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Multi-parameter deformations of $AdS_5 \times S^5$ geometry

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Outline

- Introduction
- STsTS Transformation
- 3-parameter Deformation
- BPS States, Lax Pair
- Tests of Duality
- Conclusion

Introduction

$\mathcal{N}=4$ SYM $\equiv$ IIB $AdS_5 \times S^5$ [Maldacena, 1997]

Many checks! Still a conjecture ...

2-loop agreement; 3-loop disagreement [Callan McLoughlin Swanson] ; [Serban Staudacher]

- What is the underlying reason?
- Conformal invariance, supersymmetry or integrability? Beisert's review
- Bethe ansatz for quantum strings? [Arutyunov, SF Staudacher]
- More "realistic", less supersymmetric, examples?
- Given a field theory, what is dual string theory, and vice versa?

$$\mathcal{N}=4_\beta \text{ SYM} \equiv \text{IIB } (AdS_5 \times S^5)_\beta$$

β-deformation of $\mathcal{N}=4$ SYM

β-deformed $AdS_5 \times S^5$

[Leigh,Strassler: hep-th/9503121]

[Lunin,Maldacena: hep-th/0502086]

Superpotential

$$W = \mathrm{Tr}(e^{i\pi\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\Phi_1\Phi_3\Phi_2)$$

Features:

- continuous complex deformation parameter, β
- Conformal invariance; $\mathcal{N} = 1$ supersymmetry
- no integrability for complex β [Berenstein,Cherkis]
- Non-singular background: $AdS_5 \times S^5$, $SL(3, R)$ [Lunin's talk]

Questions:

- Can we use the ideas from $\mathcal{N} = 4$ to test new duality? [BMN, FT]
- Importance of supersymmetry and integrability?
- Is string theory integrable for real β ?
- Find non-supersymmetric examples of duality?

STsTS transformation

LM used $SL(3,R)$ of IIB on T^2. [Lunin's talk]

LM transformation $\equiv$ $S_\sigma T s_\gamma T S_{-\sigma}$ [Lunin,Maldacena; SF]

$$S_\sigma : \begin{pmatrix} 1 & \sigma \\ 0 & 1 \end{pmatrix} \in SL(2,R)_s, \quad \text{this } \sigma \neq \text{ the one in LM}$$

$T s_\gamma T$ acts on torus $(\tilde{\tilde{\phi}}_1, \tilde{\tilde{\phi}}_2)$:

(i) T-duality on $\tilde{\tilde{\phi}}_1$: $\tilde{\tilde{\phi}}_1 \to \tilde{\phi}_1$; IIB $\to$ IIA;

(ii) shift s_γ: $\tilde{\tilde{\phi}}_2 \to \tilde{\phi}_2 + \gamma\tilde{\phi}_1$;

(iii) T-duality on $\tilde{\phi}_1$: $\tilde{\phi}_1 \to \phi_1$; IIA $\to$ IIB;

IIB $\to$ IIB; g_{string} remains *small*.

STsTS gives 2-par (γ, σ) deformation of the initial background.

Choose a special torus in $AdS_5 \times S^5$ $\to$ LM background.

$T^3 : (\phi_1, \phi_2, \phi_3) \subset$ initial background.

3 “natural” 2-tori: $(\phi_1, \phi_2), (\phi_2, \phi_3), (\phi_3, \phi_1)$.

6-parameter deformation

$$(S_{\sigma_1} T s_{\gamma_1} T S_{-\sigma_1}) \cdot (S_{\sigma_2} T s_{\gamma_2} T S_{-\sigma_2}) \cdot (S_{\sigma_3} T s_{\gamma_3} T S_{-\sigma_3})$$

$$S^5 : X_1 \bar{X}_1 + X_2 \bar{X}_2 + X_3 \bar{X}_3 = 1;\ X_k = \rho_k e^{i\phi_k}$$

For $AdS_5 \times S^5$, the deformed background in [SF: hep-th/0503201]

- 6-par background dual to a nonsusy marginal deformation of $\mathcal{N} = 4$ SYM.
- YM potential can be found by using associative $*$-product by LM.
- $SU(3)$ sector, $\sigma_i = 0$: dilatation operator is known [Roiban: hep-th/0312218]
- LM background: $\gamma_i = \gamma,\ \sigma_i = \sigma$

3-parameter deformation

$$\sigma_i = 0: \quad (Ts_{\gamma_1}T)\cdot(Ts_{\gamma_2}T)\cdot(Ts_{\gamma_3}T)$$

TsT in string σ-model $\Rightarrow$ relations between $\tilde{\tilde{\phi}}_i$ and ϕ_i $\Rightarrow$ BPS states; Lax pair (integrability) $\Rightarrow$ string Bethe eqs

S^5 part of string action on $AdS_5 \times S^5$

$$\tilde{\tilde{S}} = -\frac{\sqrt{\lambda}}{2}\int d\tau \frac{d\sigma}{2\pi}\left(\partial_\alpha \rho_i \partial^\alpha \rho_i + \rho_i^2 \partial_\alpha \tilde{\tilde{\phi}}_i \partial^\alpha \tilde{\tilde{\phi}}_i\right)$$

$\sqrt{\lambda} = R^2/\alpha'$; $\rho_i^2 = 1$, $i = 1, 2, 3$, $\tilde{\tilde{\phi}}_i$ are angle variables.

$Ts_{\gamma_3}T$ on $(\tilde{\tilde{\phi}}_1, \tilde{\tilde{\phi}}_2)$:

(i) T-duality on $\tilde{\tilde{\phi}}_1$: $\tilde{\tilde{\phi}}_1 \to \tilde{\phi}_1$;

(ii) shift: $\tilde{\tilde{\phi}}_2 \to \tilde{\phi}_2 + \tilde{\gamma}_3\tilde{\phi}_1$;

(iii) T-duality on $\tilde{\phi}_1$: $\tilde{\phi}_1 \to \phi_1$, $\tilde{\phi}_2 = \phi_2$;

Repeating for $(\phi_2, \phi_3);\ (\phi_3, \phi_1)$, we get in string frame

$$
\begin{aligned}
\frac{1}{R^2} ds^2_{S^5_{\gamma_i}} &= \sum_{i=1}^{3} (d\rho_i^2 + G\rho_i^2 d\phi_i^2) + G\rho_1^2\rho_2^2\rho_3^2 [d(\sum_{i=1}^{3} \tilde{\gamma}_i \phi_i)]^2 \\
B_2 &= R^2 G w_2\,, \quad w_2 \equiv \tilde{\gamma}_3 \rho_1^2 \rho_2^2 d\phi_1 d\phi_2 + \text{cyclic permut} \\
e^{\phi} &= e^{\phi_0} G^{1/2}\,, \qquad\qquad \chi = 0 \\
G^{-1} &\equiv 1 + \tilde{\gamma}_3^2 \rho_1^2 \rho_2^2 + \tilde{\gamma}_1^2 \rho_2^2 \rho_3^2 + \tilde{\gamma}_2^2 \rho_1^2 \rho_3^2
\end{aligned}
$$

- There are also C_2 and F_5.
- The background is not supersymmetric. Tachyons?
- String sigma model is integrable! [SF: 0503201]
- Lax pair can be found, and used to derive string Bethe eqs; first found for $su(2)$ sector of IIB on $AdS_5 \times S^5$ Kazakov, Marshakov Minahan, Zarembo

How TsT transforms strings?

The relations between $\tilde{\tilde{\phi}}_i$, and ϕ_i: $\tilde{\tilde{J}}_i{}^\alpha = J_i^\alpha$

$\tilde{\tilde{J}}_i{}^\alpha$, J_i^α are $U(1)$ conserved currents; $U(1)$ charges: $J_i = \int \frac{d\sigma}{2\pi} J_i^0$

J_i^0 is the momentum conjugated to ϕ_i: $p_i = J_i^0$; $\tilde{\tilde{p}}_i = \tilde{\tilde{J}}_i{}^0$

$$\tilde{\tilde{p}}_i = p_i \; , \quad \tilde{\tilde{\phi}}{}'_i = \phi'_i - \epsilon_{ijk}\gamma_j p_k \; , \quad \gamma_i = \tilde{\gamma}_i/\sqrt{\lambda}$$

ϕ_i are angles, and strings in the deformed background are closed:

$$\phi_i(2\pi) - \phi_i(0) = 2\pi n_i \; , \quad n_i \text{ are integer winding numbers}$$

Twisted boundary conditions for $\tilde{\tilde{\phi}}_i$ of the original S^5 space

$$\tilde{\tilde{\phi}}_i(2\pi) - \tilde{\tilde{\phi}}_i(0) = 2\pi(n_i - \nu_i) \; , \quad \nu_i \equiv \epsilon_{ijk}\gamma_j J_k$$

If ν_i are not integers then TsT breaks a closed string.

BPS states

If ϕ_i solve eqm for a closed string in $AdS_5 \times S^5_{\gamma_i}$ then $\tilde{\tilde{\phi}}_i$ solve those for a twisted string in $AdS_5 \times S^5$.

Virasoro constraints map to each other under TsT $\Rightarrow$

$$E_{\text{twisted string}} = E_{\text{closed string}}$$

BPS states in $AdS_5 \times S^5$: minimal energy for the given charges,

$$E = J_1 + J_2 + J_3$$

BPS state in $AdS_5 \times S^5_{\gamma_i}$ is an image of a BPS state in $AdS_5 \times S^5$ $\Rightarrow \tilde{\tilde{\phi}}_i{}' = 0,\ \ \rho_i' = 0$; $\tilde{\tilde{p}}_i = p_i = J_i$ do not depend on σ:

$$\phi_i' = \epsilon_{ijk}\gamma_j p_k = \nu_i \ , \qquad p_i = J_i$$

String in the deformed background is closed:

$$\nu_i = \epsilon_{ijk}\gamma_j J_k \quad \in Z$$

- $\nu_i \neq 0 \Rightarrow \gamma_i$ are rational: solution is a circular string [SF, Tseytlin]
- $\nu_i = 0 \Rightarrow$ solution is a point-like string: $J_i \sim \gamma_i$; $\frac{\gamma_i}{\gamma_j}$ is rational
- $(J_1, J_2, J_3) \rightarrow (J, J, J)$ in the LM case $\gamma_i = \gamma$, $J_i = J$.
- If $\nu_i = 0$, any (J_1, J_2, J_3) string in $AdS_5 \times S^5_{\gamma_i}$ can be obtained from a closed string in $AdS_5 \times S^5$; their energies are equal to each other.
- If $J_3 = 0$, then the string states belong to the 2-spin $su(2)_\gamma$ sector with BPS state $(J, 0, 0)$. Similarly, $(0, J, 0)$ and $(0, 0, J)$.

Point-like strings in $S^5_{\gamma_i}$ are governed by harmonic oscillator on S^5: Neumann system [SF, Roiban, Tseytlin: 0507021]

How to get Lax pair

- Start with usual Lax pair for sigma model on S^5
- It is not $U(1)^3$ invariant $\Rightarrow$ explicit dependence on $\tilde{\tilde{\phi}}_i$
- Make a gauge transformation to get $U(1)^3$ invariant Lax pair with dependence only on $\partial_\alpha \tilde{\tilde{\phi}}_i$
- Replace $\partial_\alpha \tilde{\tilde{\phi}}_i$ by $\partial_\alpha \phi_i$ by using $\tilde{\tilde{J}}_i{}^\alpha = J_i^\alpha$
- This gives Lax pair for sigma model on $S^5_{\gamma_i}$
- Use it to derive string Bethe eqs following

[Kazakov,Marshakov,Minahan,Zarembo; Kazakov,Zarembo; Beisert,Kazakov,Sakai; Beisert,Kazakov,Sakai,Zarembo]

It was done for $su(2)$ reduction, $\rho_3 = 0$ and the complete agreement with thermodynamic limit of spin chain Bethe eqs was found [SF, Roiban, Tseytlin: 0503192]

Tests of Duality

Scalar potential of γ_i-deformed $\mathcal{N}=4$ SYM [SF: 0503201]

$$V = \mathrm{Tr}\sum_{(i,j,k)}^{3} |e^{i\pi\gamma_k}\Phi_i\Phi_j - e^{-i\pi\gamma_k}\Phi_j\Phi_i|^2 + \mathrm{Tr}\sum_{i=1}^{3}[\Phi_i, \bar{\Phi}_i]^2$$

and similarly deformed Yukawa couplings.

One-loop dilatation operator in $su(3)$ sector of 3 holomorphic scalars, $\mathrm{Tr}(\Phi_1^{J_1}\Phi_2^{J_2}\Phi_3^{J_3}+\cdots)$ in $\mathcal{N}=4$ SYM [Minahan, Zarembo]

$$H = \frac{\lambda}{8\pi^2}\sum_{k=1}^{L} H_{k,k+1} \; ; \; H_{k,k+1} = I_{k,k+1} - P_{k,k+1}$$

is Hamiltonian of integrable spin chain of length L. [Beisert's review]

Generalization to γ_i-deformed $\mathcal{N}=4$ SYM [Roiban; Berenstein,Cherkis]

$$H_{k,k+1} = e_2^2 \otimes e_3^3 + e_3^3 \otimes e_2^2 - e^{2i\pi\gamma_1} e_2^3 \otimes e_3^2 - e^{-2i\pi\gamma_1} e_3^2 \otimes e_2^3 + \text{cyclic perm}$$

$(e_m^n)_i^j = \delta_i^m \delta_n^j;$

- It is integrable spin chain.
- Bethe ansatz is known for $su(3)$ [Roiban: 0312218]
- and for complete model [Beisert,Roiban: 0505187]
- Hamiltonian of the 6-parameter deformation obtained by using STsTS is not integrable. [Berenstein,Cherkis]

If $J_3 = 0$, $su(2)$ sector, $\mathrm{Tr}(\Phi_1^{J_1}\Phi_2^{J_2} + \cdots)$, is the same as in LM case. One-loop dilatation operator for $su(2)$ sector is integrable for complex β.

Results for $su(2)$ sector of LM background [SF, Roiban, Tseytlin: 0503192]

- Spectrum of BMN-type operators from Bethe ansatz = spectrum of fluctuations near $(J, 0)$; pp-wave;

  [Niarchos Prezas ; Lunin Maldacena]

- String Bethe equations agree with thermodynamic limit of spin chain Bethe eqs for heighest-weight states;
- Subleading $1/J$ corrections computed by using Bethe ansatz and string theory match even for complex β;
- Fast string action = Coherent state action, generalizing [Kruczenski]

Results for $su(3)$ sector of γ_i-deformed background

SF, Roiban Tseytlin : 0507021

- Point-like strings = Neumann system on S^5;
- Spectra of fluctuations near $(J, 0, 0)$ and (J_1, J_2, J_3) from strings, coherent state action and BA match;
- Fast string action = Coherent state action, nontrivial choice of equivalent H;
- E of point-like strings from FSA = Δ of quasi-BPS ops from BA;

Related results

- Integrable deformations of $\mathcal{N} = 4$ spin chain; [Beisert, Roiban: 0505187]
- Penrose limits and pp-waves; [Koch, Murugan, Smolic^2: 0505227]; [Mateos: 0505243]
- Semiclassical strings in LM background [Babev, Dimov, Rashkov: 0506063]
- Properties of LM metric [Pal: 0505257]

Conclusion

- Explained how deformed backgrounds are generated by STsTS transformation
- $SL(2,R)_s$ in STsTS breaks integrability
- TsT preserves integrability; Lax pair for γ_i-deformed background
- Found all BPS states by using TsT
- Degeneracy of states in $\mathcal{N}=4$ is lifted in deformed model
- Determined potential of γ_i-deformed SYM by using the LM $*$-product
- Discussed various successful tests of the duality
- Integrability seems to be more important then supersymmetry

Problems

- Stability of the background: tachyons in sugra spectrum?
- Dual to $SU(N)$ or $U(N)$? [Freedman, Gursoy: 0506128]; [Penati, Santambrogio, Zanon: 0506150]
- Are double-trace operators generated? [Dymarsky, Klebanov, Roiban: 0505099]
- Is γ_i-deformed SYM conformal for finite N?
- Green-Schwarz string on γ_i-deformed background, and Lax
- String Bethe equations from the Lax pair
- Bethe ansatz for quantum strings: [Arutyunov, SF, Staudacher: 0406256]
  consistency condition for dressing factor?
- $1/J$ to spectrum of fluctuations near $(J, 0, 0)$ and (J_1, J_2, J_3)
- Similar problems for the general 6-parameter deformation
- Apply STsTS to other background with $U(1)^3$ symmetry, e.g. toric manifolds or Klebanov-Strassler background