# Black hole singularities in Yang-Mills theories

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*based on*

Guido Festuccia, HL

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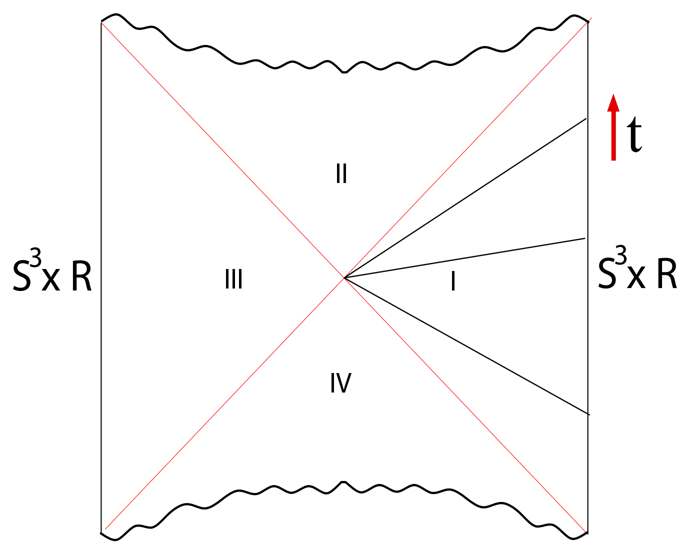
Understanding spacelike singularities is a major Challenge for string theory

Big Bang/Big Crunch, Black holes

# Schwarzschild black holes in AdS

Quantum gravity in an $AdS_5$ black hole background can be described by an $SU(N)$ Super Yang-Mills at finite temperature on $\underline{S^3}$.

Witten, Maldacena,...




AdS/CFT provides a full non-perturbative framework to address the problem.

Horowitz and Ross, Maldacena, ...

## Understanding the black hole singularity from thermal Yang-Mills ?

Classical gravity corresponds to the *large $N$* and *large 't Hooft coupling* limit of the YM theory.

To understand the singularity,

1. find manifestations of the black hole singularity in the large $N$ and large 't Hooft coupling limit of the YM theory;

2. from these manifestations, understand whether (and how)

   - finite N effects ($g_s$), or

   - finite 't Hooft coupling effects ($\alpha'$)

   resolve the singularity.

# Challenge: The singularities are hidden behind the horizons

Many people have explored how to extract physics beyond the horizon from the boundary theory correlation functions.

Balasubramanian, Ross; Louko, Marolf, Ross; Maldacena; Kraus, Ooguri, Shenker ...

Certain geodesics of the bulk geometry which go inside the horizon may be visible from boundary theory correlation functions.

In particular, Fidkowski et al found a subtle signal in YM theory of geodesics which approach the singularity.

L. Fidkowski, V. Hubeny, M. Kleban and S. Shenker

# Outline

1. Establish a direct connection between boundary momentum correlators and bulk geodesics.

2. Find signals of the singularity in momentum space correlators.

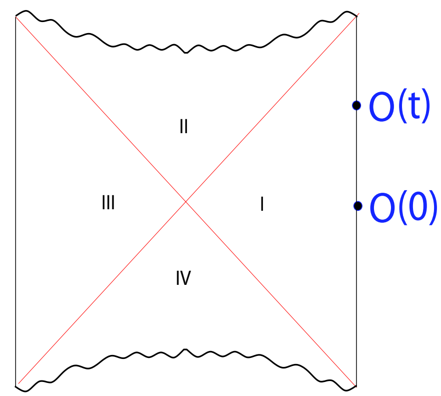
3. Introduce new gauge invariant observables in the boundary theory whose divergences reflect the presence of the bulk singularity.

4. Discuss the resolution of the singularity at finite N.

# Thermal YM correlation functions

We consider (suppressing spatial coordinates)

$$G_+(t) = \mathrm{Tr}\,[\, e^{-\beta H} O(t)\, O(0)\,] =$$



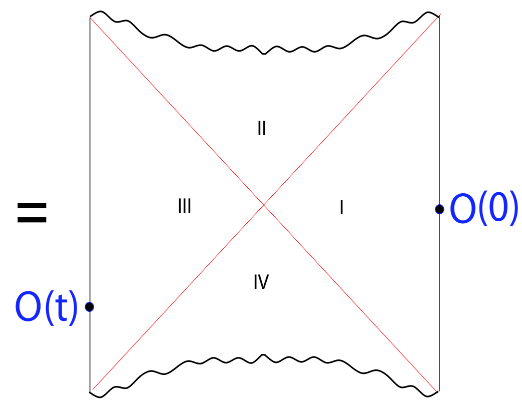

$\beta$: inverse Hawking temperature.

The boundary operator $O$ is dual to a bulk scalar field $\phi$ of mass $m$, with conformal dimension $\Delta$ of $O$ given by

$$\Delta = 2 + \nu, \qquad \nu = \sqrt{m^2 + 4}$$

One can also consider

$$G_{12}(t) = G_{+}(t - i\tfrac{\beta}{2}) =$$




We will be interested in momentum space correlators $G_{+}(\omega, l)$ with $\omega$ the frequency and $l$ angular momentum on $S^3$.

Note

$$G_{12}(\omega, l) = e^{-\frac{\beta\omega}{2}} G_{+}(\omega, l)$$

# Boundary correlators from gravity

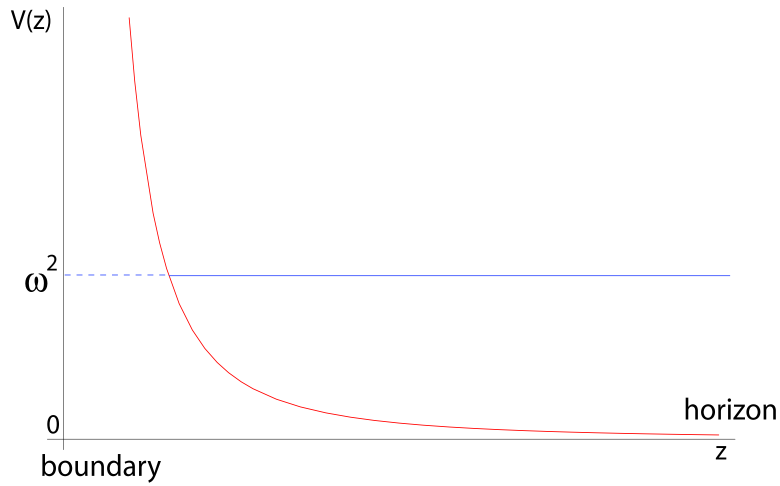
$G_+(\omega, l)$ can be obtained via AdS/CFT by solving the Laplace equation for the bulk scalar field $\phi$, which in momentum space becomes the Schrodinger equation

$$\left(-\partial_z^2 + V_l(z)\right) \phi_{\omega l}(z) = \omega^2 \phi_{\omega l}(z)$$

$z$: tortoise coordinate.

$\phi_{\omega l}$ (z): Fourier component of $\phi$.

Information about the geometry is contained in the potential $V_l$.




We consider normalizable modes $\phi_{\omega l}$ with the normalization

$$\phi_{\omega l}(z) \approx e^{-i\omega z - i\delta_\omega} + e^{i\omega z + i\delta_\omega}, \qquad z \to +\infty \ .$$

This determines

$$\phi_{\omega l}(z) \approx C(\omega, l) z^{\frac{1}{2}+\nu} + \cdots \ , \qquad z \to 0 \ .$$

Then

$$G_+(\omega, l) = \frac{(2\nu)^2}{2\omega} \frac{e^{\beta\omega}}{e^{\beta\omega} - 1} C^2(\omega, l)$$

Hard to extract information about the bulk geometry directly from $G_+(\omega, l)$.

We would like to know how the presence of the bulk singularity is reflected in $G_+(\omega, l)$.

## Analytic properties

- The boundary YM theory has a continuous spectrum despite being on $S^3$.

  This is due to the presence of the horizon in the bulk.

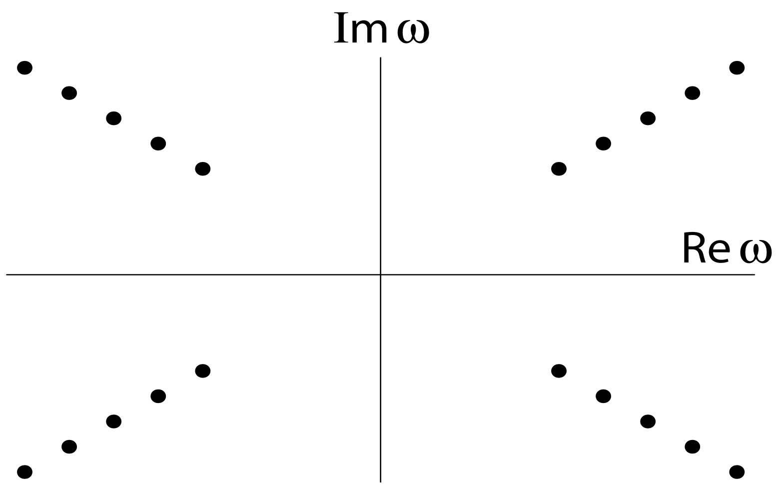
- One can analytically continue $G_+(\omega, l)$ to complex $\omega$ and $l$.

- We find that the only singularities of $G_+(\omega, l)$ in the complex $\omega$-plane are poles (quasi-normal poles of the black hole).

# Quasi-normal poles

For $l$ not too large,

Nunez and Starinets; Cardoso, Natario and Schiappa; Siopsis



# Large operator dimension limit

To make connection with the bulk geometry, consider the limit

$$\omega = \nu u, \qquad l = \nu k, \qquad \nu \gg 1, \qquad (\nu = \sqrt{m^2 + 4})$$

Then $G_+$ can be expanded in the large $\nu$ limit as

$$G_+(\nu u, \nu k) \approx 2\nu \, e^{\nu Z(u,k)} \, (1 + \cdots) + \cdots$$

$Z(u,k)$ and higher order terms of the expansion can be worked out explicitly from the Schrodinger equation.

In the large $\nu$ limit, the mass of the corresponding bulk particle is large and its propagation should approximately follow geodesics.

# Relation with bulk geodesics I

Since the bulk geometry has Killing vectors along $t$ and $S^3$ directions, a bulk geodesic can be characterized by integrals of motion $(E, q)$.

We find that $Z(u, k)$ can be identified as the Legendre transform of the geodesic distance of a bulk spacelike geodesic with $(E, q)$ starting and ending at the boundary

$$u = iE, \qquad k = iq$$

Note: $\omega = \nu u$, $l = \nu k$.

For each complex pair $(\omega, l)$

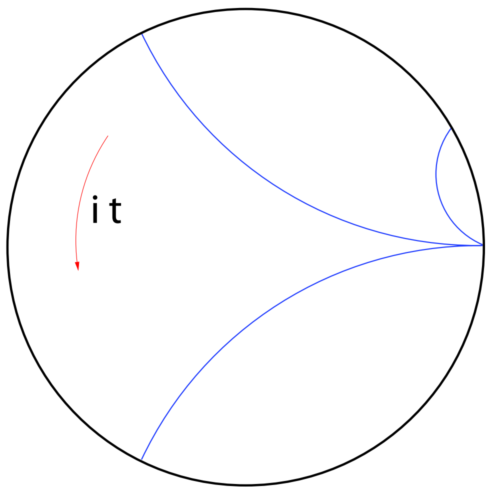
$$G_+(\omega, l) \quad \to \quad Z(u, k) \quad \to \quad \text{complex bulk geodesic}$$

In principle one can map out the full bulk geometry if $G_+(\omega, l)$ is known for all complex $(\omega, l)$ (an inverse scattering problem).

We will now look at some examples with $k = 0$.

# Relation with geodesics II

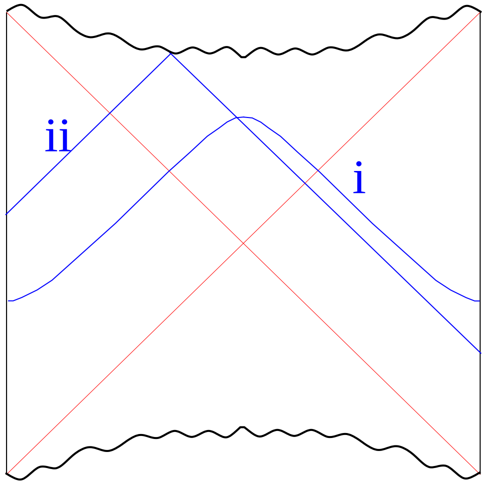
For real $\omega$, the geodesic lies in the Euclidean section of the complexified spacetime




For real $\omega \to \pm\infty$, the geodesic approaches the boundary

UV/IR connection

For $\omega$ pure imaginary, the geodesic probes the region inside the horizon.

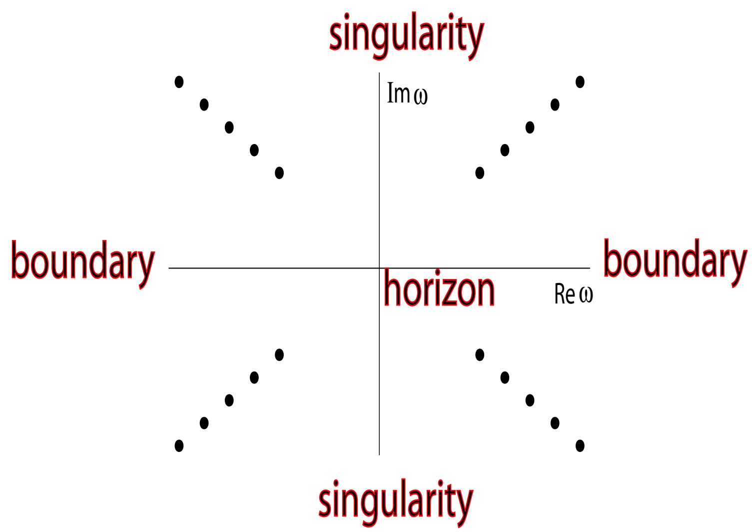



As $|\omega| \to \infty$, the geodesic *approaches the singularity*.

UV/UV connection

# Summary I

We thus find




The lines of poles of $G_+(\omega, l)$ "create" new asymptotic regions in the complex-$\omega$ plane corresponding to the regions around the singularity.

## Signal of the singularity I

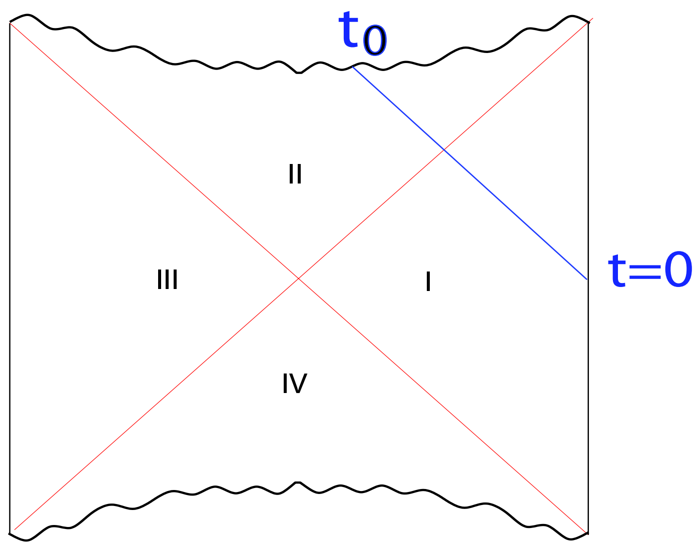
To probe the singularity, we consider $\omega \to \pm i\infty$

$$G_+(\omega, l=0) \sim \left(\mp i\frac{\omega}{2}\right)^{2\nu} e^{i\omega\left(\pm\frac{\tilde{\beta}}{2}-\frac{i\beta}{2}\right)}$$

- $G_+(\omega)$ decays exponentially along the imaginary $\omega$ axis. The decay is controlled by the parameter

$$\mathcal{B} = \tilde{\beta} + i\beta$$

- Valid at finite $\nu$.




$$4t_0 = \mathcal{B} = \tilde{\beta} + i\beta$$

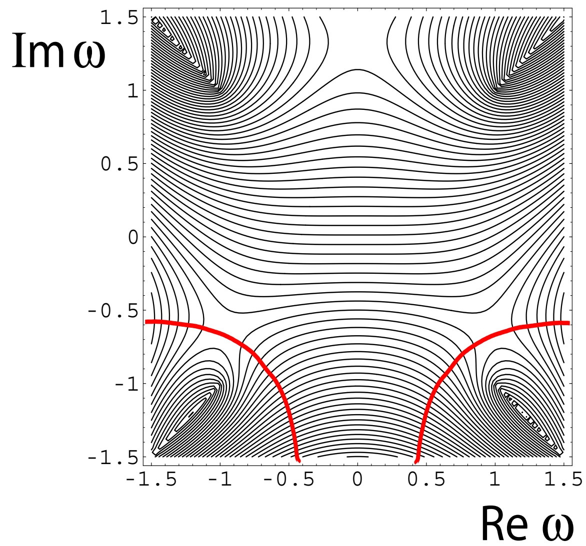
# Coordinate space correlation functions

$$G_{12}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\, G_{12}(\omega)$$
$$\sim \int du\, e^{-i\nu u t - \frac{1}{2}\nu u \beta}\, e^{\nu Z(u,k)}$$

- The Fourier integral can be evaluated using the saddle point approximation in the large $\nu$ limit.

- Bulk geodesics with end point separation given by $t$ appear as saddle points of the Fourier integral.

- Fourier integrals give a precise prescription for summing over geodesics.




Contour plot of the imaginary part of the exponent of the integrand.

The saddle on the imaginary axis corresponds to a geodesic passing inside the horizon.

The geodesic which goes inside the horizon does not contribute to the correlation function in the saddle point approximation.

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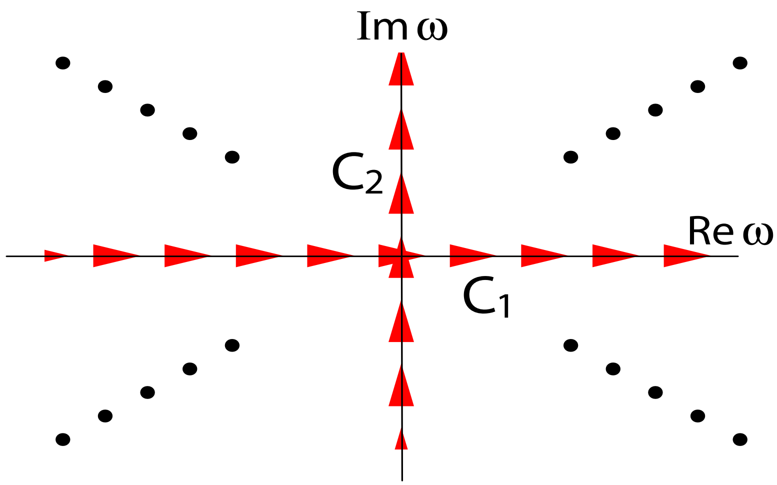
# New gauge invariant observables and signals of the singularity (II)

New observables:

$$H_{12}(\tau) = \int_{C_2} \frac{d\omega}{2\pi}\, e^{-i\omega\tau} G_{12}(\omega)$$




The divergence of $H_{12}(\tau)$ for $\tau \to \pm\frac{\tilde{\beta}}{2}$ reflects the divergence of a spacelike geodesic approaching the singularity.

# Signal amplification process

The geodesic which goes inside the horizon does *not* contribute to coordinate space correlation function $G_+(t)$ in the saddle point approximation.

However, through the process

$$G_+(t) \quad \rightarrow \quad G_+(\omega) \quad \rightarrow \quad G_{12}(\omega) \quad \rightarrow \quad H_{12}(\tau)$$

the signal of the singularity is amplified.

Recall $G_{12}(\omega, l) = e^{-\frac{\beta\omega}{2}} G_+(\omega, l)$.

# Yang-Mills theory at finite $N$ ?

At finite $N$, the YM theory on $S^3$ has a *discrete* spectrum, i.e. *finite* number of states below any given energy.

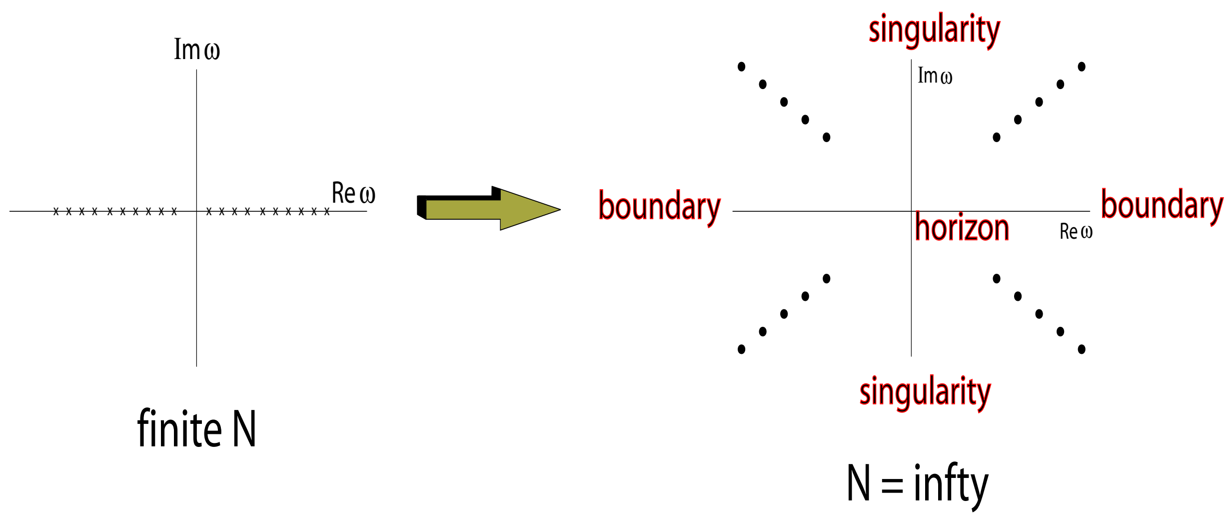
This should be true even for coupling of order $O(1)$.

In particular

$$G_+(\omega) = 2\pi \sum_{m,n} e^{-\beta E_m} \rho_{mn} \delta(\omega - E_n + E_m)$$

$m, n$ sum over the physical states of the theory.

# Resolution of the singularity at finite $N$ ?




The discrete spectrum at finite $N$ appears to be *in conflict* with the presence of the horizon.

This suggest the horizon and the singularity are *approximate* concepts valid *only in large $N$ expansion*.

# Fate of the singularity

The previous argument does not tell us in detail how the singularity is resolved. We can consider two logical possibilities:

1. The singularity is already resolved by $\alpha'$-effects in perturbative string theory, i.e. at *infinite* $N$ by finite 't Hooft coupling effects.

2. The singularity is resolved *only* at finite $N$.

If possibility 2 is realized, then the singularity should be visible in weakly coupled YM and one should be able to address the singularity problem by focusing on the large $N$ limit.

# Summary

- Wightman functions evaluated at imaginary frequencies can probe the region inside the horizon.

- The presence of the singularity implies exponential decay for $G_+(\omega, l)$ along the imaginary frequency axis.

- We constructed new observables in YM theory which appear to directly probe the singularity.

- The horizon and singularity should be resolved at finite $N$.