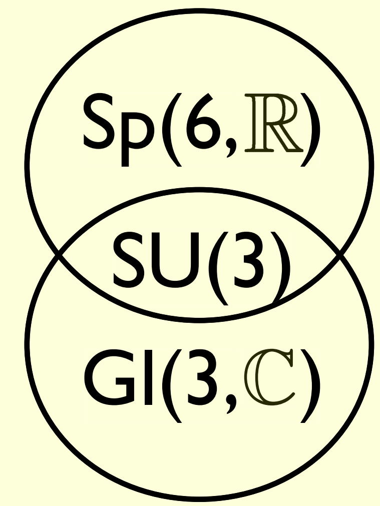
Example: SU(3) structure in six dimensions

Then

$$\Phi_+ = e^{iJ}$$
$$\Phi_- = \Omega$$

J real two-form

Ω complex three-form



compatibility $J \wedge \Omega = 0$

$$J^3/3! = i\Omega \wedge \bar{\Omega}$$

We will see more general pairs.

SU(3) structure is a topological remnant of SU(3) holonomy: $\mathcal{W}_3 = 0$, $c_1 = 0$

~~$de^{iJ} = 0$, $d\Omega = 0$ $\Rightarrow$ Calabi-Yau~~

We will not assume Φs are closed

In fact, Calabi-Yau is $\mathcal{N}=2$;

$$(d+H\wedge)\Phi_1=0$$
$$(d+H\wedge)\Phi_2=0$$

our aim is to show that for $F_{RR}\neq 0$ $\quad \mathcal{N}=1 \quad \Longleftrightarrow$

$$(d+H\wedge)\Phi_1=0$$
$$(d+H\wedge)\Phi_2=F_{RR}$$

- This result has a clear mathematical interpretation in terms of generalized complex geometry
- It implies: a topological model behind $\mathcal{N}=1$ vacua

  "half a structure" has an independent meaning

- Natural mirror map

$$\Phi_1 \longleftrightarrow \Phi_2$$
$$F_{IIA} \longleftrightarrow F_{IIB}$$

leading to mirror symmetry beyond Calabi-Yaus

Plan

- General classification of type II $\mathcal{N} = 1$ vacua
  - Mathematical interpretation
    - Mirror symmetry?

Classification

[Graña, Minasian, Petrini,AT]

Two internal spinors η_i

$$\epsilon_1^{10} = \epsilon_+^4 \otimes \eta_1 + \text{c.c.}$$
$$\epsilon_2^{10} = \epsilon_\mp^4 \otimes \eta_2 + \text{c.c.}$$

IIA or IIB

$$\eta_1 \otimes \eta_2^\dagger = \sum \cancel{\text{forms}}$$

(Clifford map; Fierz)

Examples:

η_1, η_2 everywhere parallel

$$\Phi_- = \eta_1 \otimes \eta_2^t = \cancel{\Omega} \equiv \Omega_{mnp}\gamma^{mnp}$$
$$\Phi_+ = \eta_1 \otimes \eta_2^\dagger = \cancel{e^{iJ}} \equiv 1 + iJ_{mn}\gamma^{mn} - \frac{1}{2}J_{mn}J_{pq}\gamma^{mnpq} - i\gamma^7$$

SU(3)

η_1, η_2 everywhere orthogonal

(requires $\chi = 0$!)

$$\Phi_- = \eta_1 \otimes \eta_2^t = \cancel{e^{ij} \wedge (v + iw)}$$
$$\Phi_+ = \eta_1 \otimes \eta_2^\dagger = \cancel{\omega \wedge e^{iv\wedge w}}$$

(static) SU(2)

"4d" "2d"

(distributions)

There exist intermediate cases.

[Witt; Jeschek,Witt]

$$F_{RR} = \sum_k F_k = (d + H\wedge) \sum_k C_k$$ is also naturally a bispinor

$$\delta\psi^1_m = \nabla^H_m \epsilon_1 + e^\phi F \gamma_m \epsilon^2$$

[Bergshoeff,Kallosh,Ortin,Roest, van Proeyen]

$$\mathcal{N} = 1 \quad \delta\psi = \delta\lambda = 0$$

$$\Updownarrow$$

IIA	IIB
$(d + H\wedge)(e^{2A-\phi}\Phi_+) = 0$	$(d + H\wedge)(e^{2A-\phi}\Phi_+) = dA \wedge \Phi_+^* + (a^2 - b^2)e^\phi F - i(a^2 + b^2)e^\phi * F$
$(d + H\wedge)(e^{2A-\phi}\Phi_-) = dA \wedge \Phi_-^* + (a^2 - b^2)e^\phi F - i(a^2 + b^2)e^\phi * F$	$(d + H\wedge)(e^{2A-\phi}\Phi_-) = 0$

- There is a mathematical interpretation!
- Particular cases (SU(3)): IIA $\Rightarrow de^{iJ} = 0 \Rightarrow$ symplectic; IIB $\Rightarrow d\Omega = 0 \Rightarrow$ complex
- For $F = 0$ eqns. actually describe $\mathcal{N} = 2$
- c.c. modifies: $(d + H\wedge)\Phi_+ = \sqrt{\lambda}\,\mathrm{Im}(\Phi_-)\,, \ldots$

A warping
ϕ dilaton
a, b normalizations

$$\Phi_\pm^\dagger \wedge \Phi_\pm = ab\,\mathrm{vol}$$
$$a^2 - b^2 = c\,e^{-A}$$
$$a^2 + b^2 = c'\,e^{A}$$

Generalized complex geometry

[Hitchin; Gualtieri]

Use sum of tangent and cotangent bundles $T \oplus T^*$ rather than T

natural pairing $v_m \omega^m$! no need for metric yet

At algebraic level.
(topological)

The counterpart of SU(3) structure is an SU(3,3) structure

$\sum_k \text{form}_k \sim$ bispinor $\sim Cl(T \oplus T^*)$ spinor

"gamma matrices":

contractions $v \llcorner$

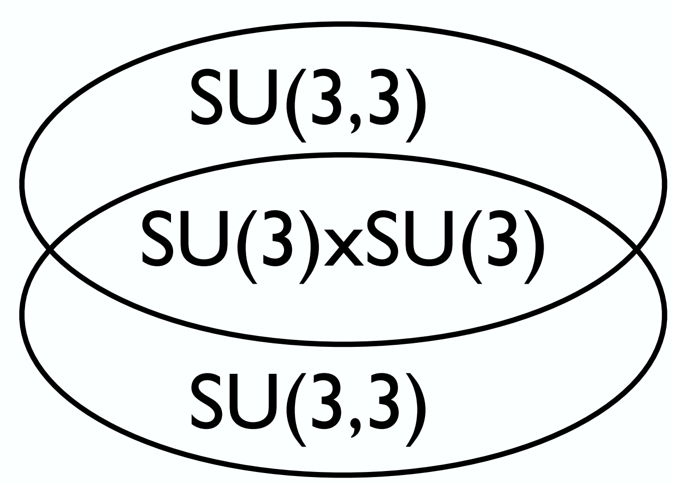
wedges $\omega \wedge$

spinor η is pure (annihilator of maximal dim)

$\Phi = \eta_1 \otimes \eta_2^\dagger$ is pure $\Rightarrow$ it defines an SU(3,3) structure

Compatible pairs (common annihilator of maximal dim) (Φ_+, Φ_-)

define an SU(3)xSU(3) structure



$\longrightarrow$ metric & B-field

At differential level.

Want to define Nijenhuis for $T \oplus T^*$

Remember $\mathrm{Nij}(J)(v,w) \sim (1-iJ)[(1+iJ)v,(1+iJ)w]_{\mathrm{Lie}}$

measures if $(1,0)$ -vectors are closed under Lie bracket

On $T \oplus T^*$ a Courant bracket exists

$$[A_1, A_2]_{\mathrm{Courant}} \equiv d \circ A_1 \circ A_2 + A_1 \circ d \circ A_2 - A_2 \circ d \circ A_1 - A_2 \circ A_1 \circ d$$
$$-(1 \leftrightarrow 2)$$

$$A_i \Phi = 0 \;\Rightarrow\; [A_1, A_2]_{\mathrm{Courant}} \Phi = (A_1 A_2 - A_2 A_1) d\Phi$$

the annihilator of Φ is closed under Courant $\Leftarrow$ set it to zero

Easy to include H :

"derived bracket" $d \longrightarrow$ other differential

$$(d + H\wedge)^2 = 0$$

(unique choice)

[Kosmann-Schwarzbach]

Interpretations

Hence $(d + H\wedge)\Phi_1 = 0 \;\Rightarrow\; \mathrm{Ann}(\Phi_1)$ closed $\Longrightarrow$ (twisted) generalized Calabi-Yau

and $(d + H\wedge)\Phi_2 = F \;\Longrightarrow\; F \sim \mathrm{Nij}(\Phi_2)$

- valid for SU(3), SU(2), intermediate (generic!) cases

- gCY $\Longrightarrow$ a topological model exists

[Kapustin; Zucchini;Zabzine; Lindstrom,Minasian,Zabzine,AT]

A	$de^{iJ} = 0$
B	$d\Omega = 0$
gen.	$d\Phi = 0$

- when $F = 0$, two $\mathrm{Ann}(\Phi_i)$ are closed $\Longrightarrow$ $\mathcal{N} = 2$

(twisted) generalized CY metric

worldsheet interpretation: (2,2) models

SU(3)	SU(2) or general
CY, $H = 0$	twisted multiplets $H \neq 0$ [Gates,Hull,Roček]

[Gualtieri]

Mirror symmetry

[Fidanza,Minasian,AT]

Mirror map:

IIA		IIB
Φ_+	$\longrightarrow$	$i\Phi_-$
Φ_-	$\longrightarrow$	$-i\Phi_+$
F	$\longrightarrow$	iF

Check: T-duality $\quad e^{i(12+34+56)} \xrightarrow{T_1} (1+i2)e^{i(34+56)}$

$T \oplus T^* \circlearrowleft \qquad \xrightarrow{T_{35}} (1+i2)(3+i4)(5+i6)$

Compare: $\Omega \leftrightarrow e^{B+iJ}\sqrt{Td}$ for branes on Calabi-Yau's

General law more involved, but:

$$
\begin{array}{ccccccc}
 & & & \Phi_+ & & & \\
 & & * & & * & & \\
 & * & & * & & * & \\
\Phi_-^* & & * & & * & & \Phi_- \\
 & * & & * & & * & \\
 & & * & & * & & \\
 & & & \Phi_+^* & & &
\end{array}
$$

expand $(d + H\wedge)\Phi_\pm$ in a "pure Hodge diamond":

(Ex.: $\langle \Phi_+, (d + H\wedge)\Phi_- \rangle$)

Then rotate the diamond

Pairs (Φ_+, Φ_-) make mirror symmetry more manifest.

Note that with $J \overset{?}{\longleftrightarrow} \Omega$

(intrinsic torsions) $\quad \begin{matrix} \Omega \llcorner dJ \\ (dJ)^{2,1}_{\rm prim} \\ \cdots \end{matrix} \longrightarrow \; ?$

- in general, T-duality will send to non-geometrical examples [Dabholkar, Hull; Hellerman, McGreevy, Williams]
- singular fibres can also be included; get topology of mirror of (CY, H) — applicable e.g. to all IIB compactifications [AT]

But: no reason in general a non-CY should be T^3-fibred

- Not cohomology but twisted K-theory is exchanged ($\leftrightarrow (d + H\wedge)$)
- Mirror symmetry as equivalence of topological models [Kontsevich]

Example: A model on symplectic $\longleftrightarrow$ B-model on complex

non-Kahler examples? [Smith, Thomas, Yau]

- as equivalence of 4d effective theories

preserved $\mathcal{N}=2$	preserved $\mathcal{N}=1$	spont. broken $\mathcal{N}=2$
$d_H\Phi_1 = 0$ $d_H\Phi_2 = 0$	$d_H\Phi_1 = 0$ $d_H\Phi_2 = F$	SU(3)xSU(3) structure

[Gurrieri, Louis, Micu, Waldram; Louis, Graña, Waldram]

must include massive modes;

massive harmonics

$[d, \Delta] = 0 \Longrightarrow$

$$d\omega_i^{(2)} = e_i^a \omega_a^{(3)} \text{ etc.}$$

gaugings; mixed with $\int_{\text{cycles}} H$

Information on how topology talks to $d\Phi$s?

Conclusions

- $\mathcal{N} = 1$ vacua are generalized Calabi-Yau
  - RR fields as a generalized Nijenhuis tensor
    - Mirror symmetry as exchange of pairs (Φ_+, Φ_-)