Noncritical M-Theory in $2+1$ Dimensions as a Nonrelativistic Fermi Liquid

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UC Berkeley & LBNL

Strings 2005, Toronto

work with Cindy Keeler

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Outline of the talk

I. Introducing noncritical M-theory

Motivation and definition

Two-dimensional Type 0A, 0B as solutions

The space of all vacua

II. (2+1)-dimensional vacuum of M-theory

Exact vacuum energy

Exact free energy at finite temperature

Symmetries, observables, bosonization

III. Spacetime physics as hydrodynamics

Emergent spacetime: Hydrodynamics of the Fermi surface

Time-dependent solutions: $e.g.$, interpolating

(Type 0 vacuum) $\rightarrow$ (M-vacuum) $\rightarrow$ (Type 0 vacuum)

IV. Conclusions

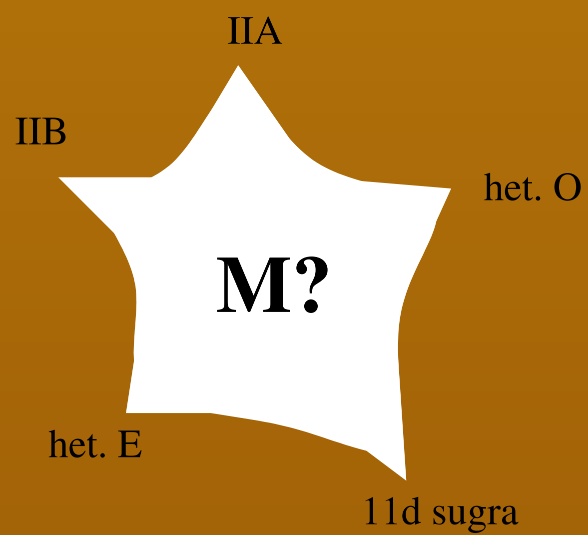
I.

Noncritical M-theory for noncritical strings

Motivation

10 years since the second string revolution, string theory is a unique theory, but not (always) of strings.

The starfish diagram:



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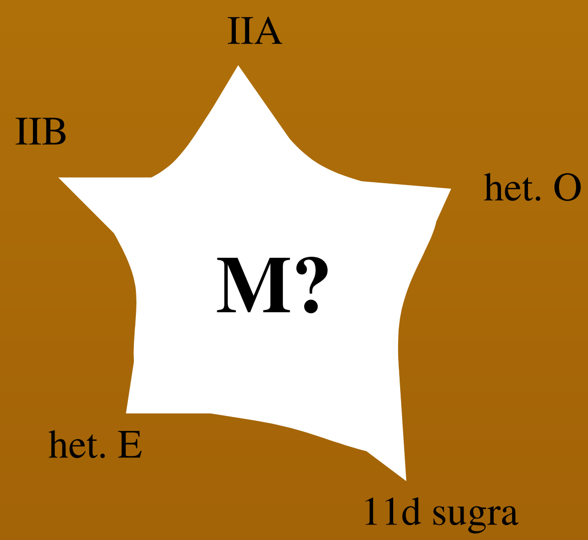

(narrow vs. broad sense of "M-theory")

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. . . but the degrees of freedom of M-theory remain mysterious.

It would be desirable to have a complete understanding of the starfish diagram (the “whole elephant”, or moduli space of solutions), and understand the underlying degrees of freedom in the process.

We will adress this goal in the highly controlled context of two-dimensional noncritical strings.

Type 0A and 0B strings in two dimensions

Type 0B theory:

Gauged $U(N)$ matrix model of an $N \times N$ matrix M,

$$S_{0\mathrm{B}} = \beta N \int dt \, \mathrm{Tr} \left(\frac{1}{2}(D_t M)^2 + \frac{1}{4\alpha'} M^2 + \ldots \right)$$

in a double scaling limit, involving $N \to \infty$.

Eigenvalues λ of M act as free fermions, theory can be formulated in terms of a second-quantized nonrelativistic Fermi field $\psi(\lambda, t)$ in $1+1$ dimensions. Double-scaling limit: $\epsilon_F \to 0$ with

$$\mu = N\epsilon_F \sim \frac{1}{g_s}$$

fixed and playing the role of the 0B string coupling.

Type 0A theory:

A $U(N) \times U(N+q)$ quiver matrix model of an $N \times (N+q)$ matrix M,

$$S_{0\mathrm{B}} = \beta N \int dt \,\mathrm{Tr} \left((D_t M)^\dagger D_t M + \frac{1}{2\alpha'} M^\dagger M + \ldots \right)$$

in the double scaling limit. N fermions in $1+1$ dimensions, in a potential modified by q.

q has several physical meanings:

- net D0-brane charge (0A has stable D0's & anti-D0's)
- background RR flux, of a RR two-form field strength

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- net D0-brane charge (0A has stable D0's & anti-D0's)
- background RR flux, of a RR two-form field strength
- angular momentum on a plane.

Noncritical M-theory for noncritical strings

We are looking for "noncritical M-theory".
Some desired properties may include:

- D0-charge reinterpreted as momentum along an extra, compact dimension (S^1) of M-theory ($i.e.$, expect the RR 1-form to be a KK gauge field);
- The string coupling g_s related to the radius of the extra S^1;
- Type 0A, 0B, . . . string theories should be solutions;
- Expect $2+1$ dimensional solutions, beyond $\hat{c} = 1$ string theory.

Such a theory indeed exists.

How to find it: In the matrix model language? In the string effective action language?

Our philosophy: The $\hat{c} = 1$ string theories are fully defined via the double-scaled Fermi theory. Look for the nonperturbative formulation of M-theory in the same language.

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Wait, isn't this proposal counterintuitive?

Intuition from critical M-theory: $R_{11} \sim g_s$; large radius corresponds to strong string coupling.

In $\hat{c} = 1$, string coupling grows towards the origin, and the weakly coupled regime at infinity, where R_3 is large.

String coupling vs. the radius R_3

A parable:

Consider the Einstein-Hilbert action in $D+1$ dimensions,

$$\frac{1}{\ell_{D+1}^{D-1}} \int d^{D+1}X \sqrt{G}\, \mathcal{R}(G),$$

reduce to the string-frame effective action,

$$\int d^D x \sqrt{g} e^{-2\phi} \left(\mathcal{R}(g) + \ldots \right).$$

For $D = 2$, we get

$$R_3 = \frac{\ell_3}{g_s^2}$$

Hence, strong string coupling corresponds to large radius.

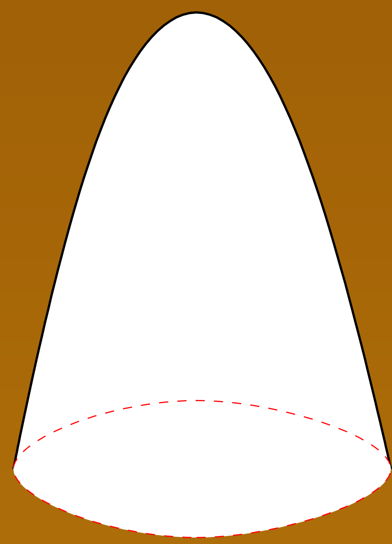
Nonperturbative noncritical M-theory as a Fermi liquid in 2+1 dimensions

Start with a nonrelativistic Fermi field theory of $\Psi(t,\lambda_1,\lambda_2)$ (a spinless field), on flat R^3 (parametrized by t,λ_i, $i=1,2$), in the upside-down harmonic oscillator potential,

$$S_M = \int dt\, d^2\lambda \left(i\Psi^\dagger\partial_t\Psi + \frac{1}{2}\Psi^\dagger(\Delta + \omega_0^2\lambda^2 + \ldots)\Psi \right).$$

Noncritical M-theory:= double-scaling limit of this system.

Careful double-scaling limit involves a regulating cutoff, by non-universal pieces in the potential. We will regulate by placing an infinite wall at some distance $\sqrt{2\Lambda}$ from the origin in the λ_i plane ($\omega_0 = 1$).



The moduli space of all solutions

Theory easily quantized, using info from double-scaled $1+1$ fermions.
Two natural representations:

Cartesian coordinates

Expansion in products of wavefunctions of Type 0B theory,

$$\Psi(t,\lambda_i) = \int d^2E\, a_{\alpha_1\alpha_2}(E_1,E_2)e^{i(E_1+E_2)t}\psi_{\alpha_1}(E_1,\lambda_1)\psi_{\alpha_2}(E_2,\lambda_2)$$

with anticommutation relations:

$$\{a_{\alpha_1\alpha_2}(E_1,E_2), a^{\dagger}_{\alpha'_1\alpha'_2}(E'_1,E'_2)\} = \delta^2(E_i - E'_i)\delta_{\alpha_1\alpha'_1}\delta_{\alpha_2\alpha'_2}$$

Polar coordinates λ, θ

Expansion in Type 0A wavefunctions,

$$\Psi(t,\lambda,\theta) = \sum_{q\in\mathbf{Z}} e^{iq\theta} \int dE\, a_q(E) e^{iEt} \psi_q(E,\lambda)$$

(again, q is the Type 0A RR-flux).

More on the double-scaling limit

Two parts:

- Introduce cutoff Λ, N fermions, take $N \to \infty$;
- Simultaneously, identify the scaling variable (typically, a product of N with a conserved quantity, such as energy or angular momentum).

General philosophy on the moduli space of solutions:

A state is specified by declaring how each canonical pair of oscillators acts on it. Typically, two such different states are not in each other's Hilbert spaces; represent (excitations of) different vacua.

Most states do not have a “smooth” hydrodynamic description (such as in terms of the bosonic profile of a Fermi surface).

Spacetime is an emergent property: For states that do have a hydrodynamic description (bosonization).

Time-dependent solutions emerge on an equal footing with static solutions.

Thus, the free fermions define the starfish diagram.

Type 0A and 0B string vacua as solutions

Type 0A strings in the linear dilaton vacuum

Pick a value of q. Define $|0\mathrm{A}, \mu\rangle$ by filling only that sector, up to $-\mu$:

$$a_q(E)|0\mathrm{A}, \mu\rangle = 0, \quad E > -\mu,$$
$$a_q^\dagger(E)|0\mathrm{A}, \mu\rangle = 0, \quad E < -\mu,$$

and

$$a_{q'}(E)|0\mathrm{A}, \mu\rangle = 0, \quad q' \neq q, \text{ for all } E.$$

(Note: This is not equivalent to just naively sending $\mu \to \infty$ in sectors of $q' \neq q$ after the double-scaling limit.)

Type 0B strings in the linear dilaton vacuum

Pick a value of E_2. Define $|0\mathrm{B},\mu\rangle$ by filling only that sector, up to $-\mu$:

$$\begin{aligned} a_{\alpha_1\alpha_2}(E_1,E_2)|0\mathrm{B},\mu\rangle &= 0, \quad E_1 > -\mu, \\ a^\dagger_{\alpha_1\alpha_2}(E_1,E_2)|0\mathrm{B},\mu\rangle &= 0, \quad E_1 < -\mu, \end{aligned}$$

and

$$a^\dagger_{\alpha_1\alpha_2}(E_1,E_2')|0\mathrm{B},\mu\rangle = 0, \quad E_2' \neq E_2, \ \text{for all } E_1.$$

Comments:

• Unlike in Type 0A, the choice of E_2 does not add a new parameter – shift of E_1 equivalent to shift in μ.

• T-duality between Type 0A and 0B is non-obvious in this M-theory framework (just as it is not obvious between IIA and IIB in critical M-theory).

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II.

The (2+1)-dimensional vacuum

The 2+1 dimensional vacuum of noncritical M-theory

The regulated theory at finite N has an obvious solution $|\mathrm{M},\mu\rangle$: Fill every state up to a uniform Fermi energy $-\mu$, (irrespective of q, etc.).

$$a_q(E)|\mathrm{M},\mu\rangle = 0, \quad E > -\mu, \text{ all } q$$

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We will refer to this state as the "M-theory vacuum."
This is the noncritial analog, for strings in the linear dilaton background, of the 11-dimensional Minkowski solution of critical M-theory.

Let us explore some of the properties of $|\mathrm{M},\mu\rangle$.

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Vacuum energy of noncritical M-theory

We shall calculate the exact vacuum energy of this solution.

Recall first how it works in string theory (Type 0 or bosonic in the linear dilaton background):

Define the density of states $\rho(\mu)$. In terms of $\rho(\mu)$, the energy of the ground state is given by

$$F_0 \sim \int \mu\rho(\mu)\, d\mu.$$

Asymptotic expansion in weak string coupling $1/\mu$:

$$\rho(\mu) \sim \ln(\mu/\Lambda) + \mathcal{O}(1/\mu^2).$$

This gives for the ground-state energy

$$F_0 \sim \mu^2 \ln(\mu/\Lambda) + \ln(\mu/\Lambda) + \mathcal{O}(1/\mu^2).$$

This is reinterpreted as the sum over connected Riemann surfaces.

$\ln(\mu/\Lambda)$ is the volume of the Liouville dimension.

All terms in this series are nontrivial, and given in terms of Bernoulli numbers.

Vacuum energy: Leading order in $1/N$

For the M-theory vacuum, look first at the leading order in $1/N$, identify the scaling variable.

$N \to \infty$ is the semiclassical limit. We have

$$N = \int \frac{d^2 p d^2 \lambda}{(2\pi\hbar)^2} \theta(\frac{1}{2}p^2 - \frac{1}{2}\lambda^2 - \epsilon)$$

The density of states is

$$\rho(\epsilon) = \hbar \frac{\partial N}{\partial \epsilon} \sim \int_{\sqrt{2\epsilon}}^{\sqrt{2\Lambda}} \lambda \, d\lambda \sim (-\epsilon + \Lambda)/\hbar.$$

This gives

$$F_0 \sim \frac{\epsilon_F^3}{\hbar^3} + \ldots$$

Hence:

- The scaling variable is $\mu \equiv -\epsilon_F N$, just as in $\hat{c} = 1$ string theory;

- The leading log of string theory disappears; instead, the leading ("tree-level") term in the vacuum energy scales as μ^3. (We shall write this as κ^{-2}.)

- The natural expansion parameter is $1/\mu \sim \kappa^{2/3}$. Recall heterotic M-theory in 11d!

- The volume dependence is absent from the leading term in F_0

- Particle-hole duality predicts that ρ should be an even function!

Vacuum energy: The double-scaling limit

Thus, the double-scaling limit is defined as in string theory, $N \to \infty$ and $\epsilon_F \to 0$ while $\mu = i\epsilon_F N$ is fixed.

The double-scaled density of states in M-theory can be evaluated (in several equivalent ways),

$$\rho(\mu) \sim \int_0^\infty dT \cos(\mu T) \frac{1}{\sinh(T/2)} \sum_q e^{-|q|T/2} = \int_0^\infty dT \cos(\mu T) \frac{1/2}{\sinh^2(T/4)}$$

This is an exact formula.

The expression diverges in a trivial way, calculate $\partial\rho/\partial\mu$ instead, or regulate by Λ. (The divergence is μ-independent.)

Vacuum energy: The weak coupling expansion

$1/\mu$ is a dimensionless (in $\omega_0 = 1$ units) coupling constant in M-theory, analogous to the string coupling.

We can expand in small $1/\mu$. We get:

$$\frac{\partial \rho}{\partial \mu} = -\frac{1}{2} + \mathcal{O}(1/\mu).$$

This reproduces our WKB expectation, $\rho \sim \mu$.

Now look at higher orders in $1/\mu$. They can be viewed as an infinite sum of nonzero, multiloop contributions to the vacuum energy of Type 0A with all possible values of the RR flux q. They all sum to zero!

Hence, the perturbative vacuum energy is one-loop exact in perturbation theory,

$$F_0 \sim -\frac{\mu^3}{6} + c_2\mu^2 + c_1\mu + C_0.$$

There could be nonperturbative corrections, which we now determine.

Vacuum energy: The strong coupling expansion

Expand our exact formula

$$\int_0^\infty dT \cos(\mu T) \frac{1/2}{\sinh^2(T/4)}.$$

in powers of μ. One gets

$$\rho(\mu) \sim \sum_{m=1}^{\infty} (2\pi\mu)^{2m} \frac{B_{2m}}{(2m)!} + c.$$

Using the definition of Bernoulli numbers,

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = \frac{x}{e^x - 1},$$

and properties of Bernoulli numbers, we get

$$\rho(\mu) = \frac{\mu}{e^{2\pi\mu} - 1} - B_1 \mu = \mu \left(\frac{1}{e^{2\pi\mu} - 1} + \frac{1}{2} \right) + c.$$

This looks very suggestive of a dual bosonic DoF (the Planck distribution)!

μ plays effectively the role of the Debye frequency ω_D in this bosonic dual.

Vacuum energy: The exact formula

This exact formula can now be integrated to get the vacuum energy F_0, which then can be re-expanded in powers of $1/\mu$:

$$F_0 \sim -\frac{\mu^3}{6} + c\mu^2 + \sum_{k=1}^{\infty} \left(\frac{\mu^2}{2\pi k} + \frac{\mu}{2\pi^2 k^2} + \frac{1}{4\pi^3 k^3} \right) e^{-2\pi k \mu}.$$

This is an infinite series of nonperturbative instanton-like contributions, each of which is one-loop exact, and starts at order $\kappa^{2/3}$ compared to the perturbative leading term!

M-theory at finite temperature

The calculation can be extended to Euclidean compactified time. The free energy $\Gamma(\mu, R)$ is given by

$$\frac{\partial^2\Gamma}{\partial\mu^2} \sim \frac{\frac{1}{2\pi R}\frac{\partial}{\partial\mu}}{\sin\left(\frac{1}{2\pi R}\frac{\partial}{\partial\mu}\right)}\frac{\pi\mu}{\tanh(\pi\mu)}.$$

Unlike the string-theory free energy, this does not exhibit any obvious T-duality. (Just as in the case of critical M-theory.)

Symmetries of noncritical M-theory

The following quantities

$$a_1 = (p_1 - \lambda_1)e^t, \qquad b_1 = (p_1 + \lambda_1)e^{-t},$$

$$a_2 = (p_2 - \lambda_2)e^t, \qquad b_2 = (p_2 + \lambda_2)e^{-t}$$

are conserved. They represent elementary constituents of the symmetry algebra, which is generated by

$$W_{m_1 m_2 n_1 n_2} = a_1^{m_1} a_2^{m_2} b_1^{n_1} b_2^{n_2},$$

for $n_i, m_i = 0, 1, \ldots$. This is an M-theory generalization of the w_∞ symmetry algebras of $\hat{c} = 1$ string theories.

(Up to) quadratic charges form a closed subalgebra. Both H and J are such quadratic charges.

Fermi surface (=solution) preserved by only a subalgebra. Other symmetries spontaneously broken by the solution.

Notice that the Fermi surface of the M-theory vacuum defines AdS_3 in the phase space $R^{2,2}$.

Observables of noncritical M-theory

A natural bosonic observable: Density of eigenvalues

$$\rho(t,\lambda_i) = \Psi^\dagger \Psi(t,\lambda_i).$$

M-theory analog of the massless tachyon (& RR scalar) of $\hat{c} = 1$ strings:

$$W(t,x_1,x_2) = \int d^2\lambda e^{-x\cdot\lambda} \Psi^\dagger(t,\lambda_i)\Psi(t,\lambda_j)$$

In fact, in the M-theory vacuum it is convenient to define

$$\tilde{W}(t,r,\theta) = \int_{\sqrt{2\mu}}^{\infty} d\lambda e^{-r|\lambda|} \Psi^\dagger(t,\lambda_i)\Psi(t,\lambda_j),$$

with θ the angle on the eigenvalue plane. This is very reminiscent of formulas for multidimensional bosonization from condensed matter.

In some states, W behaves semiclassically. For those states, it is likely that a dual, spacetime description in terms of an effective (hydrodynamic) action for W can be found.

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III.

Spacetime physics as hydrodynamics of the Fermi surface

Hydrodynamic equations for the Fermi surface

can be derived just as in noncritical string theory.

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Choose, for example, $p_1 = P(p_2, \lambda_1, \lambda_2, t)$ as the dependent variable. Then the EoM is

$$\partial_t P = \lambda_1 - (\lambda_2 \partial_{p_2} + p_2 \partial_{\lambda_2})P - P\partial_{\lambda_1} P.$$

(Of course, sometimes it is useful to switch to another representation, such as in polar coordinates.)

Various static as well as time-dependent solutions can be easily found.

Some static solutions

A simple modification $|\tilde{\mathrm{M}}, \mu\rangle$ of the M-theory vacuum:
Fill up to $-\mu$ for even q, and down to $-\mu$ for odd q.

This is an example of a slightly exotic state, which nevertheless seems hydrodynamical (but slightly outside the EoM for the Fermi surface).

Another example:

Define a family of Fermi surfaces via

$$\frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 - \frac{1}{2}\lambda_1^2 - \frac{1}{2}\lambda_2^2 + \Omega(p_1\lambda_2 - p_2\lambda_1) = -\mu.$$

(Ω: angular velocity). Playing with Ω and μ, one can interpolate between Fermi surfaces set by H and J.

An interesting duality

Consider the Fermi surface given by

$$p_1\lambda_2 - p_2\lambda_1 = q.$$

A linear canonical transformation maps this to

$$\frac{1}{2}\tilde{p}_1^2 + \frac{1}{2}\tilde{\lambda}_1^2 - \frac{1}{2}\tilde{p}_2^2 - \frac{1}{2}\tilde{\lambda}_2^2 = -\tilde{\mu}$$

Under this map, $H \leftrightarrow \tilde{J}$, $J \leftrightarrow \tilde{H}$.

Duality to the Hamiltonian of the thermofield dynamics of free fermions in the rightside-up harmonic oscillator potential.

Contact with the Itzhaki-McGreevy, . . . string theory (and its M-theory lift).

Some time-dependent solutions

Use the time-dependent charges to find time-dependent solutions for the EoM of the Fermi surface.

For example,

$$\frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 - \frac{1}{2}\lambda_1^2 - \frac{1}{2}\lambda_2^2 + c_1(p_1 + \lambda_1)e^{-t} + c_2(p_2 - \lambda_2)e^{t} = -\mu$$

(with c_1, c_2 constants) is a typical such solution.

It represents a Type 0B string theory in the far past, evolving via the M-theory phase, and decaying into another Type 0B theory in the future.

Novelty compared to string theory: well-defined macroscopic initial and final states.

Tachyon scattering can be studied in this time-dependent background. Is it possible to define a unitary S-matrix?

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IV.

Conclusions

Lessons for string/M-theory

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• Moduli space of solutions fully defined via fermions
• Spacetime emergent as a hydrodynamic description of only some states
• DoF of M-theory (nonrelativistic fermions/D-branes)
Recall [P.H., hep-th/0502006]: K-theory classifies stable Fermi surfaces.
• Remarkable "moral" similarities with M(atrix) theory
• and holographic field theory.

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Lessons for quantum gravity

• Two successful approaches to quantum gravity in $2+1$ dimensions so far:
Chern-Simons theory, critical string/M-theory. Noncritical M-theory might be a third. Indications that the topological nature of the CS formulation is combined with propagating DoF.
• Vacuum energy ($\approx$ cosmo. constant) has fascinating features;

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The End