Going Beyond Bekenstein and Hawking

Exact and Asymptotic Degeneracies of Small Black Holes

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- ***A. Dabholkar*** 0409148

PRL 94, 2005

- ***A. D., Kallosh, & Maloney*** 0410076

JHEP 0412:059,2004

- ***A. D., Denef, Moore, & Pioline*** 0502157

JHEP, 2005

- ***A. D., Denef, Moore, & Pioline*** 0507014

S(Q) = k log Ω (Q) ?

- For supersymmetric black holes with large classical area, one can explain the thermodynamic entropy in terms of microscopic counting. For example,

$$\frac{A(Q_1, Q_5, P)}{4} = 2\pi\sqrt{Q_1 Q_5 P}$$

Bekenstein-Hawking *Strominger-Vafa*

Can we compute Corrections?

- **Macroscopic** (from effective action)

$$S = a_0 A(Q) + a_1 \log A(Q) + \frac{a_2}{A(Q)} + \ldots$$

- **Microscopic** (from brane configurations)

$$k \log \Omega(Q) = b_0 A(Q) + b_1 \log A(Q) + \frac{b_2}{A(Q)} + \ldots$$

- For a class of special BPS black holes in 4d, N=4 string theories, one can compute both $\{a_i\}$ and $\{b_i\}$ exactly to all orders to find that

$$a_i = b_i \qquad\qquad i = 0, 1, 2, 3, \ldots$$

- On the microscopic side, one should be able to count the states exactly.
- On the macroscopic side, one must take into account higher derivative corrections to Einstein-Hilbert action, solve the equations, compute the corrections to the entropy.
- To make a comparison for finite area corrections, one must decide on the statistical ensemble.

Ingredients

Counting: *Heterotic perturbative BPS states*

Action: *N=2 sugra, topological string*

Entropy: *Bekenstein-Hawking-Wald*

Solution: *Attractor mechanism*

Ensemble: *Mixed OSV ensemble*

Small Black Holes

- **Microscopics easy.**

  Counting can be done ***exactly*** because these are perturbative states.

- **Classical area vanishes**

  $\mathbf{A_c = 0.}$

- **Macroscopics difficult.**

  Quantum corrections to sugra essential.

Results

- Heterotic on $\mathbf{M}^5$ **£** $\mathbf{S}^1$.
- Take a string wrapping w times and carrying momentum n along $\mathbf{S}^{1.}$ It is **BPS** if in the right-moving ground state but can have arbitrary left-moving oscillations.

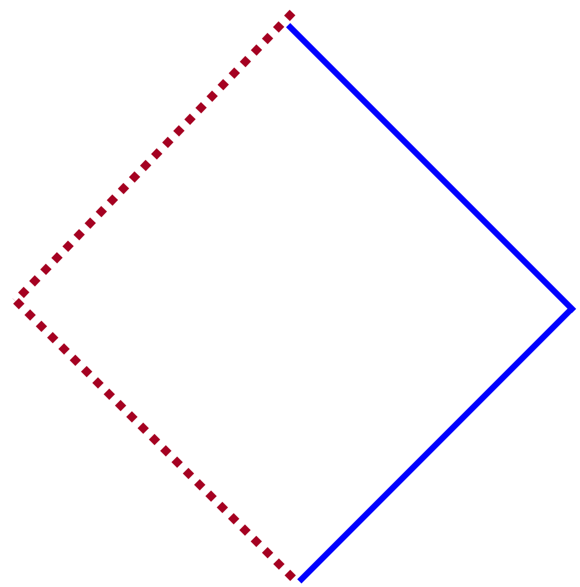
Dabholkar & Harvey

- Going back to the pre D-brane idea of \`Elementary Strings as Black Holes'

Sen; Susskind; Horowitz & Polchinski

Macroscopic side

- Classical spacetime is singular. There is a mild, null singularity. Classical area $\mathbf{A_c = 0}$.



Stringy Cloak for a Null Singularity

- Once we include quantum corrections, the singularity is `cloaked' and we obtain a spacetime with a regular horizon.
- The quantum corrected entropy to leading order is given by

$$S = 4\pi\sqrt{Q^2/2}$$

- In the corrected geometry, the entropy is finite. The counting is governed by the number of abelian vector fields $\mathbf{n_v}$ in the low energy N=2 supergravity and is also given by a Bessel function

$$\Omega_{macro}(Q) \sim I_{\frac{nv+2}{2}}(4\pi\sqrt{Q^2/2})$$

The overall normalization cannot yet be fixed.

Microscopic Counting

- **Microscopic** partition function that counts these states with charge vector **Q** is generically a modular form of negative weight w. The asymptotics are governed by a (hyperbolic) Bessel function

$$\Omega_{micro}(Q) \sim I_{|w|+1}(4\pi\sqrt{Q^2/2})$$

$$\sim \exp\left(4\pi\sqrt{nw}\right)$$

The asymptotic expansion is given by

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}}\{1 - \frac{(\mu - 1^2)}{1!(8z)} + \frac{(\mu - 1^2)(\mu - 3^2)}{2!(8z)^2} - \frac{(\mu - 1^2)(\mu - 3^2)(\mu - 5^2)}{3!(8z)^3} + \ldots\}$$

with $\mu = 4\nu^2$. For us, $z = 4\pi\sqrt{Q^2/2}$.

Action

- We have a charged state that couples to gravity, abelian vector fields, and moduli.
- In N=2 supergravity, the effective action for these fields including an infinite number of F-type higher derivative is determined entirely by a single holomorphic function.

$$F(X^I, W^2) = \sum_{h=0}^{\infty} F_h(X^I) W^{2h}$$

- Here $\mathbf{W^2}$ is the graviphoton field strength and $\mathbf{X^I}$ are the vector multiplet moduli.
- The functions $\mathbf{F_h}$ are computed by the topological string amplitudes at h-loop.

Witten; Bershadsky, Cecotti, Ooguri, Vafa; Antoniadis, Gava, Narain, Taylor

Entropy

- Consider an action $\int \mathcal{L}$ with arbitrary higher derivative corrections of the form R^2 etc in addition to the Einstein Hilbert term. These corrections are expected to modify not only the black hole solution but the Bekenstein-Hawking area formula itself.

Bekenstein-Hawking-Wald Entropy

There is an elegant formal expression for the entropy of a black hole given such a general higher derivative action.

$$S = 2\pi \int_{S^2} \epsilon_{ab}\epsilon_{cd} \frac{\delta \mathcal{L}}{\delta R_{abcd}},$$

such that the first law of thermodynamics is satisfied.

$$\delta M = T\delta S - \delta W.$$

Solution: Attractor Geometry

- Using supersymmetry, the geometry and the values of the vectors fields and scalars near the black hole horizon are determined by solving *algebraic* equations

$$p^I = \mathrm{Re}[X^I] \qquad q_I = \mathrm{Re}[\frac{\partial F}{\partial X^I}],$$

- The attractor metric is determined by the central charge of the supersymmetry algebra evaluated at the attractor values.
- The Wald entropy can then be evaluated for this attractor geometry.

Cardoso, de Wit, Mohaupt (Kappeli)

Ferrara, Kallosh, Strominger

Ensemble

- The Wald entropy thus computed can be viewed as a Legendre transform

$$S_{\mathsf{BH}}(q,p) = \mathcal{F}(\phi,p) - \phi\frac{\partial}{\partial\phi}\mathcal{F}(\phi,p).$$

- This defines a `free energy' in a *mixed* ensemble in which magnetic charges **p** and electric potentials ϕ conjugate to **q** are held fixed.

Ooguri, Strominger, Vafa

where the `free energy' is defined by

$$\mathcal{F}(\phi, p) = -\pi \mathrm{Im}\left[F\left(p + \frac{i}{\pi}\phi, 256\right)\right].$$

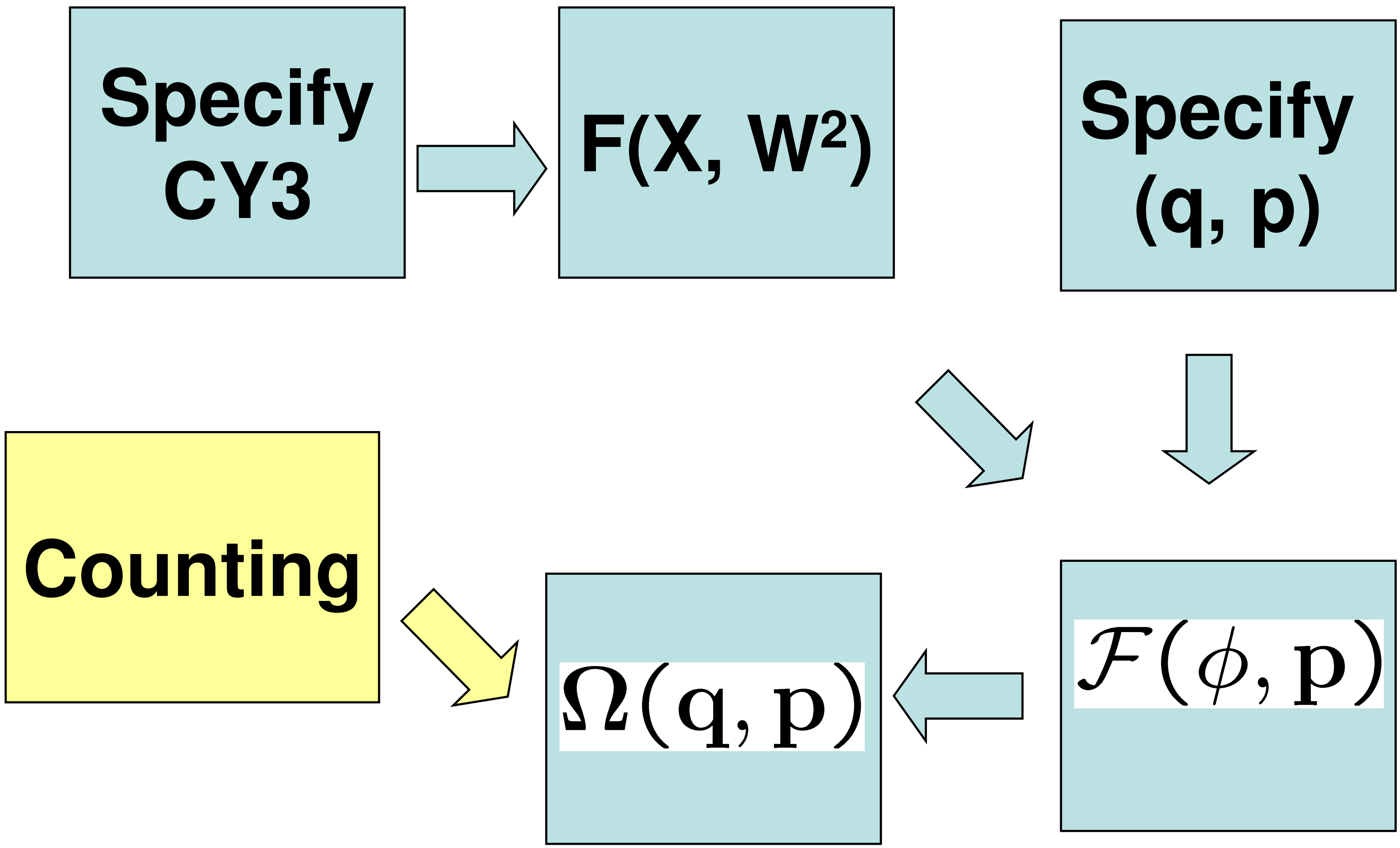
and the potentials ϕ are determined by

$$q = -\frac{\partial}{\partial \phi}\mathcal{F}(\phi, p).$$

- Consider now the partition function

$$
\begin{aligned}
Z_{\mathsf{BH}}(\phi, p) &= e^{\mathcal{F}(\phi,p)} \\
&\equiv \sum_q \Omega(q,p) e^{-\phi q},
\end{aligned}
$$

- Note that given $\mathcal{F}$ this *defines* black hole degeneracies $\Omega(p,q)$ by inverse Laplace transform.


Specify CY3
$F(X, W^2)$
Specify (q, p)
Counting
$\Omega(q, p)$
$\mathcal{F}(\phi, p)$

- Executing each of these arrows is generically not possible in practice.
- To actually get numbers out, need to make a clever choice of the **CY3** and black hole charges **(q, p)** such that all ingredients are under computational control.

- IIA on **K3£ T**$^{\mathbf{2}}$ $\mathbf{n_v} = 24$
- IIA on **K3£ T**$^{\mathbf{2}}$ orbifold $\mathbf{n_v} = 16$

Type-IIA on K3 £ T^2

- Dual to Heterotic on **T^4 £ S^1 £ S^1**
- A heterotic state **(n, w)** with winding **w** and momentum **n** in right moving ground state.
- We can excite 24 left-moving oscillators with total oscillator number **N_L** subject to Virasoro constraint

$$N_L -1 = n\ w \text{ ´ } N$$

- Partition function

$$Z(\beta) = \sum \Omega(N) e^{-2\pi\beta N},$$

- For 24 left-moving bosons we get

$$\begin{aligned} Z(\beta) &= \frac{1}{q \prod (1-q^n)^{24}} \\ &= \frac{1}{\eta^{24}(q)}, \end{aligned}$$

- Here q = exp(-2π β)

- Modular property and asymptotics

$$
\begin{aligned}
Z(\beta) &= \beta^{12} Z(-1/\beta) \\
&\sim \beta^{12} \exp \frac{2\pi}{\beta}
\end{aligned}
$$

- Follows from the fact that **$\eta^{24}(q)$** is a modular form of weight **12** & **$\eta^{24}(q)$ » q**, for small q. round state energy is -1.

- Microscopic degeneracy

$$\Omega(N) = \frac{1}{i} \int_C d\beta \, e^{2\pi\beta N} \, Z(\beta).$$

- Large N asymptotics governed by high temperature, β ! 0 limit

$$\Omega(N) = \frac{1}{i} \int_C d\beta \, \beta^{12} \, e^{2\pi\beta N + 2\pi/\beta}.$$

This is an integral of Bessel type.

From this integral representation we see

$$\boxed{\Omega(nw) \sim I_{13}(4\pi\sqrt{nw})}$$

Thus the Boltzmann entropy is given by

$$\begin{aligned}\log\Omega(nw) \quad \sim \quad & 4\pi\sqrt{nw} - \frac{27}{2}\log\sqrt{nw} \\ & - \log\sqrt{2} - \frac{675}{32\pi\sqrt{nw}} \\ & - \frac{675\times 9}{2048\pi^2 nw} - \dots\end{aligned}$$

Prepotential can be computed exactly.

$$F_0 = -\frac{1}{2} C_{ab} X^a X^b (\frac{X^1}{X^0})$$

$$F_1 = \frac{i}{128\pi} \log \eta^{24} (\frac{X^1}{X^0}) \sim -\frac{1}{64} (\frac{X^1}{X^0})$$

$$F_h = 0, \quad h > 1.$$

- The charge assignment is
  q_0 = n and **p^1 = w,** all other charges zero.

- Compute free energy $\mathcal{F}$
- Compute the inverse Laplace transform over the 24 electric potentials.
- 22 integrals Gaussian,1 integral over a constant, 1 integral of Bessel type. The index of Bessel is ($\mathbf{n_v}$ + 2)/2 =13 $\mathbf{n_v}$ =24

$$\Omega(n,w) \sim I_{13}(4\pi\sqrt{nw})$$

(*overall normalization cannot be fixed and naively has a w^2 factor in front*)

Type-IIA on K3 £ T^2 orbifold

- Heterotic dual is a $\mathbf{Z_2}$ orbifold.
- The orbifold action α flips the two E_8 factors and carries out a half-shift along a circle. This reduces the rank by 8 giving us $\mathbf{n_v}$ = 16 instead of 24.
- For a state with charge vector Q now have

$$\Omega(Q^2) = I_9(4\pi\sqrt{Q^2/2})$$

Partition Function

$$Z(\beta) = \mathbf{Tr}\,\frac{1+\alpha}{2}e^{-2\pi\beta N}$$

$$= \frac{1}{2}\Big[\frac{1}{\eta^{24}(q)} + \frac{1}{\eta^{8}(q)\eta^{8}(q^{2})}\Big]$$

$$I_{13}(4\pi\sqrt{Q^2/2}) \qquad\qquad I_9(2\pi\sqrt{Q^2/2})$$

$$wrong\ index \qquad\qquad wrong\ argument$$

- Mistake? The physical charge operator invariant under orbifold projection is

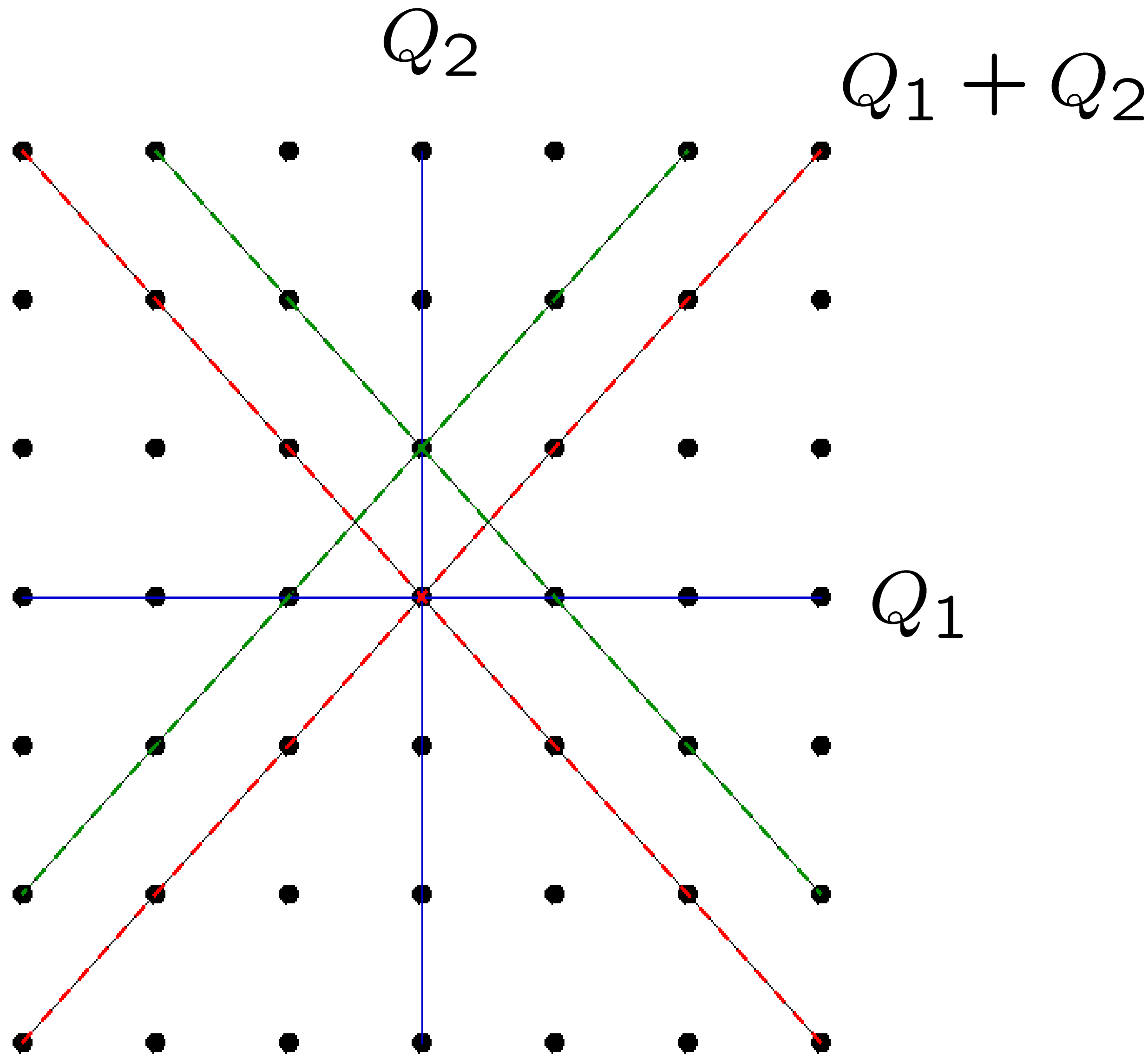
$$Q = Q_1 + Q_2$$

- Some states that are charged under the E_8's in the original theory are neutral in the orbifold.

$$Q(|P, -P > + | - P, P >) = 0$$

- We should sum over these additional states which gives an $\mathbf{E_8}$ theta function at level 2.

$$\Theta_{E_8}(q^2)$$


Q_2
$Q_1 + Q_2$
Q_1

- Write the product of two $\mathbf{E_8}$ lattices in terms of sum of products of shifted level 2 lattices. Check theta identity

$$\Theta^2_{E_8[1]}(q) = \sum_{\mathbf{p}} \Theta^2_{E_8[2];\mathbf{p}}(q)$$

- Here {**p**} are the fundamental weights of 1, 248, 3875. Their orbits under Weyl group give the 248 points in the unit cell $\mathbf{E_8/\ 2E_8}$.

$$Z(\beta) \;=\; \frac{1}{2}[\frac{\Theta_{E_8}(q^2)}{\eta^{24}(q)} + \frac{1}{\eta^8(q)\eta^8(q^2)}]$$

$$I_9(4\pi\sqrt{Q^2/2})$$

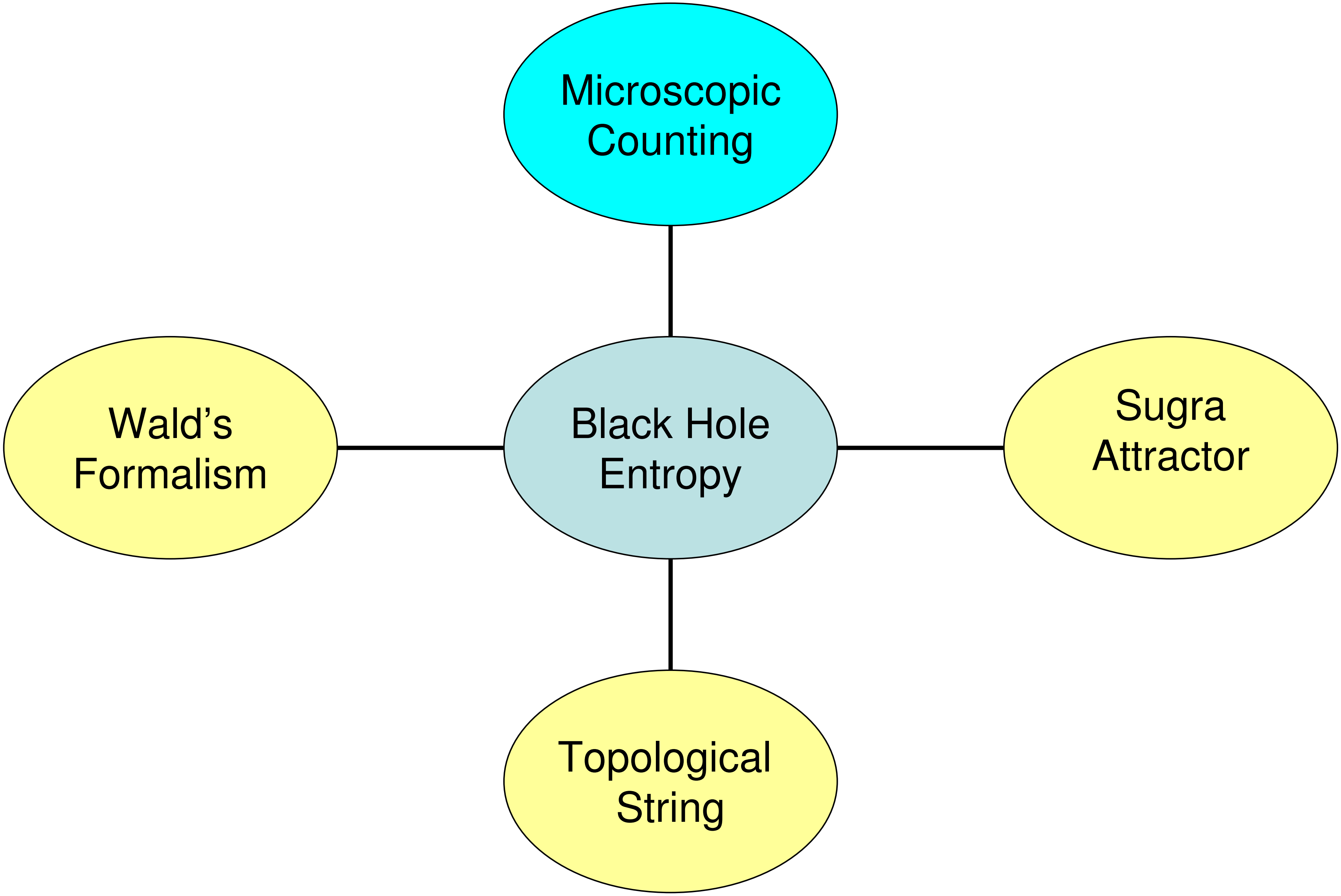
- Similarly we uniformly obtain $I_9(4\pi\sqrt{Q^2/2})$ for all possible choices of the charge vector Q in the untwisted, twisted, or shifted lattice.

Conclusions

- Precise accounting of macroscopic entropy in terms of microstate counting to all orders in an asymptotic expansion.
- It is remarkable how several independent formalisms such as the Wald formula, attractor geometry, and the topological string are incorporated into a coherent structure in string theory in precise agreement with its quantum spectrum.

- Cloaking a null singularity.

  Examples with Euclidean signature are well-known such as orbifolds, conifolds where the stringy quantum geometry is nonsingular even though Riemannian geometry is singular. Here we have an example with Lorentzian signature where stringy corrections modify the classical geometry significantly.


Microscopic Counting
Wald's Formalism
Black Hole Entropy
Sugra Attractor
Topological String

Comments

- Index = absolute number in all these examples. BPS short multiplets cannot combine into long multiplets and the spectrum is stable.
- The Wald entropy is field redefinition independent even though the area of the horizon is not.

Open problems

- Understand the role of D-terms. CFT?
- Generalizations:

  Dyons, Higher dimensions, Spin, Rings..
- Resolve some of the puzzles for perturbative winding states of Type-II.
- Beyond asymptotic expansion?
- Holomorphic anomaly and the measure.